ANALYSIS ON PROPERTIES OF VECTOR SPACES OVER PRE A*-ALGEBRAS

JONNALAGADDA VENKATESWARA RAO\(^1\), T. NAGESWARA RAO\(^2\), S.R. RAVI KUMAR EMANI\(^3\),
M.N. SRINIVAS\(^4,\)∗, B.J. BALAMURUGAN\(^5\)

\(^1\)Department of Mathematics, School of Science and Technology, United States International University, Nairobi,
Kenya
\(^2\)Department of Mathematics, Koneru Lakshmaiah Education Foundation, Green Fields, Vaddeswaram-522502,
Guntur, Andhra Pradesh, India
\(^3\)V.R.Siddhartha Engineering College, Vijayawada, Andhra Pradesh, India
\(^4\)Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore—632014,
Tamilnadu, India
\(^5\)Division of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai—600127,
Tamilnadu, India

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Abstract. In this work the perception of vector space is initiated over Pre A*-algebras. This article discusses the
basic properties of Pre A*-vector spaces, the notion of norm and their worth while representations.

Keywords: pre A*-algebra; pre A*-vector space; normed pre A*-vector space; Boolean pre A*-ring; R-module;
pre A*-metric space; Boolean semiring.

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∗Corresponding author
E-mail address: mnsrinivaselr@gmail.com
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1. **Introduction and Preliminaries**


**Definition 1.1** [7]: A Pre A*-algebra is a system \((A, \land, \lor, (\sim))\) satisfying, for \(x, y, z\) in \(A\):

(a) \(x\sim\sim = x\) (double tilde rule)

(b) \(x \land x = x\) (idempotent rule respecting \(\land\))

(c) \(x \land y = y \land x\) (commutative rule respecting \(\land\))

(d) \((x \land y)\sim = x\sim \lor y\sim\) (De Morgan’s rule)

(e) \(x \land (y \land z) = (x \land y) \land z\) (associative rule respecting \(\land\))

(f) \(x \land (y \lor z) = (x \land y) \lor (x \land z)\) (\(\land\) is distributive over \(\lor\))

(g) \(x \land y = x \land (x\sim \lor y)\) (representation).

**Example 1.1** [7]: A three element Pre A* algebra \(3 = \{0, 1, 2\}\) by means of \(\land, \lor, (\sim)\) described as:

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<tr>
<th>(\land)</th>
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Note 1.1 [7]: From the above (Example 1.1) we note the following: (a) 2 is merely the self-tilde element. (b) 1 is the $\land$ identity element. (c) 0 is the $\lor$ identity element. (d) 2 is the uncertain element.

Example 1.2 [7]: The two element Pre A* algebra ($2 = \{0, 1\}$) by means of $\land, \lor, (-)\sim$ described as:

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2. **PRE A*- VECTOR SPACES (RESULTS AND DISCUSSIONS)**

Definition 2.1: Let V be an abelian group under addition, also A be a Pre A*-algebra. V is named a Pre A*-vector space over A if there exists a mapping from, $A \times V \rightarrow V$ such that, $\forall u, v \in V$ and $a, b \in A$,

(i) $a \cdot (u + v) = a \cdot u + a \cdot v$

(ii) $a \cdot (b \cdot v) = (a \land b) \cdot v$

(iii) If $a \land b = 0$, then $(a \lor b) \cdot v = a \land v + b \cdot v$

(iv) $1 \cdot v = v$ for all $v \in V$.

Note 2.1: We note the product $a \cdot v$ from the ordered pairs of the above as scalar multiplication.

Theorem 2.1:

Let $A$ be Pre A*-algebra. For all $a, b \in A$, $a + b = (a \land b\sim) \lor (a\sim \land b)$ and $a \cdot b = a \land b$.

Then $(A, +, \cdot)$ exists as Boolean Pre A*-ring.

Proof: By the expression,

$(a \land b\sim) \lor (a\sim \land b) = (a \lor (a\sim \land b)) \land (b\sim \lor ((b\sim) \lor a\sim))$

$= (a \lor b) \land (a \lor b)\sim$

Hence, $a + b = b + a$, follows by above

Consider, $(a + b) + c = (a \land b\sim \land c\sim) \lor (a\sim \land b \land c\sim) \lor (a\sim \land b\sim \land c) \lor (a \land b \land c)$

The above is symmetric in $a, b, c$ and therefore, + is associate and commutative.

For any $a \in A$, consider $a + 0 = (a \land 0\sim) \lor (a\sim \land 0)$

$= (a \land 1) \lor (a\sim \land a)$ (since, $a\sim \land 0 = a\sim \land a$)

$= a \land (1 \lor a\sim) = a \land (a\sim \lor 1) = a \land 1$ (by representation) = $a$. 
Similarly, we can see that $0 + a = a$. Hence, 0 is the additive identity in $A$.

Further, note that, $a + (a \land a \sim) = [a \land (a \land a \sim) \lor (a \sim \land (a \land a \sim))]$

\[= [a \land (a \lor a)] \lor [(a \sim \land a \sim) \land a] \]

\[= (a \land a) \lor (a \sim \land a) = a \lor (a \sim \land a) = a \lor a = a.\]

This leads to $a + (a \land a \sim) = a$ for each $a$ in $A$.

Similarly, we can verify that $(a \land a \sim) + a = a$ for each $a$ in $A$.

By above, we conclude that $a + 0 = a = a + (a \land a \sim)$ and hence, $a \land a \sim = 0$, the additive identity for each $a$ in $A$.

To prove that every element of $A$ has additive inverse:

Consider, $a + b = (a \land b \sim) \lor (a \sim \land b)$. Put $b = a$.

Then, $a + a = (a \land a \sim) \lor (a \sim \land a) = a \land a \sim = 0$, the additive identity for each $a$ in $A$ (by above).

Hence, $a$ is additive inverse of $a$ in $A$. Therefore, $(A, +)$ is an abelian group.

Clearly, the multiplication is associative in $A$ (since, $\land$ is associative in $A$).

To prove verify the distributive laws in $A$.

Let $a, b, c \in A$.

Consider, $a \cdot (b + c) = a \land [(b \land c \sim) \lor (b \sim \land c)]$

\[= [a \land (b \land c \sim)] \lor [a \land (b \sim \land c)] \]

\[= [(a \land b) \land c \sim] \lor [(a \land c) \sim] \lor [(a \land c) \sim] \lor [(a \land c) \land b \sim] \]

\[= [(a \land b) \land (a \land c)] \lor [(a \land b) \land (a \land c)] \lor [(a \land b) \land (a \land c)] \lor [(a \land b) \land (a \land c)] \]

\[= [(a \land 0) \land b] \lor [(a \land b) \land (a \land c)] \lor [(a \land 0) \land c] \lor [(a \land c) \land b \sim] \]

\[= [((a \land b) \land (a \land b)) \lor [(a \land b) \land (a \land c)] \lor [(a \land c) \land (a \land c)] \lor [(a \land c) \land (a \land c)] \]

(since, $a \land 0 = a \land a \sim$)

\[= [a \land b] \land ((a \land b) \lor (a \land c)) \lor [(a \land c) \land ((a \land c) \lor (a \land b))].\]
\[(a \land b) \land c \Rightarrow [(a \land c) \land b] \quad (2)\]

By (1) and (2), \(a \cdot (b + c) = a \cdot c + a \cdot c\). Since, \(\cdot\) is commutative (as \(\land\) is so), we have the other distributive law. Thus, \((A, +, \cdot)\) is a Pre A*-ring with identity 1.

Since, \(a \cdot a = a \land a = a\) for all \(a\) in \(A\), \((A, +, \cdot)\) is a Boolean Pre A*-ring in which 0 and 1 as required.

**Example 2.1:** Let \(A\) be any Pre A*-algebra and \(V\) be the additive group of the resultant Pre A*-ring as in the 2.1 theorem. Then \(V\) is an A-vector space if for \(a \in A\) and \(v \in V\), \(av \in A\).

**Theorem 2.2:** Let \(R\) be any ring with 1. Suppose that there is defined a subset \(A\) of \(R\) as \(A = \{r \in R / r^2 = r\} \land (\forall s \in R\}\), set of central idempotents. Then, \((A, \lor, \land, (-)\lor)\) stands as Pre A*-algebra, through operations: \(x \lor y = x + y - x \cdot y; x \land y = x \cdot y\) and \(x\lor = 1 - x\), for all \(x, y \in A\).

**Proof:** For that entire \(x, y\) in \(A\), we verify the postulates as required.

(i) \(x\lor = (x\lor)\lor = (1 - x)\lor = 1 - (1 - x) = 1 - 1 + x = x\).

(ii) and (iii) are clear.

(iv) \((x \land y)\lor = (x \cdot y)\lor = 1 - x \cdot y\).

Also consider \(x\lor \lor y\lor = (1 - x) + (1 - y) - (1 - x)(1 - y) = 1 - x \cdot y\).

(v) Clearly \(\land\) is associative.

(vi) Consider, \(x \land (y \lor z) = x \cdot y + x \cdot z - x \cdot y \cdot z \quad (I)\)

Also consider, \((x \land y) \lor (x \land z) = (x \cdot y) \lor (x \cdot z) = x \cdot y + x \cdot z - x \cdot y \cdot z \quad (II)\)

(since \(x^2 = x\)).

Hence, by (I) and (II), the result follows as required.

(vii) Consider, \(x \land (x\lor \lor y) = x \cdot (1-x) + x \cdot y - x \cdot (1-x) \cdot y\). Hence, the result follows as required.

Therefore, \((A, \lor, \land, (-)\lor)\) is an algebra as required.

**Illustration 2.2:** Let us consider the Pre A*-algebra \(A\) and \(R\) as in the above 2.2 theorem. If \(V\) is the additive group of the ring \(R\), then \(V\) is a Pre A*-vector space over \((A, \lor, \land, (-)\lor)\) with the similar scalar product as discussed above.

**Illustration 2.3:** Let \((A = P(S), \land, \lor, (-)\lor)\) be the Pre A*-algebra of all subsets of a set \(S\) (\(A = P(S)\), power set of \(S\)) and \(V = \{v / v: S \rightarrow G\}\), the functions of \(S\) into a group \(G\) with respect to addition; any \(u, v \in V; a \in A\) (\(a = \text{subset of } S\)), define, \((u + v) (p) = u (p) + v (p)\) for all \(p \in S\).
and \( (av)(p) = v \) if \( p \in a \); \( (av)(p) = 0 \) if \( p \not\in a \). At that juncture \( V \) is a Pre \( A^* \)-vector space over \( A \).

**Illustration 2.4:** An illustration of a Pre \( A^* \)-vector space is \( L_n(A) = A^n \), where \( A^n = A \times \cdots \times A \) (\( n \) factors). In this instance, we define, the vector addition and scalar multiplication defined as follows:

(i) \((a_1, \ldots, a_n) + (b_1, \ldots, b_n) = ((a_1 \wedge b_1) \vee (a_1 \wedge b_1), \ldots, (a_n \wedge b_n) \vee (a_n \wedge b_n)) \) for all \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n \) and

(ii) \( a \cdot (b_1, \ldots, b_n) = (a \wedge b_1, \ldots, a \wedge b_n) \), for all \( a \in A \) and \((b_1, \ldots, b_n) \in A^n \).

Here, + is a binary operation on \( A^n \) and \( \cdot \) (scalar multiplication) is a map from \( A \times A^n \rightarrow A^n \).

**Verification:** Left to the reader as it is straightforward verification.

**Theorem 2.3:** Let \( A^n \) be a Pre \( A^* \)-vector space over \( A \). Then \( A^n \) is a Pre \( A^* \)-algebra.

**Proof:** Let \( u, v \in L_n(A) \).

Define, \( \sim u \vee v = (u_1, u_2, \ldots, u_n) \vee (v_1, v_2, \ldots, v_n) = (u_1 \vee v_1, u_2 \vee v_2, \ldots, u_n \vee v_n) \);

\( u \wedge v = (u_1, u_2, \ldots, u_n) \wedge (v_1, v_2, \ldots, v_n) = (u_1 \wedge v_1, u_2 \wedge v_2, \ldots, u_n \wedge v_n) \) and

\[ (u)^\sim = (u_1, u_2, \ldots, u_n)^\sim = (u_1^\sim, u_2^\sim, \ldots, u_n^\sim). \]

1. Consider \( (u)^\sim = u^\sim = ((u_1^\sim, u_2^\sim, \ldots, u_n^\sim))^\sim = (u_1, u_2, \ldots, u_n) = u \), for all \( u \in A^n \).

2. Consider \( u \wedge u = (u_1, u_2, \ldots, u_n) \wedge (u_1, u_2, \ldots, u_n) = (u_1, u_2, \ldots, u_n) = u \), for all \( u \in A^n \).

3. Let \( u, v \in L_n(A) \). Consider \( u \wedge v = (u_1, u_2, \ldots, u_n) \wedge (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) \wedge (u_1, u_2, \ldots, u_n) = v \wedge u \), for all \( u, v \in L_n(A) \).

4. Consider, \( (u \wedge v)^\sim \)

\[ = (u_1^\sim, u_2^\sim, \ldots, u_n^\sim) \vee (v_1^\sim, v_2^\sim, \ldots, v_n^\sim) \]

\( = u^\sim \vee v^\sim \), for all \( u, v \in A^n \).

5. Consider, \( u \wedge (v \wedge w) = ((u_1, u_2, \ldots, u_n) \wedge (v_1, v_2, \ldots, v_n)) \wedge (w_1, w_2, \ldots, w_n) \)

\( = (u \wedge v) \wedge w \), for all \( u, v, w \in A^n \).

6. Consider, \( u \wedge (v \lor w) = (u_1, u_2, \ldots, u_n) \wedge ((v_1, v_2, \ldots, v_n) \lor (w_1, w_2, \ldots, w_n)) \)

\( = ((u_1, u_2, \ldots, u_n) \wedge (v_1, v_2, \ldots, v_n)) \lor ((v_1, v_2, \ldots, v_n) \wedge (w_1, w_2, \ldots, w_n)) \)
= (u \land v) \lor (u \land w), \text{ for all } u, v, w \in A^n.

(7) Consider, \ u \land (u\lor v) = (u_1, u_2, \ldots, u_n) \land ((u_1, u_2, \ldots, u_n)\lor (v_1, v_2, \ldots, v_n))
\ = (u_1, u_2, \ldots, u_n) \land (v_1, v_2, \ldots, v_n) = u \land v.

Thus, (A^n, \land, \lor, (\sim)\sim) \text{ is an algebra as required.}

**Lemma 2.1:** Let \( V \) be an arbitrary Pre A*-vector space over a Pre A*-algebra. For all \( v \in V \) and a in \( A \), \( 0 \lor v = 0 \) and \( a \lor 0 = 0 \).

**Proof:** Let us consider \( v = 1 \lor v = (0 \lor 1) \lor v = 0 \lor 1 \lor v = 0 \lor v \lor v \). Hence, as required.

Also the second result is obvious. Hence, \( a \lor 0 = 0 \).

**Lemma 2.2:** Let \( V \) be an arbitrary Pre A*-vector space over \( A \).

Then, \( a (-v) = -a \lor v \) for all a in \( A \) and \( v \) in \( V \).

**Proof:** Consider \( 0 = a \lor 0 = a \lor (v +(-v)) = a \lor v + a \lor (-v) \). Hence, as required.

**Note 2.2 [8]:** Henceforth, to enable the subsequent consequences, we consider \( a, b \in A \) such that \( a \lor b = 1 \) (so that \( a \lor a\sim = 1 \) and \( a \land a\sim = 0 \) in \( A \)).

**Lemma 2.3:** Let \( V \) be an arbitrary Pre A*-vector space over \( A \). If \( a, b \in A \) such that \( a \lor b = 1 \) and \( v \in V \), then (i) \( a\sim v = v - a \lor v \) and (ii) \( (a \lor b) \lor v = a \lor v + b \lor v - a \land b \lor v \).

**Proof:** (i) Consider \( v = 1 \lor v = (a \lor a\sim) \lor v = a \lor v + a\sim \lor v \). Hence, the result follows.

(ii) Consider, \( (a \lor b) \lor v = [a \lor (b \land a\sim)] \lor v \) (since, \( a \lor b = a \lor (b \land a\sim) \))
\ = a \lor v + (b \land a\sim) \lor v \) (since, \( a \land (b \land a\sim) = 0 \))
\ = a \lor v + b \land (a\sim \lor v) = a \lor v + b \lor (v + (-a \lor v)) = a \lor v + b \lor v - a \land b \lor v.

Hence, result as required.

**Theorem 2.4:** Let \( V \) be a Pre A*-vector space over a \( A \), such that \( a \lor b = 1 \), for all \( a, b \in A \); and let \( R = (R, +, \cdot) \) be a Boolean Pre A*-ring corresponding to \( A \). Then the necessary and sufficient condition for \( V \) is a module over \( R \) is \( v + v = 0 \) for all \( v \in A \).

**Proof:** Let \( a, b \in R \) and \( v \in V \). Let us observe, \( (a + b) \lor v = (a \lor a\sim) \lor v = a \lor v + a\sim \lor v \)
\ = a \lor v - b \lor v \lor v - a \land b \lor v \)
\ = a \lor (v - b \lor v) \lor v - a \land b \lor v \)
\ = a \lor v + b \lor (v - a \lor v) = a \lor v + b \lor v - 2 a \land b \lor v.

Successively, \( V \) is an \( R \)-module equivalently \( 2 a \land b \lor v = 0 \) for all \( a, b \in A \) and \( v \in V \), or correspondingly, \( v + v = 0 \) for all \( v \in A \).
**Definition 2.2:** A Pre A*-vector space $V$ over $A$ is said to be Pre-A*-normed if and only if there exists a mapping $\|\cdot\|: V \rightarrow A$ such that (1) $\|v\| = 0$ if and only if $v = 0$ and (2) $\|a v\| = a \|v\|$ for all $a \in A$ and $v \in V$.

**Note 2.3:** The Pre A*-vector spaces of above examples 2.1 and 2.3 are normed.

**Theorem 2.5:** For a Pre A*-vector space $V$ over $A$ (with $a \lor b = 1$ for all $a, b$ in $A$), the subsequent are equivalent: (1) $V$ is Pre A*-normed (2) To each $v \in V$, there relates an element $a_v \in A$ such that (i) $a_v v = v$ and (ii) if $b \in A$ and $b v = v$, then $b a_v = a_v$. ($a_v$, for a specified $a$, is exceptional).

**Proof:** Suppose that (1) holds. So $V$ is A-normed. Let $a_v = \|v\|$.

(i) Consider, $\|v - a_v v\| = \|a_v \sim v\| = a_v \sim \|v\| = a_v = 0$. Hence, $a_v v = v$.

(ii) Let $b \in A$ and $b v = v$. Consider, $a_v = \|v\| = \|b v\| = b \|v\| = b a_v$. Hence, $b a_v = a_v$.

Suppose that (2) holds.

Suppose $c \in A$, $v \in V$ and $c v = 0$. Then consider, $c \sim v = v - c v = v$ (as $c v = 0$). Hence, $c \sim v = v$.

Then, $c \sim a_v = a_v$ (By hypothesis). This indicates, $c c \sim a_v = c a_v$. Hence, $c a_v = 0$ (as $c c \sim a_v = 0$).

Hence, if $b \in A$ and $b (c v) = c v$, then, $b \sim (c v) = c v - b (c v) = c v - c v = 0$ (as $b (c v) = c v$). Therefore, $b \sim (c v) = 0$ and hence, $b c a_v = 0$.

Consider $(c a_v) (c v) = c c a_v v = c v$. Thus, $(c a_v) (c v) = c v$ (X) Also, consider, $(a_v c) (c v) = c v$ (Y)

We conclude that $a_c v = c a_v$.

Let us define $\|v\| = a_v$. By above, $a_v = \|c v\| = c \|v\|$.

So, therefore, the mapping $\|\cdot\|$ describes as required.

**Corollary 2.1:** If $V$ is a Pre A*-normed vector space (over $A$), then $\|u + v\| \leq \|u\| \lor \|v\|$ for all $u, v \in V$.

**Proof:** By above results, we are considering $\|v\| = a_v$ (so that $\|v\| v = a_v v = v$).

Observe that $(\|u\| \lor \|v\|) (u + v) = \|u\| (u + v) + \|v\| (u + v) - (\|u\| \land \|v\|) (u + v) = \|u\| u + \|u\| v + \|v\| u + \|v\| v - \|u\| (\|v\| (u) + \|v\| (v))$.

Therefore, $\|u + v\| = \|u\| \lor \|v\| (u + v) \| = (\|u\| \lor \|v\|) \|u + v\|$.

Here, by the partial order on the Pre A*-algebra $A$ [4], we can observe as required.
Corollary 2.2: If $V$ is a Pre A*-normed vector space, then $d(u, v) = \|u - v\|$ defines Pre A*-metric on $V$.

Proof: (i) Suppose that $d(u, v) = 0$ if and only if $\|u - v\| = 0$ if and only if $u - v = 0$ if and only if $u = v$.

(ii) Consider, $d(u, v) = \|u - v\| = \|(-1)(v - u)\| = \|(v - u) - (-1)\sim(v - u)\|$

(Since, $a v = v - a\sim v$, for all $a \in A$ and $v \in V$, by above lemma)

$= \|v - u\| = d(v, u)$. Hence, $d(u, v) = d(v, u)$ for all $u, v \in V$.

As the two expressions are symmetric in $u$ and $v$. Hence, $d(u, v) = d(v, u)$.

(iii) Consider $d(u, w) = \|u - w\| \leq \|u - v\| \lor \|v - w\| = d(u, v) \lor d(v, w)$.

Thus, $d$ becomes a metric as required.

Definition 2.3 [5]: A system $(R, +, .)$ is called a Boolean semiring if it satisfies:

(i) $(R, +)$ is an additive abelian group.

(ii) $(R, .)$ is a semigroup of idempotents in the sense, $a a = a$, for all $a \in R$

(iii) $a \cdot (b + c) = a \cdot b + a \cdot c$ and

(iv) $a b c = b a c$ for all $a, b, c \in R$.

Theorem 2.6: Let $V$ be a normed Pre A*-vector space over $A$ and let, for $u, v$ in $V$, $u v = \|u\| v$. Then $(V, +, .)$ is a Boolean semiring.

Proof: $(V, +, .)$ is a Boolean semiring because of the following:

(1) Note that $(V, +)$ is an additive abelian group;

(2) To verify that $(V, .)$ is a semigroup of idempotents:

For any $u, v, w \in V$, consider $(u v) w = \|u v\| w = \|u\| \|v\| w$.

Also consider, $u (v w) = \|u\| (v w) = \|u\| \|v\| w$. Hence, $(u v) w = u (v w)$ for all $u, v, w \in V$.

For any $u \in V$, $u \cdot u = \|u\| \cdot u = u$

(as by previous lemma, $a_v v = v$, and by $a_v = \|v\|$, $\|v\| v = v$).

(3) For any $u, v, w \in V$, let us consider, $u \cdot (v + w) = \|u\| \cdot (v + \|u\| w$.

Also $u v + u w = \|u\| v + \|u\| w$. Hence, $u(v + w) = u v + u w$ for all $u, v, w \in V$.

(4) For any $u, v, w \in V$, consider $(u v) w = \|u v\| w = \|u\| \|v\| w$. Also consider, $(v u) w$

$= \|v u\| w = \|v\| \|u\| w = \|u\| \|v\| w$ (since, $\|u\|$, $\|v\| \in A$ implies, $\|u\| \lor \|v\| = \|v\| \lor \|u\|$ and hence, we follow that $\|u\| \|v\| = \|v\| \|u\|$).
**Theorem 2.7:** If \( v \in V \), uniquely as \( v = a_1v_1 + a_2v_2 + \ldots + a_nv_n \), where \( v_1, v_2, \ldots, v_n \in V \) and \( a_1, a_2, \ldots, a_n \in A \), then \( a = a_1v_1 \lor a_2v_2 \lor \ldots \lor a_nv_n \) (where \( a_i \lor a_j = a_i \) if \( i = j \) and is 0 if \( i \neq j \)) is the duplicator of \( v \) such that \( a_i = b a_i \).

**Proof:** To verify that \( a v = v \). Consider, \( a v = (a_1v_1 + a_2v_2 + \ldots + a_nv_n) \)

\[
= (a_1v_1 + \cdots + a_nv_n) a_1v_1 + \cdots + (a_1v_1 + \cdots + a_nv_n) a_nv_n
\]

\[= a_1(a_1v_1) + a_2(a_1v_1) + \cdots + a_n(a_1v_1) + \cdots + a_1(a_nv_n) + a_2(a_nv_n) + \cdots + a_n(a_nv_n)
\]

\[= a_1v_1 + a_2v_2 + \cdots + a_nv_n \quad (a_i \lor a_j = a_i \text{ if } i = j \text{ and is 0 if } i \neq j)
\]

Suppose that \( b v = v \) for some \( b = b_1v_1 \lor b_2v_2 \lor \ldots \lor b_nv_n \), similarly taken as \( a = a_1v_1 \lor a_2v_2 \lor \ldots \lor a_nv_n \). Then, \( v = b v = b_1v_1 + b_2v_2 + \cdots + b_nv_n \).

This implies, \( a_i = b a_i \) for all \( i \) (by the uniqueness of \( v \)).

**Definition 2.4:** A finite subset of nonzero elements \( \{v_1, v_2, \ldots, v_n\} \in V \) is named linearly independent over \( A \) if and only if \( a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0 \) and \( a_1, a_2, \ldots, a_n \neq 0 \) imply that \( v_1 + v_2 + \ldots + v_n = 0 \). A subset of nonzero elements of \( V \) is called linearly independent over \( A \) if and only if every limited subset of \( S \) is linearly independent.

**Definition 2.5:** A subset \( S \) of \( V \) spans \( V \) if and only if each \( v \in V \) can be written as a finite sum \( v = \sum g \in S a_g g \), \( a_g a_h = 0 \) for \( g \) different from \( h \) and \( a_g = 0 \) for nearly all \( g \) in \( S \).

**Definition 2.6:** A basis of \( V \) is (i) linearly independent subset of \( V \); and (ii) spans \( V \).

**Example 2.5:** Let \( V \) be a Pre A*-vector space over \( A \) as in 2.3 example. Let \( K \) be the set of all nonzero constant maps in \( V \). Then, \( K \) is a basis of \( V \). Let \( K = \{f_1, f_2, \ldots, f_n\} \subseteq V \). To verify that \( \{f_1, f_2, \ldots, f_n\} \) is linearly independent. Suppose that \( f_1a_1 + f_2a_2 + \ldots + f_na_n = 0 \) and \( f_1, f_2, \ldots, f_n \neq 0 \). Then, \( a_1 + a_2 + \ldots + a_n = 0 \) (as each \( f_i \) is a constant function).

Hence, \( K = \{f_1, f_2, \ldots, f_n\} \) is linearly independent. Let \( v_1 \in V \) and \( a_v \in A \) such that \( a_v u = v \) if \( u = v \) and 0 if \( u \neq v \). Then we can see that \( v_1 = a_v v_1 + a_v v_2 + \ldots + a_v v_n \). Therefore, \( K \) is a basis of \( V \).

**Lemma 2.4:** Let \( V \) be a normed Pre A*-vector space and \( G^* \) be a basis of \( V \). If \( g \in G^* \), then, (i) \(-g \in G^* \), (ii) if \( g, h \in G^* \) in addition \( g + h \neq 0 \), \( g + h \in G^* \).

**Proof:** As \( G^* \) spans \( V \), \( -g = \sum k \in G^* a_k k \), where, \( a_k a_h = 0 \) for \( k \neq h \) also \( a_k = 0 \), nearby all \( k \in G^* \). As, \( g \neq 0 \), \( a_k \neq 0 \) for some \( k (= m, \text{say}) \) in \( G^* \). At that point \(-a_m g = a_m (-g) = a_m m \).
Hence, \( a_m (g + m) = 0 \). As, \( g, m \in G^* \), \( a_m \neq 0 \), in addition to \( G^* \) is independent, \( g + m = 0 \) and therefore, \(-g = m \in G^*\).

If \( g, h \in G^* \) in addition to \( g + h \neq 0 \), we similarly observe that \( a_k (g + h) = a_k k \) for some \( k \in G^* \) plus \( a_k \neq 0 \). This implies \( a_k g + a_k h + a_k (-k) = 0 \). As, \( k \in G^* \) infers, \( -k \in G^* \), \( g + h = k \in G^* \).

**Theorem 2.8:** If \( G^* \) is a basis of \( V \), then \( G^* \) is an additive subgroup \( G \) of \( V \).

**Lemma 2.5:** If \( G \in G^* \), then \( \|g\| = 1 \).

**Proof:** If \( \|g\| = a \), then \( a^\sim g = g - a g = g - \|g\| g = g - g = 0 \). This implies, \( a^\sim g = 0 \). Since, \( g \neq 0 \), we must have \( a^\sim = 0 \). Then by above, \( 0 g = g - a g \), so, \( a g = g \). From this, it follows that \( a = 1 \) and hence, \( \|g\| = 1 \).

**Lemma 2.6:** If \( u = \sum_{i=1}^{n} a_i u_i \), where \( a_i a_j = 0 \) for \( i \neq j \), then \( \|u\| = \sqrt{\sum_{i=1}^{n} a_i \|u_i\|} \).

**Proof:** Suppose the result is true for \( n-1 \). Let \( v = \sum_{i=1}^{n} a_i u_i \) and \( b = \|v\| \).

Then \( b = \| \sum_{i=1}^{n} a_i u_i \| = \sqrt{\sum_{i=1}^{n} a_i \|u_i\|} \) and \( u = a_1 u_1 + v \) (since, \( u = \sum_{i=1}^{n} a_i u_i \)).

Also, \( a_1 v = a_1 (\sum_{i=1}^{n} a_i u_i) = a_1 a_2 u_2 + a_1 a_3 u_3 + \cdots + a_1 a_n u_n = 0 \) (if \( a_i a_j = 0 \) for \( i \neq j \)).

Hence, \( a_1 u = a_1 u_1 \) (by above, since, \( a v = 0 \)).

Then, \( \|v\| = \|u - a_1 u_1\| = \|u - a_1 u\| \) (since, \( a_1 u = a_1 u_1 \) = \( a_1^\sim u = \|a_1^\sim u\| = \|a_1 u\| \).

Hence, \( \|v\| = a_1^\sim \|u\| \).

Thus, \( \|u\| = 1 \|u\| = (a_1 \lor a_1^\sim) \|u\| = a_1 \|u\| \lor a_1^\sim \|u\| = a_1 \|u\| \lor b = \sqrt{\sum_{i=1}^{n} a_i \|u_i\|} \).

**Corollary 2.3:** If \( u = \sum_{i=1}^{n} a_i u_i \), where, where \( a_i a_j = 0 \) for \( i \neq j \) and \( u_1, u_2, \ldots, u_n \in G^* \), then \( \|u\| = \sqrt{\sum_{i=1}^{n} a_i} \).

**Proof:** By above results, the proof is immediate.

**Concluding Remarks**

This work made a stand to study vector spaces over algebra and its useful characterizations as well. The Pre A* -vector space is initiated and observed its various representations. An n-factored set \( L_n (A) (\cong A^n = A \times A \times \cdots \times A \text{ (n-factors)}) \) is observed as a vector space over \( A \) and such a Pre A* -vector space is identified as a Pre A* -algebra as well. The notion of normed Pre A* -vector space is initiated and studied its properties. The method of construction of a Boolean semiring from a normed Pre A* -vector space is obtained. It is noted that the basis of the Pre A* -vector space forms a subgroup of the Pre A* -vector space.
CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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