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FIXED POINT THEOREMS FOR (α,β)-(ϕ,ψ)-RATIONAL CONTRACTIVE TYPE MAPPINGS

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Abstract. In this paper, we introduce the concept of (α,β)-(ϕ,ψ)-rational contractive mapping in b-metric spaces. We establish some fixed point theorems for such mappings and also give an application supporting our results.

Keywords: b-metric space; cyclic (α,β)-admissible mapping; (α,β)-(ϕ,ψ)-contractive mapping.

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1. Introduction and Preliminaries

Fixed point theory occupies a large area in mathematical analysis due to its wide applications in various fields like engineering, computer science, biological science, economics, etc. Banach fixed point theorem is considered as most important results in the field of analysis. Due to its applicability it is extended in different directions.

The concept of b-metric space is introduced by Bakhtin [1] by generalizing the definition of metric space. Further, this concept is extensively used by S. Czerwick [2, 3]. The notion of cyclic (α,β)-admissible mapping was introduced by Alizadeh et. al. [4] and prove some

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fixed point results for mappings in the setting of complete metric spaces. Oratai and Wuthipol [5] established some fixed point theorems for (α,β)-(ψ,ϕ)-contractive mappings in b-metric space. For more information on different types of contractive mappings one can see in [6, 7, 8, 9, 10, 11, 12].

In this paper, we introduce the concept of (α,β)-(ψ,ϕ)-rational contractive mappings and prove some fixed point theorems in the setting of b-metric space.

Definition 1.1. [6] *The function $\phi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are hold:*

i): *ϕ is continuous and non-decreasing;*

ii): *$\phi(t) = 0$ if and only if $t = 0$.*

Example 1. [6] *Let $\phi : [0,\infty) \to [0,\infty)$ defined by $\phi(t) = \sinh^{-1} t$ for all $t \in [0,\infty)$. Then ϕ is an altering distance function.*

Definition 1.2. [1] *Let X be a nonempty set. A b-metric on X is a function $d : X^2 \to [0,\infty)$ if there exists a real number $b \geq 1$ such that the following conditions holds for all $x,y \in X$*

(i): *$d(x,y) = 0$ if and only if $x = y$,*

(ii): *$d(x,y) = d(y,x)$*

(iii): *$d(x,z) \leq b[d(x,y) + d(y,z)]$*

The pair (X,d) is called a b-metric space.

Definition 1.3. [7] *Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*

(i): *b-convergent if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.*

(ii): *b-Cauchy if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.*

Proposition 1.1. [7] *In a b-metric space (X,d), the following assertions hold:*

(p_1): *a b-convergent sequence has a unique limit;*

(p_2): *each b-convergent is b-Cauchy;*

(p_3): *in general, a b-metric is not continuous.*

Lemma 1.2. [4] *Let (X,d) be a b-metric space with coefficient $b\geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x,y\in X$, respectively. Then we have*

$$\frac{1}{b^2}d(x,y)\leq \lim_{n\to\infty}\inf d(x_n,y_n)\leq \lim_{n\to\infty}\sup d(x_n,y_n)\leq b^2 d(x,y).$$

In particular, if $x=y$, then we have $\lim_{n\to\infty} d(x_n,y_n)=0$. Moreover, for each $z\in X$, we have

$$\frac{1}{b}d(x,y)\leq \lim_{n\to\infty}\inf d(x_n,z)\leq \lim_{n\to\infty}\sup d(x_n,z)\leq bd(x,y).$$

Definition 1.4. [7] *Let (X,d) and (X',d') be two b-metric spaces.*

i): *The space (X,d) is b-complete if every b-Cauchy sequence in X is b-convergent.*

ii): *A function $f:X\to X'$ is b-continuous at a point $x\in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.*

Definition 1.5. [7] *Let Y be a nonempty subset of a b-metric space (X,d). The closure $\bar{Y}$ and Y is the set of limits of all b-convergent sequences of points in Y, i.e,*

$$\bar{Y}=\{x\in X: \textit{there exists a sequence } \{x_n\} \textit{ in } Y \textit{ so that } \lim_{n\to\infty}x_n=x.\}$$

Definition 1.6. [7] *Let (X,d) be a b-metric space. Then a subset $Y\subseteq X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which b-converges to an element x, we have $x\in Y$ (i.e. $\bar{Y}=Y$).*

Definition 1.7. [4] *Let X be a nonempty set, f be a self-mapping on X and $\alpha,\beta:X\to[0,\infty)$ be two mappings. We say that f is a cyclic (α,β)-admissible mapping if*

$$x\in X \textit{ with } \alpha(x)\geq 1\Rightarrow \beta(fx)\geq 1$$

and

$$x\in X \textit{ with } \beta(x)\geq 1\Rightarrow \alpha(fx)\geq 1.$$

2. Main Results

Let (X,d) be a b-metric space with coefficient $b\geq 1$ and $f:X\to X$ be a self-mapping. Throughout this paper, unless otherwise stated, for all $x,y\in X$, let

$$M_b(x,y) := \max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fx)d(y,fy)}{1+d(x,y)},\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}.$$

Note that $M_b(x,y) = M_b(y,x)$. If $b = 1$, we write $M(x,y)$ instead $M_b(x,y)$ that is

$$M(x,y) \;:=\; \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}.$$

Definition 2.1. *Let (X,d) be a b-metric space with coefficient $b \geq 1$ and let $\alpha, \beta : X \to [0,\infty)$ be two given mappings. We say that $f : X \to X$ is an (α,β)-(ϕ,ψ)-rational contractive type mapping if the following condition holds*

$$\begin{aligned} x,y \in X \text{ with } \alpha(x)\beta(y) &\geq 1 \\ \Rightarrow \psi(b^2 d(fx,fy)) &\leq \psi(M_b(x,y)) - \phi(M_b(x,y)) \end{aligned} \tag{1}$$

where $\psi, \phi : [0,\infty) \to [0,\infty)$ are altering distance functions.

Theorem 2.1. *Let (X,d) be a complete b-metric space with co-efficient $b \geq 1$, $\alpha, \beta : X \to [0,\infty)$ be two mappings and $f : X \to X$ be an (α,β)-(ϕ,ψ)-rational contractive type mapping. Suppose that*

- **(1):** *one of the following condition holds:*
  - **(1.1):** *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$.*
  - **(1.2):** *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$.*
- **(2):** *f is continuous.*
- **(3):** *f is a cyclic (α,β)-admissible mapping.*

Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in condition (1.1) and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converges to a fixed point of f.

Proof. Let $x_0 \in X$ such that $\alpha(x_0) \geq 1$. Then we construct an iterative sequence $\{x_n\}$, where $x_{n+1} = fx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then x_0 is a fixed point of f, and the proof is finished. Hence, we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $d(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Since f is a cyclic (α,β)-admissible mapping, we have

$$\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) = \beta(fx_0) \geq 1 \Rightarrow \alpha(x_2) = \alpha(fx_1) \geq 1.$$

Continuing this way, we have

$$\alpha(x_{2k}) \geq 1 \quad \text{and} \quad \beta(x_{2k+1}) \geq 1 \tag{2}$$

for all $k \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_2)\beta(x_1) \geq 1$, we get

$$\begin{aligned} \psi(d(fx_0,fx_1)) &\leq \psi(b^3 d(fx_0,fx_1)) \\ &\leq \psi(M_b(x_0,x_1)) - \phi(M_b(x_0,x_1)) \end{aligned}$$

Again, since $\alpha(x_2)\beta(x_1) \geq 1$, we have

$$\begin{aligned} \psi(d(fx_1,fx_2)) &= \psi(d(fx_2,fx_1)) \\ &\leq \psi(b^3 d(fx_2,fx_1)) \\ &\leq \psi(M_b(x_2,x_1)) - \phi(M_b(x_2,x_1)) \\ &= \psi(M_b(x_1,x_2)) - \phi(M_b(x_1,x_2)) \end{aligned}$$

By similar process, we have

$$\begin{aligned} \psi(d(fx_n,fx_{n+1})) &\leq \psi(b^3 d(fx_n,fx_{n+1})) \\ &\leq \psi(M_b(x_n,x_{n+1})) - \phi(M_b(x_n,x_{n+1})) \end{aligned} \tag{3}$$

for each $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} M_b(x_n,x_{n+1}) &= \max\Bigg\{ d(x_n,x_{n+1}), d(x_n,fx_n), d(x_{n+1},fx_{n+1}), \\ &\quad \frac{d(x_n,fx_n)d(x_{n+1},fx_{n+1})}{1+d(x_n,x_{n+1})}, \frac{d(x_n,fx_n)d(x_{n+1},fx_{n+1})}{1+d(fx_n,fx_{n+1})} \Bigg\} \\ &= \max\Bigg\{ d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \\ &\quad \frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})}, \frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_{n+2})} \Bigg\} \\ &= \max\{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})\} \end{aligned}$$

From (3) and the properties of ψ and ϕ, it follows that

$$\begin{aligned} \psi(d(fx_n, fx_{n+1})) &\leq \psi(\max\{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})\}) \\ &\quad -\phi(\max\{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})\}) \\ &< \psi(\max\{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})\}) \end{aligned} \tag{4}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} = d(x_{n+1},x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$, then by (4), we have

$$\begin{aligned} \psi(d(fx_n, fx_{n+1})) &\leq \psi(d(x_{n+1},x_{n+2}) - \phi(d(x_{n+1},x_{n+2}) \\ &< \psi(d(x_{n+1},x_{n+2})) \end{aligned}$$

a contradiction. Therefore,

$$\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} = d(x_n,x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$. By (4), we get

$$\begin{aligned} \psi(d(x_{n+1},x_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\ &\leq \psi(d(x_n,x_{n+1})) - \phi(d(x_n,x_{n+1})) \\ &< \psi(d(x_n,x_{n+1})) \end{aligned} \tag{5}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since ψ is a non-decreasing mapping the sequence $\{d(x_n,x_{n+1})\}$ is decreasing and bounded from below. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. Letting $n \to \infty$ in (5), we have

$$\psi(r) \leq \psi(r) - \phi(r) \leq \psi(r).$$

This implies that $\phi(r) = 0$ and thus $r = 0$. Consequently,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{6}$$

Next we claim that $\{x_n\}$ is a b-Cauchy sequence in X, that is, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_m,x_n) < \varepsilon$ for all $m,n \geq k$.

Assume, to the contrary, that there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$, $m(k)$ is even and $n(k)$ is odd,

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{7}$$

and $n(k)$ is the smallest number such that (7) holds. From (7), we get

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon \tag{8}$$

By the triangle inequality, (7) and (8), we obtain that

$$\begin{aligned} \varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq b[d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})] \\ &< b[\varepsilon + d(x_{n(k)-1}, x_{n(k)})] \end{aligned} \tag{9}$$

Taking limit supremum as $k \to \infty$ in (9), by using (6), we get

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq b\varepsilon. \tag{10}$$

From the triangle inequality, we get

$$d(x_{m(k)}, x_{n(k)}) \leq b[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})] \tag{11}$$

and

$$d(x_{m(k)}, x_{n(k)+1}) \leq b[d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})] \tag{12}$$

Taking limit supremum as $k \to \infty$ in (11) and (12), from (6) and (10), we obtain that

$$\varepsilon \leq b\big(\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1})\big)$$

and

$$\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq b^2\varepsilon.$$

This implies that

$$\frac{\varepsilon}{b} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq b^2\varepsilon \tag{13}$$

Again, using above process, we get

$$\frac{\varepsilon}{b} \leq \lim_{k\to\infty}\sup d(x_{n(k)},x_{m(k)+1}) \leq b^2\varepsilon \tag{14}$$

Finally, we obtain that

$$d(x_{m(k)},x_{n(k)+1}) \leq b[d(x_{m(k)},x_{m(k)+1})+d(x_{m(k)+1},x_{n(k)+1})]. \tag{15}$$

Taking limit supremum as $k\to\infty$ in (15), from (6) and (13), we obtain that

$$\frac{\varepsilon}{b^2} \leq \lim_{k\to\infty}\sup d(x_{m(k)+1},x_{n(k)+1}). \tag{16}$$

Similarly, we have

$$\lim_{k\to\infty}\sup d(x_{m(k)+1},x_{n(k)+1}) \leq b^3\varepsilon \tag{17}$$

By (16) and (17), we get

$$\frac{\varepsilon}{b^2} \leq \lim_{k\to\infty}\sup d(x_{m(k)+1},x_{n(k)+1}) \leq b^3\varepsilon. \tag{18}$$

Relation (2), implies that

$$\alpha(x_{m(k)})\beta(x_{n(k)}) \geq 1.$$

From (1), we have

$$\begin{aligned} \psi(b^3 d(x_{m(k)+1},x_{n(k)+1})) &= \psi(b^3 d(fx_{m(k)},fx_{n(k)})) \\ &\leq \psi(M_b(x_{m(k)},x_{n(k)}))-\phi(M_b(x_{m(k)},x_{n(k)})), \end{aligned} \tag{19}$$

where

$$\begin{aligned} M_b(x_{m(k)},x_{n(k)}) &= \max\Bigg\{d(x_{m(k)},x_{n(k)}),d(x_{m(k)},fx_{m(k)}),d(x_{n(k)},fx_{n(k)}), \\ &\quad \frac{d(x_{m(k)},fx_{m(k)})d(x_{n(k)},fx_{n(k)})}{1+d(x_{m(k)},x_{n(k)})},\frac{d(x_{m(k)},fx_{m(k)})d(x_{n(k)},fx_{n(k)})}{1+d(fx_{m(k)},fx_{n(k)})}\Bigg\} \\ &= \max\Bigg\{d(x_{m(k)},x_{n(k)}),d(x_{m(k)},x_{m(k)+1}),d(x_{n(k)},x_{n(k)+1}), \\ &\quad \frac{d(x_{m(k)},x_{m(k)+1})d(x_{n(k)},x_{n(k)+1})}{1+d(x_{m(k)},x_{n(k)})},\frac{d(x_{m(k)},x_{m(k)+1})d(x_{n(k)},x_{n(k)+1})}{1+d(x_{m(k)+1},x_{n(k)+1})}\Bigg\}. \end{aligned}$$

Taking limit supremum as $k\to\infty$ in above equation and using (6), (10), (13) and (14), we obtain

$$\begin{aligned}
\varepsilon &= \max\left\{\varepsilon,\frac{\varepsilon}{b},\frac{\frac{\varepsilon}{b}\cdot\frac{\varepsilon}{b}}{1+\varepsilon},\frac{\frac{\varepsilon}{b}\cdot\frac{\varepsilon}{b}}{1+\frac{\varepsilon}{b^2}}\right\}=\max\left\{\varepsilon,\frac{\varepsilon}{b},\frac{\varepsilon^2}{b^2(1+\varepsilon)},\frac{\varepsilon^2}{b^2+\varepsilon}\right\}\\
&\le \lim_{k\to\infty}\sup M_b(x_{m(k)},x_{n(k)})\\
&\le \max\left\{b\varepsilon,b^2\varepsilon,\frac{b^2\varepsilon.b^2\varepsilon}{1+b\varepsilon},\frac{b^2\varepsilon.b^2\varepsilon}{1+b^3\varepsilon}\right\}\\
&\le \max\left\{b^2\varepsilon,b^3\varepsilon,b\varepsilon\right\}=b^3\varepsilon.
\end{aligned}$$

Also, we can show that

$$\begin{aligned}
\varepsilon &= \max\left\{\varepsilon,\frac{\varepsilon}{b},\frac{\frac{\varepsilon}{b}\cdot\frac{\varepsilon}{b}}{1+\varepsilon},\frac{\frac{\varepsilon}{b}\cdot\frac{\varepsilon}{b}}{1+\frac{\varepsilon}{b^2}}\right\}=\max\left\{\varepsilon,\frac{\varepsilon}{b},\frac{\varepsilon^2}{b^2(1+\varepsilon)},\frac{\varepsilon^2}{b^2+\varepsilon}\right\}\\
&\le \lim_{k\to\infty}\inf M_b(x_{m(k)},x_{n(k)})\\
&\le \max\left\{b\varepsilon,b^2\varepsilon,\frac{b^2\varepsilon.b^2\varepsilon}{1+b\varepsilon},\frac{b^2\varepsilon.b^2\varepsilon}{1+b^3\varepsilon}\right\}\\
&= b^3\varepsilon.
\end{aligned}$$

Taking limit supremum as $k\to\infty$ in (19) and using (18), it follows that

$$\begin{aligned}
\psi(b\varepsilon) &= \psi(b^4(\frac{\varepsilon}{b^2}))\\
&\le \psi(b^3\lim_{k\to\infty}\sup d(x_{m(k)+1},x_{n(k)+1}))\\
&\le \psi(\lim_{k\to\infty}\sup M_b(x_{m(k)},x_{n(k)}))-\phi(\lim_{k\to\infty}\inf M_b(x_{m(k)},x_{n(k)}))\\
&\le \psi(b\varepsilon)-\phi(\varepsilon).
\end{aligned}$$

This implies that $\phi(\varepsilon)=0$ and then $\varepsilon=0$, which is a contradiction. Therefore, $\{x_n\}$ is a b-Cauchy sequence.

By the completeness of the b-metric space X, there exists $x\in X$ such that

$$\lim_{n\to\infty}d(x_n,x)=0$$

and hence

$$\lim_{n\to\infty}d(fx_n,x) = \lim_{n\to\infty}d(x_{n+1},x)=0. \tag{20}$$

By the continuity of f, we get

$$\lim_{n\to\infty} d(fx_n, fx) = 0.$$

From the triangle inequality, we have

$$d(x,fx) \leq b[d(x,fx_n) + d(fx_n, fx)]. \tag{21}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequality, we obtain that $d(x, fx) = 0$, that is, x is a fixed point of f.

If we assume that there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$, Then proceeding in a similar manner as above, we can obtain the same conclusion. □

Corollary 2.1. *Let (X,d) be a complete b-metric space with co-efficient $b \geq 1$, $\alpha, \beta : X \to [0,\infty)$ and $f : X \to X$ be mappings such that*

$$x, y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \Rightarrow b^3 d(fx, fy) \leq kM_b(x,y),$$

where $k \in [0,1)$. Suppose that

- **(1):** *one of the following condition holds:*
  - **(1.1):** *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$.*
  - **(1.2):** *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$.*
- **(2):** *f is continuous.*
- **(3):** *f is a cyclic (α, β)-admissible mapping.*

Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in condition (1.1) and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converges to a fixed point of f.

Proof. The result follows from Theorem 2.1 by taking $\psi(t) = t$ and $\phi(t) = (1-k)t$ for all $t \in [0,\infty)$. □

Corollary 2.2. *Let (X,d) be a complete b-metric space with co-efficient $b \geq 1$, and $f : X \to X$ be a continuous mapping such that*

$$\psi(b^3 d(fx,fy)) \leq \psi(M_b(x,y)) - \phi(M_b(x,y)),$$

for all $x,y \in X$, where $\alpha,\beta : [0,\infty) \to [0,\infty)$ are altering distance functions. Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of f.

Proof. The result follows from Theorem 2.1 by taking $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$. □

Corollary 2.3. *Let (X,d) be a complete b-metric space with co-efficient $b \geq 1$, and $f : X \to X$ be a continuous mapping such that*

$$b^3 d(fx,fy) \leq kM_b(x,y),$$

for all $x,y \in X$, where $k \in [0,1)$. Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of f.

Proof. It follows from Theorem 2.1 by taking $\psi(t) = t$, $\phi(t) = (1-k)t$ for all $t \in [0,\infty)$ and $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$. □

Taking $b = 1$ we obtain the following fixed point results in the framework of classical metric spaces:

Corollary 2.4. *Let (X,d) be a complete metric space, $\alpha,\beta : X \to [0,\infty)$ and $f : X \to X$ be three mappings such that*

$$x,y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \Rightarrow \psi d(fx,fy) \leq \psi(M(x,y)) - \phi(M(x,y)),$$

where $\psi,\phi : [0,\infty) \to [0,\infty)$ are altering distance functions. Suppose that

(1): *one of the following condition holds:*

(1.1): *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$.*

(1.2): *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$.*

(2): *f is continuous.*

(3): *f is a cyclic (α,β)-admissible mapping.*

Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in condition (1.1) and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converges to a fixed point of f.

Corollary 2.5. *Let (X,d) be a complete metric space , $\alpha,\beta : X \to [0,\infty)$ and $f : X \to X$ be three mappings such that*

$$x,y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \Rightarrow d(fx,fy) \leq kM(x,y),$$

where $k \in [0,1)$. Suppose that

(1): *one of the following condition holds:*

(1.1): *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$.*

(1.2): *there exists $y_0 \in X$ such that $\beta(y_0) \geq 1$.*

(2): *f is continuous.*

(3): *f is a cyclic (α,β)-admissible mapping.*

Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in condition (1.1) and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in condition (1.2), then $\{x_n\}$ and $\{y_n\}$ converges to a fixed point of f.

Corollary 2.6. *Let (X,d) be a complete metric space, and $f : X \to X$ be a continuous mapping such that*

$$\psi(d(fx,fy)) \leq \psi(M(x,y)) - \phi(M(x,y)),$$

for all $x,y \in X$, where $\psi,\phi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of f.

Corollary 2.7. *Let (X,d) be a complete metric space, and $f : X \to X$ be a continuous mapping such that*

$$d(fx,fy) \leq kM(x,y),$$

for all $x,y \in X$, where $k \in [0,1)$. Then f has a fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that $x_0 \in X$ is an initial point, then $\{x_n\}$ converges to a fixed point of f.

3. Application

In this section, we apply our main results to prove a fixed point theorem involving a cyclic mapping.

Definition 3.1. [8] *Let A and B be nonempty subsets of a set X. A mapping $f : A \cup B \to A \cup B$ is called cyclic if $f(A) \subseteq B$ and $f(B) \subseteq A$.*

Definition 3.2. *Let (X,d) be a complete b-metric space with co-efficient $b \geq 1$. We say that a mapping $f : A \cup B \to A \cup B$ is a (A,B)-(ψ,ϕ)-contractive mapping if*

$$\psi(b^3 d(fx,fy)) \quad \leq \quad \psi(M_b(x,y)) - \phi(M_b(x,y)) \tag{22}$$

for all $x \in A$ and $y \in B$, where $\psi,\phi : [0,\infty) \to [0,\infty)$ are altering distance functions.

Theorem 3.1. *Let A and B be two nonempty subsets of the complete b-metric space (X,d) with coefficient $b \geq 1$ and $f : A \cup B \to A \cup B$ be a b-continuous cyclic mapping which is an (A,B)-(ψ,ϕ)-contractive mapping. Then f has a fixed point in $A \cap B$.*

Proof. Define mappings $\alpha,\beta : A \cup B \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & x \in A; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \beta(x) = \begin{cases} 1, & x \in B; \\ 0, & \text{otherwise.} \end{cases}$$

For $x,y \in A \cup B$ such that $\alpha(x)\beta(y) \geq 1$, we get $x \in A$ and $y \in B$. Then we have

$$\psi(b^3 d(fx,fy)) \leq \psi(M_b(x,y)) - \phi(M_b(x,y)),$$

and thus the condition (1) holds. Therefore, f is an (α,β)-(ψ,ϕ)-rational contractive mapping. It is easy to see that f is a cyclic (α,β)-admissible mapping. Since A and B are nonempty subsets, there exists $x_0 \in A$ such that $\alpha(x_0) \geq 1$ and there exists $y_0 \in B$ such that $\beta(y_0) \geq 1$. Now, all the conditions of Theorem 2.1 hold, so f has a fixed point in $A \cup B$, say z. If $z \in A$, then $z = fz \in B$. Similarly, if $z \in B$, then we have $z \in A$. Hence, $z \in A \cup B$. This completes the proof. □

Corollary 3.1. *Let A and B be two nonempty subsets of the complete b-metric space (X,d) with coefficient $b \geq 1$ and $f : A \cup B \to A \cup B$ be a b-continuous cyclic mapping. Assume that*

$$b^3 d(fx, fy) \leq kM_s(x,y)$$

for all $x \in A$ and $y \in B$, where $k \in [0,1)$. Then f has a fixed point in $A \cap B$.

Proof. It follows from Theorem 3.1 by taking $\psi(t) = t$ and $\phi(t) = (1-k)t$ for all $t \in [0,\infty)$. □

Competing Interests

The authors declare that they have no competing interests.

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