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# FIXED POINT THEOREMS OF GENERALISED $\alpha$-RATIONAL CONTRACTIVE MAPPINGS ON RECTANGULAR $b$-METRIC SPACES

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**Abstract.** In this paper, we introduce the concept of generalised $\alpha$-rational contractive mappings in the framework of rectangular $b$-metric spaces. Using this new contractive mapping we prove some fixed point theorems. Various results are also derived and presented as corollaries. Obtained results are verified with suitable examples. The result also applied in integral equation.

**Keywords:** fixed point; $b$-metric space; rectangular $b$-metric space.

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## 1. Introduction and Preliminaries

There are various generalisations of metric space. The main aim is to generalised the famous Banach Contraction Principle. Some of the generalisations of metric space which will be of interest to our study are $b$-metric space [1], rectangular metric space [2], rectangular $b$- metric space [3, 4] etc.

Samet et al. [5] introduced the concept of $\alpha$-admissible mappings. The concept of $\alpha$-admissible is generalised by different authors [6, 7, 8, 9, 10, 11, 12, 13, 14].

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In this paper, we introduce the concept of generalised $\alpha$-rational contractive mappings in the framework of rectangular $b$-metric spaces. By using this new contractive mapping we prove some fixed point theorems.

**Definition 1.** [1] *Let $X$ be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric provided that, for all $x,y,z \in X$, the following conditions hold:*

*($b_1$) $d(x,y) = 0$ if and only if $x = y$,*

*($b_2$) $d(x,y) = d(y,x)$,*

*($b_3$) $d(x,z) \leq s[d(x,y) + d(y,z)]$.*

*Then the pair $(X,d)$ is called a b-metric space with parameter s.*

**Definition 2.** [2] *Let $X$ be a nonempty set, and let $d : X \times X \to [0,\infty)$ be a mapping such that for all $x,y \in X$ and all distinct points $u,v \in X$, each distinct from $x$ and $y$:*

$(r_1)$ *$d(x,y) = 0$ if and only if $x = y$,*

$(r_2)$ *$d(x,y) = d(y,x)$,*

$(r_3)$ *$d(x,z) \leq d(x,u) + d(u,v) + d(v,y)$ (rectangular inequality).*

*Then $(X,d)$ is called a rectangular or a generalized metric space (g.m.s.).*

**Definition 3.** [3, 4] *Let $X$ be a nonempty set, $s \geq 1$ be a given real number, and let $d : X \times X \to [0,\infty)$ be a mapping such that for all $x,y,z \in X$, and all distinct points $u,v \in X$, each distinct from $x$ and $y$:*

$(b_{r1})$ *$d(x,y) = 0$ if and only if $x = y$,*

$(b_{r2})$ *$d(x,y) = d(y,x)$,*

$(b_{r3})$ *$d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ (rectangular inequality).*

*Then $(X,d)$ is called a b-rectangular or a b-generalized metric space (b-g.m.s.).*

**Definition 4.** [3] *Let $(X,d)$ be a rectangular b-metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then*

**(a):** *The sequence $\{x_n\}$ is said to be convergent in $(X,d)$ and converges to $x$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

**(b):** *The sequence $\{x_n\}$ is said to be Cauchy sequence in $(X,d)$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.*

**(c):** *$(X,d)$ is said to be a complete rectangular b-metric space if every Cauchy sequence in $X$ converges to some $x \in X$.*

**Lemma 5.** [6] *Let $(X,d)$ be a rectangular b-metric space and $T : X \to X$ be a given mapping. Suppose that $T$ is continuous at $u \in X$. Then, for all sequence $\{x_n\}$ in $X$ such that $x_n \to u$, we have $Tx_n \to Tu$, that is, $\lim_{n\to\infty} d(Tx_n, Tu) = 0$.*

In 2012, Samet et al. [5] introduced the concept of $\alpha$-admissible mapping.

**Definition 6.** [5] *For a nonempty set $X$, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that the self mapping $T$ on $X$ is $\alpha$-admissible if for all $x, y \in X$, we have*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1 \tag{1}$$

Take $s \geq 1$ and denote $\mathbb{N}$ the set of positive integers and $\Psi$, the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying:

**(i):** $\psi$ is nondecreasing,

**(ii):** $\Sigma_{n=1}^{\infty} s^n \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where $n$ is the $n^{th}$ iterate of $\psi$.

Now we introduce the following definition

**Definition 7.** *Let $(X,d)$ be a b-rectangular metric space and $T : X \to X$ be a given mapping. We say that $T$ is generalised $\alpha$-rational contraction if there exists $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$, such that, for all, $x, y \in X$ with $\alpha(x,y) \geq 1$, we have*

$$d(Tx,Ty) \leq \psi(M(x,y)), \tag{2}$$

*where*

$$\begin{aligned} M(x,y) &= \max\Big\{ d(x,y), d(x,Tx), d(y,Ty), \\ &\qquad \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)} \Big\}. \end{aligned} \tag{3}$$

## 2. Main Results

In this section, we shall state and prove our main results. We start with the following useful proposition which we find its analogous for rectangular $b$-metric space. We give its proof for convenience of readers.

**Proposition 8.** *Suppose that $\{x_n\}$ is Cauchy sequence in a rectangular b-metric space such that*

$$\lim_{n\to\infty} d(x_n,u) = \lim_{n\to\infty} d(x_n,z) = 0,$$

*where $u,z \in X$. Then $u = z$.*

*Proof.* Suppose, in the contrary , that $u \neq z$. If $x_n = z$ for arbitrary large values of $n$, so necessarily $u = z$. So, we assume that $x_n \neq z$ for infinitely many $n$. So, we may assume that $x_n \neq x_m \neq u$ and $x_n \neq x_m \neq z$ for all $m,n \in \mathbb{N}$ with $n \neq m$. Thus, by the quadrilateral inequality

$$d(z,u) \leq sd(z,x_n) + sd(x_n,x_{n+1}) + sd(x_{n+1},u),$$

for all $n \in \mathbb{N}$.

Since $\{x_n\}$ is Cauchy in $(X,d)$, so $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Then, letting $n \to \infty$, we get $d(z,u) = 0$ which is a contradiction. □

**Theorem 9.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$. Assume that $T$ is an $\alpha$-rational contraction. Suppose also that*

- **(i):** *$T$ is $\alpha$-admissible,*
- **(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$,*
- **(iii):** *$T$ is continuous on $(X,d)$.*

*Then there exists $u \in X$ such that $u$ is fixed point of $T$.*

*Proof.* By assumption $(ii)$, there exists a point $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$. Take the sequence $\{x_n\}$ such that

$$x_n = Tx_{n-1}, \tag{4}$$

for all $n = 1,2,\dots$ . Suppose that $x_k = x_{k+1} = Tx_k$ for some $k$. So, $x_k$ is a fixed point of $T$. Now, assume that

$$x_n \neq x_{n+1}, \tag{5}$$

for all $n = 0,1,\dots$ .

We have $\alpha(x_0,x_1) \geq 1$ and $\alpha(x_0,x_2) \geq 1$. Since $T$ is $\alpha$-admissible, so

$$\begin{aligned} \alpha(x_1,x_2) &= \alpha(Tx_0,Tx_1) \geq 1, \\ \text{and } \alpha(x_1,x_3) &= \alpha(Tx_1,Tx_2) \geq 1. \end{aligned}$$

Similarly to above, we obtain

$$\alpha(x_n,x_{n+1}) \geq 1 \text{ and } \alpha(x_n,x_{n+2}) \geq 1, \tag{6}$$

for all $n = 0,1,\dots$ .

**Step 1:** We shall prove

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{7}$$

From (2) and (3), we find that

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq \psi(M(x_{n-1},x_n)), \tag{8}$$

for all $n \geq 1$,

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\Big\{ d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n), \\ &\quad \frac{d(x_n,Tx_{n-1})d(x_n,Tx_n)}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{1+d(Tx_{n-1},Tx_n)} \Big\} \\ &\leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

If for some $n$, $M(x_{n-1},x_n) = d(x_n,x_{n+1})$, then we obtain from (8)

$$0 < d(x_n,x_{n+1}) \leq \psi(d(x_n,x_{n+1})) < d(x_n,x_{n+1}),$$

which is a contradiction. Hence $M(x_{n-1},x_n) = d(x_{n_1},x_n)$, for all $n \in \mathbb{N}$, and (8) becomes

$$d(x_n,x_{n+1}) \leq \psi(d(x_{n-1},x_n)), \tag{9}$$

for all $n = 1,2,\dots$ .

This yields

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n), \tag{10}$$

for all $n = 1,2,\dots$ .

By (9), we find that

$$d(x_n,x_{n+1}) < \psi^n d(x_0,x_1), \tag{11}$$

for all $n \in \mathbb{N}$.

Since $\sum_n s^n \psi^n < \infty$, then $\sum_n \psi^n < \infty$ and hence $\lim_{n\to\infty} \psi^n(d(x_0,x_1)) = 0$. So, $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$.

**Step 2:** We shall prove

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0. \tag{12}$$

From (2) and (3), we find that

$$d(x_n,x_{n+2}) = d(Tx_{n-1},Tx_{n+1}) \leq \psi(M(x_{n-1},x_{n+1})), \tag{13}$$

where

$$\begin{aligned} M(x_{n-1},x_{n+1}) &= \max\Big\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n+1},x_{n+2}), \\ &\quad \frac{d(x_{n-1},x_n)d(x_{n+1},x_{n+2})}{1+d(x_{n-1},x_{n+1})}, \frac{d(x_{n-1},x_n)d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+2})}\Big\} \\ &\leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

We have

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}) < d(x_{n-1},x_n).$$

Then

$$M(x_{n-1},x_{n+1}) \;=\; \max\left\{d(x_{n-1},x_{n+1}),d(x_{n-1},x_n),[d(x_{n-1},x_n)]^2\right\}.$$

Let $a_n = d(x_n,x_{n+2})$, $b_n = d(x_n,x_{n+1})$.

Thus, $M(x_{n-1},x_{n+1}) \leq \max\{a_{n-1},b_{n-1},[b_{n-1}]^2\}$, for all $n$.

If $M(x_{n-1},x_{n+1}) \leq b_{n-1}$ or $M(x_{n-1},x_{n+1}) \leq [b_{n-1}]^2$, since $b_n \to 0$ as $n \to \infty$ then from (13) we have

$$\begin{aligned}\lim_{n\to\infty} d(x_n,x_{n+2}) &\leq \lim_{n\to\infty} \psi(M(x_{n-1},x_{n+1}))\\ &= 0.\end{aligned}$$

If $M(x_{n-1},x_{n+1}) \leq a_{n-1}$, then we see that

$$d(x_n,x_{n+2}) \leq \psi(d(x_{n-1},x_{n+1})) < d(x_{n-1},x_{n+1}).$$

Thus, the sequence $\{d(x_n,x_{n+2})\}$ is a decreasing sequence of non-negative real numbers and hence $d(x_n,x_{n+2}) \to 0$ as $n \to \infty$.

**Step 3:** We shall prove that $x_n \neq x_m$ for all $n \neq m$.

Suppose that $x_n = x_m$ for some $n > m$, so we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$. By continuing in this fashion, we obtain $x_{n+k} = x_{m+k}$ for all $k \in \mathbb{N}$.

We have

$$\begin{aligned}0 < d(x_m,x_{m+1}) &= d(x_n,x_{n+1})\\ &= d(Tx_{n-1},Tx_n) \leq \psi(M(x_{n-1},x_n)) \leq \psi(d(x_{n-},x_n))\\ &< d(x_{n-1},x_n) < d(x_{n-2},x_{n-1})\\ &\;.\\ &\;.\\ &\;.\\ &< d(x_m,x_{m+1}),\end{aligned}$$

which is a contradiction. Thus, in that follows, we can assume that $x_n \neq x_m$ for all $n \neq m$.

**Step 4:** We shall prove that $\{x_n\}$ is a Cauchy sequence, that is, $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p \in \mathbb{N}$. the cases $p = 1$ and $p = 2$ are proved. Now, take $p \geq 3$ arbitrary. We distinguish the two cases.

**Case(1):** Let $p = 2m$ where $m \geq 2$. By quadrilateral inequality, we find

$$\begin{aligned}
d(x_n, x_{n+2m}) &\leq sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + sd(x_{n+3}, x_{n+2m}) \\
&\leq sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) \\
&\quad + s\big\{sd(x_{n+3}, x_{n+4}) + sd(x_{n+4}, x_{n+5}) + sd(x_{n+5}, x_{n+2m})\big\} \\
&\leq sd(x_n, x_{n+2}) + sd(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) \\
&\quad + s^2 d(x_{n+4}, x_{n+5}) + s^2 d(x_{n+5}, x_{n+2m}) \\
&. \\
&. \\
&. \\
&\leq sd(x_n, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_{n+4}) \\
&\quad + s^5 d(x_{n+4}, x_{n+5}) + \ldots + s^{2m} d(x_{n+2m-1}, x_{n+2m}) \\
&= sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^{k-n+1} d(x_k, x_{k+1}) \\
&\leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^{k-n+1} \psi^k d(x_0, x_1) \\
&\leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{n+2m-1} s^k \psi^k d(x_0, x_1) \\
&\leq sd(x_n, x_{n+2}) + \sum_{k=n+2}^{\infty} s^k \psi^k d(x_0, x_1)
\end{aligned}$$

Since, $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$ and $\sum_{k=n+2} s^n \psi^n d(x_0, x_1) < \infty$, then

$$\lim_{n\to\infty} d(x_n, x_{n+2m}) = 0.$$

**Case(2):** Let $p = 2m+1$ where $m \geq 1$. By rectangular inequality, we find

$$
\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + sd(x_{n+2}, x_{n+2m+1}) \\
&\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) \\
&\quad + s\left\{sd(x_{n+2}, x_{n+3}) + sd(x_{n+3}, x_{n+4}) + sd(x_{n+4}, x_{n+2m+1})\right\} \\
&\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) \\
&\quad + s^2 d(x_{n+3}, x_{n+4}) + s^2 d(x_{n+4}, x_{n+2m+1}) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) \\
&\quad + s^3 d(x_{n+3}, x_{n+4}) + \ldots + s^{2m} d(x_{n+2m}, x_{n+2m+1}) \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\
&\quad + s^4 d(x_{n+3}, x_{n+4}) + \ldots + s^{2m+1} d(x_{n+2m}, x_{n+2m+1}) \\
&= \sum_{k=n}^{n+2m} s^{k-n+1} d(x_k, x_{k+1}) \\
&\leq \sum_{k=n}^{n+2m} s^{k-n+1} \psi^k d(x_0, x_1) \\
&\leq \sum_{k=n}^{n+2m} s^k \psi^k d(x_0, x_1) \\
&\leq \sum_{k=n}^{\infty} s^k \psi^k d(x_0, x_1) \to 0 \text{ as} n \to \infty.
\end{aligned}
$$

Finally, we get

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0,$$

for all $p \in \mathbb{N}$.

We conclude that $\{x_n\}$ is a Cauchy sequence in $(X,d)$. Since $(X,d)$ is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n,u) = 0. \tag{14}$$

Since $T$ is continuous, we obtain that $\lim_{n\to\infty} d(Tx_n,Tu) = 0$. So, $\lim_{n\to\infty} d(x_n,Tu) = 0$. By proposition 8, we conclude that $Tu = u$, that is, $u$ is fixed point of $T$.

In the following, we state some consequences and corollaries of our obtained result.

□

**Corollary 10.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$ be a given continuous mapping. Suppose there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$, such that*

$$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)), \tag{15}$$

*for all $x,y \in X$. Suppose also that*

- **(i):** *$T$ is $\alpha$-admissible,*
- **(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$,*
- **(iii):** *$T$ is continuous on $(X,d)$.*

*Then there exists $u \in X$ such that $u$ is fixed point of $T$.*

*Proof.* Let $x,y \in X$ such that $\alpha(x,y) \ge 1$. Then, if (15) holds, we have

$$d(Tx,Ty) \le \alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)).$$

Then, the proof is concluded by theorem 9.

□

**Corollary 11.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$ be a given continuous mapping. Suppose there exist a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(M(x,y)),$$

*for all $x,y \in X$. Then, $T$ has a fixed point.*

*Proof.* It suffices to take $\alpha(x,y) = 1$ in corollary 10. $\square$

**Corollary 12.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$ be a given continuous mapping. Suppose there exist a constant $k \in (0,1)$ such that*

$$d(Tx,Ty) \leq kM(x,y),$$

*for all $x,y \in X$. Then $T$ has a fixed point.*

*Proof.* It suffices to take $\psi(t) = kt$, with $k \in (0,1)$, in corollary 11. $\square$

Considering $s = 1$ in theorem 9, we have

**Corollary 13.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$ be a given continuous mapping. Assume that $T$ is an $\alpha$-contraction. Suppose also that*

**(i):** *$T$ is $\alpha$-admissible,*

**(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$,*

**(iii):** *$T$ is continuous on $(X,d)$.*

*Then there exists $u \in X$ such that $u$ is a fixed point of $T$.*

**Corollary 14.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$ such that*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)),$$

*for all $x,y \in X$.*

*Suppose also that*

**(i):** *$T$ is $\alpha$-admissible,*

**(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$,*

**(iii):** *$T$ is continuous on $(X,d)$.*

*Then there exists $u \in X$ such that $u$ is a fixed point of $T$.*

**Remark 15.** *We may get many other consequences of theorem 9 when considering for example some cases of $\alpha$, that is, we may recover the cyclic contractions and we can also obtain some fixed point results involving a partial order.*

We may replace the continuity hypothesis of $T$ by the following

$(H)$ : if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k$.

**Theorem 16.** *Let $(X,d)$ be a complete rectangular b-metric space and $T : X \to X$. Assume that $T$ is an $\alpha$-rational contraction. Suppose also that*

- **(i):** *$T$ is $\alpha$-admissible,*
- **(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$,*
- **(iii):** *$(H)$ holds.*

*If $\psi(t) < \frac{t}{s}$ for all $t > 0$, then $T$ has a fixed point.*

*Proof.* Proceeding as in the proof of Theorem 9, we construct a sequence $\{x_n\}$ with $x_n \neq x_m$ for all $n \neq m$ in $X$ such that $x_n \to u$ as $n \to \infty$. We show that $Tu = u$. Suppose, on the contrary, $Tu \neq u$. So, we may assume that $x_n \neq x_m \neq u$ and $x_n \neq x_m \neq Tu$ for all $m, n \in \mathbb{N}$ with $n \neq m$. Then, for all $k \in \mathbb{N}$, by assumption $(iii)$, (2) and by quadrilateral inequality, we have

$$\begin{aligned}
0 < d(u,Tu) &\leq sd(u,x_{n(k)}) + sd(x_{n(k)},x_{n(k)+1})) + sd(x_{n(k)+1,Tu}) \\
&= sd(u,x_{n(k)}) + sd(x_{n(k)},x_{n(k)+1})) + sd(Tx_{n(k),Tu}) \\
&\leq sd(u,x_{n(k)}) + sd(x_{n(k)},x_{n(k)+1}) + s\psi(M(x_{n(k)},u)),
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n(k)},u) &= \max\Big\{ d(x_{n(k)},u), d(x_{n(k)},u), d(u,Tu), \\
&\quad \frac{d(x_{n(k)},x_{n(k)+1})d(u,Tu)}{1+d(x_{n(k)},u)}, \frac{d(x_{n(k)},x_{n(k)+1})d(u,Tu)}{1+d(x_{n(k)+1},Tu)} \Big\}.
\end{aligned}$$

We have

$$\lim_{k\to\infty} d(x_{n(k)},u) = \lim_{k\to\infty} d(x_{n(k)},x_{n(k)+1}) = 0.$$

This implies that there exists $n \in \mathbb{N}$ such that

$$\max\{d(x_{n(k)},u), d(x_{n(k)},x_{n(k)+1}), d(u,Tu), \frac{d(x_{n(k)},x_{n(k)+1})d(u,Tu)}{1+d(x_{n(k)},u)}, \frac{d(x_{n(k)},x_{n(k)+1})d(u,Tu)}{1+d(x_{n(k)},u)} = d(u,Tu), \text{ for all } k \geq \mathbb{N}.$$

Then, for $k \geq \mathbb{N}$, we obtain

$$0 < d(u,Tu) \leq sd(u,x_{n(k)}) + sd(x_{n(k)},x_{n(k)+1}).$$

From $\psi(t) < \frac{t}{s}$, letting $k \to \infty$, we get

$$0 < d(u,Tu) \leq s\psi(d(u,Tu)) < d(u,Tu),$$

which is a contradiction. Hence we find that $u$ is a fixed point of $T$, that is, $Tu = u$.

□

**Corollary 17.** *Let $(X,d)$ be a complete rectangular b-metric space and $T: X \to X$ be a given mapping. Suppose there exists a function $\psi \in \Psi$, such that*

$$d(Tx,Ty) \leq \psi(M(x,y)),$$

*for all $x,y \in X$. If $\psi(t) < \frac{t}{s}$ for all $t > 0$, then $T$ has a fixed point.*

*Proof.* It suffices to take $\alpha(x,y) = 1$ in theorem 16. □

Considering $s = 1$ in theorem 16, we have

**Corollary 18.** *Let $(X,d)$ be a complete rectangular b-metric space and $T: X \to X$. Assume that $T$ is an $\alpha$-contraction. Suppose that*

**(i):** *$T$ is $\alpha$-admissible,*

**(ii):** *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$,*

**(iii):** *$(H)$ holds.*

*Then $T$ has a fixed point.*

We present the following example.

**Example 19.** *Consider the set* $X = A \cup [1,2]$*, where* $A = \{0\} \cup \{\frac{1}{n} : n = 2,3,...,6\}$*. Define* $d : X \times X \to [0,\infty)$ *as follows*

$$d\left(0,\frac{1}{2}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{6}\right) = 0.09,$$
$$d\left(0,\frac{1}{3}\right) = d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.04,$$
$$d\left(0,\frac{1}{4}\right) = d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.16,$$
$$d\left(0,\frac{1}{5}\right) = d\left(\frac{1}{2},\frac{1}{6}\right) = d\left(\frac{1}{3},\frac{1}{6}\right) = 0.25,$$
$$d\left(0,\frac{1}{6}\right) = d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = 0.36,$$

*with* $d(x,x) = 0$*,* $d(x,y) = d(y,x)$ *for all* $x,y \in X$*, and* $d(x,y) = (x-y)^2$ *if* $\{x,y\} \cap [1,2] \neq \phi$*. Obviously, d is neither a metric nor a rectangular metric. But d is a rectangular b-metric space with* $s = 3$ *and* $(X,d)$ *is complete. Consider now the mapping*

$$Tx = \begin{cases} \frac{1+x}{2}, & \text{if } x \in [1,2]; \\ \frac{1}{3}, & \text{if } x \in A. \end{cases}$$

*Take* $\psi(t) = \frac{1}{4}t$ *for* $t \in [0,\infty)$ *and*

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [1,2]; \\ 0, & \text{otherwise.} \end{cases}$$

*We show that* $T$ *is an* $\alpha$*-contraction. Let* $x,y \in X$ *such that* $\alpha(x,y) \geq 1$*. So,* $x,y \in [1,2]$*. We have*

$$Tx = \frac{1+x}{2} \text{ and } Ty = \frac{1+y}{2}$$

*Then*

$$\begin{aligned} d(Tx,Ty) &= (Tx-Ty)^2 = \left(\frac{1+x}{2}-\frac{1+y}{2}\right)^2 \\ &= \frac{1}{4}(x-y)^2 \\ &= \psi(d(Tx,Ty)) \\ &\leq \psi(M(x,y)). \end{aligned}$$

*Moreover, there exists* $x_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$ *and* $\alpha(x_0,T^2x_0) \geq 1$. *In fact, for* $x_0 = 2$*, we have*

$$\alpha(2,T2) = \alpha(2,\frac{3}{2}) = 1$$
$$\alpha(2,T^22) = \alpha(2,\frac{5}{4}) = 1.$$

*Notice that* $T$ *is* $\alpha$*-admissible. To show this , assume that* $x,y \in X$ *such that* $\alpha(x,y \geq 1)$. *It yields that* $x,y \in [1,2]$. *Owing to the definition of* $T$*, we have* $Tx,Ty \in [1,2]$*, and hence* $\alpha(Tx,Ty) \geq 1$. *Thus,* $T$ *is* $\alpha$*-admissible. It is clear that* $T$ *is not continuous. Now, we show that* $(H)$ *is verified. Let* $\{x_n\}$ *be a sequence in* $X$ *such that* $\alpha(x_{n,x_{n+1}}) \geq 1$ *for all* $n \in \mathbb{N}$*, then* $\{x_n \subset [1,2]\}$. *If* $x_n \to u$ *as* $n \to \infty$*, we have* $d(x_n,u) = (x_n-u)^2 \to 0$ *as* $n \to \infty$. *Then,* $|x_n-u| \to 0$ *as* $n \to \infty$. *Hence,* $u \in [1,2]$ *and hence,* $\alpha(x_n,u) = 1$.

*Note that for* $s = 3$ *and* $\psi(t) = \frac{1}{4}t$*, we have* $\sum_{n=1}^{\infty} s^n\psi^n(t) = t\sum_{n=1}^{\infty}(\frac{3}{4})^n < \infty$ *and* $\psi(t) < \frac{t}{3}$ *for all* $t > 0$.

*Hence, all condition of 9 are satisfied. Here,* $\{\frac{1}{3},2\}$ *is the set of fixed points of* $T$*, that is, we have two fixed points.*

In example 19, mention that we have two fixed points of $T$ which are 2 and $\frac{1}{3}$. Note that $\alpha(2,\frac{1}{3}) < 1$. So, the uniqueness of the fixed point, we need the additional condition.

$(U)$ : For all $x,y \in Fix(f)$, we have $\alpha(x,y) \geq 1$, where $Fix(f)$ denotes the set of fixed points of $T$.

**Theorem 20.** *Adding condition* $(U)$ *to the hypothesis of theorem 9 (respectively theorem 16), we obtain that* $u$ *is the unique fixed point of* $T$.

*Proof.* We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Tu$ and $v = Tv$ with $u \neq v$. By assumption $(U)$, we have $\alpha(u,v) \geq 1$ so by (2)

$$0 \leq d(u,v) = d(Tu,Tv) \leq \psi(M(u,v)),$$

where

$$\begin{aligned} M(u,v) &= \max\Big\{ d(u,v), d(u,Tu), d(v,Tv) \\ &\quad \frac{d(u,Tu)d(v,Tv)}{1+d(u,v)}, \frac{d(u,Tu)d(v,Tv)}{1+d(Tu,Tv)} \Big\} \\ &= d(u,v). \end{aligned}$$

So, $0 < d(u,v) \leq \psi(M(u,v)) = \psi(d(u,v)) < d(u,v).$ which is a contradiction. Thus, $u = v$, so the uniqueness of the fixed of $T$. □

We give the following example.

**Example 21.** *We go back to example 19 when $X = \{0,1,2,3\}$ and $d$ is a rectangular b-metric on $X$ with complete. Define the map $T : X \to X$ by*

$$T0 = T1 = T2 = 0 \text{ and } T3 = 1.$$

*Let $\psi(t) = \frac{t}{4}$. For $s = 2$, we have*

$$\sum_{n=1}^{\infty} s^n \psi^n(t) = t \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

*and $\psi(t) < \frac{t}{2}$ for all $t > 0$. We shall show that*

$$d(Tx,Ty) \leq \tfrac{1}{4} M(x,y), \text{ for all } x,y \in X$$

*For this, we consider the following cases:*

***Case I:*** $x, y \in \{0,1,2\}$. *We have*

$$d(Tx,Ty) = d(0,0) = 0 \leq \frac{1}{4} M(x,y)$$

***Case II:*** $x, y \in \{0,1,2\}$, $y = 3$. *We have*

*If $x = 0$, then*

$$\begin{aligned} d(Tx,Ty) &= d(0,1) = 1 \leq \frac{7}{4} \\ &= \frac{1}{4}\max\left\{d(0,3), d(0,0), d(3,1), \frac{d(0,0), d(3,1)}{1+d(0,3)}, \frac{d(0,0), d(3,1)}{1+d(0,1)}\right\}. \end{aligned}$$

*If $x = 1$, then*

$$\begin{aligned} d(Tx,Ty) &= d(0,1) = 1 \leq \frac{3}{2} \\ &= \frac{1}{4}\max\left\{d(1,3), d(1,0), d(3,1), \frac{d(1,0)d(3,1)}{1+d(1,3)}, \frac{d(1,0)d(3,1)}{1+d(0,1)}\right\} \end{aligned}$$

*If $x = 2$, then*

$$\begin{aligned} d(Tx,Ty) &= d(0,1) = 1 \leq 2 \\ &= \frac{1}{4}\max\left\{d(2,3), d(2,0), d(3,1), \frac{d(2,0)d(3,1)}{1+d(2,3)}, \frac{d(2,0)d(3,1)}{1+d(0,1)}\right\} \end{aligned}$$

***Case III:*** $x = y = 3$, *We have*

$$d(Tx,Ty) = d(0,0) = 0 \leq \frac{1}{4}M(3,3).$$

*Then all the require hypothesis of corollary 17 are satisfied. Here, $x = 0$ is the unique fixed point of $T$.*

## 3. Applications

In this section, we give an existence theorem for a solution of the following integral equation

$$x(t) = g(t) + \int_0^t k(t,s,x(s))dr, \; t \in [0,1], \tag{16}$$

where $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are continuous mapping. Let $X = C([0,1], \mathbb{R})$ be the set of all continuous functions defined on $[0,1]$.

Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \|(x-y)^2\|,$$

where $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Then $(X,d)$ is a complete rectangular $b$-metric space . Now prove following result.

**Theorem 22.** *Suppose the following hypothesis hold:*

**(i):** *there exist* $k \in (0, \frac{1}{2})$ *and* $\beta : X \times X \to [0,\infty)$ *such that for all* $x,y \in X$ *with* $x(t) \leq y(t)$ *for all* $t \in [0,1]$ *and for every* $s \in [0,1]$ *, we have*

$$\begin{aligned} 0 &\leq \mid k(t,sx(t)) - k(t,s,y(t)) \mid \\ &\leq \beta(t,s)(y(s) - s(s)) \end{aligned}$$

*and*

$$\sup_{t\in[0,1]} \int_0^1 \beta(t,s)ds = \sqrt{k};$$

**(ii):** *$k$ is non-decreasing with respect to its third variable;*

**(iii):** *there exists* $x_0 \in X$ *such that for all* $t \in [0,1]$ *we have*

$$\begin{aligned} x_0(t) &\leq g(t) + \int_0^1 k(t,s,x(s))ds \\ \text{and } x_0(t) &\leq g(t) + \int_0^1 k(t,s,g(t) + \int_0^1 k(t,r,x_r)dr)ds. \end{aligned}$$

*Then the integral equation* (16) *has a solution in* $X$.

*Proof.* For $x \in X$ and $t \in [0,1]$ ,define the mapping

$$T_x(t) = g(t) + \int_0^1 k(t,r,s(r))dr.$$

Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \leq y; \\ 0, & \text{otherwise.} \end{cases}$$

Take $\psi(t) = kt$, so $\psi(t) < \frac{t}{s}$ for all $t > 0$ (since $s = 2$ ).

We define $x,y \in X$, $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0,1]$ where $\leq$ denotes the usual order of real numbers. Let $x,y \in X$ for all $t \in [0,1]$. Thus, by condition $(i)$.

$$
\begin{aligned}
|T(x)(t) - T(y)T(t)| &\le \int_0^1 |K(t,s,x(t))| ds \\
&\le \int_0^1 \beta(t,s)|x(s) - y(s)| ds \\
&= \int_0^1 \beta(t,s)\sqrt{(x(s)-y(s))^2} ds \\
&\le \int_0^1 \beta(t,s)\sqrt{\|(x-y)^2\|} ds \\
&\le \sqrt{k}\sqrt{\|(x-y)^2\|}.
\end{aligned}
$$

We deduce that for all $x, y \in X$ such that $x \le y$

$$
d(T_x, T_y) \le kd(x,y) = \psi(d(x,y)) \le \psi(M(x,y)). \tag{17}
$$

Since $K$ is non-decreasing with respect to its third variable, so for all $x, y \in X$ with $x \le y$ we get $T(x)(t) \le T(y)(t)$ for all $t \in [0,1]$, that is if $\alpha(x,y) \ge 1$, we obtain $\alpha(Tx, Ty) \ge 1$. Moreover, the condition $(ii)$ yields that there exists $x_0 \in X$ such that $\alpha(x_0, T^2x_0) \ge 1$. The hypothesis $(H)$ also hold easily. Therefore, all conditions of the Theorem 16 are verified with $s = 2$ and hence the operator $T$ has a fixed point, which is a solution of the integral equation (16) in $X$.

□

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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