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J. Math. Comput. Sci. 11 (2021), No. 2, 1305-1311

<https://doi.org/10.28919/jmcs/5301>

ISSN: 1927-5307

## FIXED-POINT THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACE

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**Abstract:** In this paper, two coincident fixed point theorems for pair of mappings in intuitionistic fuzzy metric spaces are proved and quoted its application in color image filtering process.

**Keywords:** fuzzy metric space; contraction mapping; fixed point; coincidence point.

**2010 AMS Subject Classification:** 47H10, 54H25.

### 1. INTRODUCTION

Fuzzy Topology is an active field of study in present due to its wide applications in various fields of engineering. The conception of fuzzy set was introduced by L.A. Zadeh [16] in 1965. Kramosil and Michalek [10] generalized the probabilistic metric space concept to fuzziness and introduced fuzzy metric space, after which several authors have studied the disposition of fuzzy metric spaces in numerous ways. In 1983, Banach and Edelstein contraction principle were extended in fuzzy metric space by M. Grabiec [6]. George and Veeramani [5] modified the concept of fuzzy metric spaces with continuous t-norm, then several authors introduced and analysed this concept in different generalized forms in fuzzy 2-metric, fuzzy 3-metric and in intuitionistic fuzzy metric. The fixed point theory in the fuzzy metric space was studied by many authors and fixed point theorems have been obtained by using the contractive condition of self

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Received December 04, 2020

mappings. Gregori and Sapena [7] investigated fixed point theorems for fuzzy contractive mappings defined on fuzzy metric spaces. Zikic [17] proved a fixed point theorem for mappings on fuzzy metric space which improved the result of Gregori and Sapena [7]. The notion of intuitionistic fuzzy sets as a generalization of fuzzy sets was introduced and studied by Atanassov [3] and later on there has been advancement in the study of intuitionistic fuzzy sets to a large extent. Gregori et al. [8] presented few new examples of fuzzy metrics and put forth its application in colour image processing. Park [12] defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces. The intuitionistic fuzzy metric space is four dimensional notion in which two fuzzy sets and continuous t-conorm are considered. Zhenhua Jiao [9] proved two theorems by extending Gregori-Sapena's fixed point theorem [7] and Zikic's fixed point theorem [17] in fuzzy metric space to complete intuitionistic fuzzy metric spaces. Cemil et al. [15] proved new version of Hind Theorem and Sierpinski Theorem for intuitionistic fuzzy metric space.

In 2012, M. Verma and R. S. Chandel [14] established theorem for absorbing mappings in complete intuitionistic fuzzy metric space. In 2014, Manandhar and Jha [11] introduced compatible mappings of type K in intuitionistic fuzzy metric space and proved common fixed point theorem with example. By overlooking the condition of continuity Tripathi et al. [13]. improved the results of Alaca et al. [2]. In this paper two more theorems are presented on the same argument.

## 2. PRELIMINARIES

**Definition 2.1:** [4] Let  $X$  be a non-empty fixed set. An intuitionistic fuzzy set  $A$  is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$$

where the functions  $\mu_A: X \rightarrow [0,1]$  and  $\nu_A: X \rightarrow [0,1]$  denote the degree of membership and the degree of non-membership of each element  $x \in X$  to the set  $A$  respectively and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \text{ for each } x \in X$$

**Definition 2.2:** [2, 12] A 5-tuple  $(X, M, N, *, \diamond)$  is said as intuitionistic fuzzy metric space if  $X$  is considered an arbitrary set,  $*$  is a continuous t-norm,  $\diamond$  is a continuous t-conorm,  $M$  and  $N$  are fuzzy sets on  $X^2 \times [0, \infty)$  with the following conditions:

$$(a) M(x, y, t) + N(x, y, t) \leq 1$$

$$(b) M(x, y, 0) = 0$$

- (c)  $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$
- (d)  $M(x, y, t) = M(y, x, t)$
- (e)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ , for all  $x, y, z \in X$ , and  $s, t > 0$
- (f)  $M(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$  is left continuous
- (g)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1$  for all  $x, y \in X$
- (h)  $N(x, y, 0) = 1$
- (i)  $N(x, y, t) = 0$  for all  $t > 0$  if and only if  $x = y$
- (j)  $N(x, y, t) = N(y, x, t)$
- (k)  $N(x, y, t) \diamond N(y, z, s) \leq N(x, z, t + s)$ , for all  $x, y, z \in X$ , and  $s, t > 0$
- (l)  $N(x, y, \cdot): [0, \infty) \rightarrow [0, 1]$  is left continuous
- (m)  $\lim_{t \rightarrow \infty} N(x, y, t) = 1$  for all  $x, y \in X$

then  $(M, N)$  is called an intuitionistic fuzzy metric on  $X$ . The functions  $M(x, y, t)$  and  $N(x, y, t)$  represent the extent of nearness and non-nearness between  $x$  and  $y$ , respectively with respect to  $t$ .

**Definition 2.3:** [2] Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy metric space. Then

- (a) A sequence  $\{x_n\}$  in  $X$  is said to be Cauchy sequence if, for  $p > 0$  and  $\forall t > 0$

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_{n+p}, x_n, t) = 0$$

- (b) A sequence  $\{x_n\}$  in  $X$  is said to be convergent to a point  $x \in X$  if,  $\forall t > 0$

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \text{ and } \lim_{n \rightarrow \infty} N(x_n, x, t) = 0$$

**Definition 2.4:** [6] Fuzzy Banach Contraction Theorem

Let the complete fuzzy metric space be denoted by  $(X, M, *)$  such that

- (a)  $\lim_{t \rightarrow \infty} M(x, y, t) = 1 \quad \forall x, y \in X$
- (b)  $M(Tx, Ty, kt) \geq M(x, y, t)$

For  $0 < k < 1$  and  $\forall x, y \in X$ . Then  $T$  has unique fixed point

**Example 2.1:** [8] Let  $X$  be the non-empty set,  $M$  is a function defined on  $X \times X \times R^+$  with values  $(0, 1]$ . Let  $f: X \rightarrow R^+$  be one-one and  $g: R^+ \rightarrow [0, \infty)$  be an increasing continuous function  $\alpha, \beta > 0$  and  $M$  is defined by

$$M(x, y, t) = \left[ \frac{(\min(f(x), f(y)))^\alpha + g(t)}{(\max(f(x), f(y)))^\alpha + g(t)} \right]^\beta$$

Then,  $(M, \cdot)$  is a fuzzy metric on  $X$ .

### 3. MAIN RESULTS

**Theorem 3.1:** Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy metric space and  $Y$  a non-empty set.

Suppose  $f, g: Y \rightarrow X$  are two mappings satisfying the following conditions:

- (i)  $f(Y) \subset g(Y)$  and either  $f(Y)$  or  $g(Y)$  is complete
- (ii)  $\exists t \in (0, 1)$  such that  $M(fx, fy, t) \geq c(M(gx, gy, t))$  and

$$N(fx, fy, t) \leq c'(N(gx, gy, t)) \quad \forall x, y \in Y, \forall t > 0$$

where  $c: [0, 1] \rightarrow [0, 1]$  and  $c': [0, 1] \rightarrow [0, 1]$  are continuous functions of  $t$  such that  $c(t) > t$  and  $c'(t) < t \quad \forall t \in (0, 1)$  and  $c(1) = 1, c(0) = 0$ .

Then  $f$  and  $g$  have coincident point.

Proof: Let  $x_0 \in Y$ . Since  $f(Y) \subset g(Y)$ , hence by theorem 2.1 [13] we can construct a sequence  $\{x_n\}$  such that  $f(x_n) = g(x_{n+1})$ ,

$$M(fx_n, fx_{n+1}, t) \geq c(M(gx_n, gx_{n+1}, t)) = c(M(fx_{n-1}, fx_n, t)) \text{ and}$$

$$N(fx_n, fx_{n+1}, t) \leq c'(N(gx_n, gx_{n+1}, t)) = c'(N(fx_{n-1}, fx_n, t))$$

Now from the definition of  $c$  and  $c'$ ,  $M(fx_n, fx_{n+1}, t) > M(gx_n, gx_{n+1}, t)$  and

$$N(fx_n, fx_{n+1}, t) < N(gx_n, gx_{n+1}, t)$$

Thus  $\{M(fx_n, fx_{n+1}, t)\}$  is an increasing sequence in  $[0, 1]$  and  $\{N(fx_n, fx_{n+1}, t)\}$  is a decreasing sequence in  $[0, 1]$ . Also,

$$M(fx_n, fx_{n+1}, t) \rightarrow l \leq 1 \text{ and } N(fx_n, fx_{n+1}, t) \rightarrow l' \geq 0.$$

If  $l < 1$  then by taking  $n \rightarrow \infty$  in (1) we get  $l > 1$  which is a contradiction.

Therefore, we get  $l = 1$  and similarly  $l' = 0$ . Now for any positive integer  $p$  and  $0 < t < 1$  we have,

$$M(fx_n, fx_{n+p}, t) \geq M\left(fx_n, fx_{n+1}, \frac{t}{p}\right) * \dots * \dots * (p \text{ times}) * M\left(fx_{n+p-1}, fx_{n+1}, \frac{t}{p}\right) \text{ and}$$

$$N(fx_n, fx_{n+p}, t) \leq N\left(fx_n, fx_{n+1}, \frac{t}{p}\right) \diamond \dots \diamond \dots \diamond (p \text{ times}) \diamond N\left(fx_{n+p-1}, fx_{n+1}, \frac{t}{p}\right)$$

Since,  $M(fx_n, fx_{n+1}, t) = 1$  and  $N(fx_n, fx_{n+1}, t) = 0$  as  $n \rightarrow \infty$ , it follows that

$$M(fx_n, fx_{n+p}, t) \geq 1 * \dots * 1 \text{ and } N(fx_n, fx_{n+p}, t) \geq 0 \diamond \dots \diamond 1, \text{ by the definition 2.3,}$$

$\{gx_n\} = \{fx_{n-1}\}$  is a Cauchy sequence.

Suppose  $g(Y)$  is complete. Then there exists  $p \in g(Y)$  such that  $\{gx_n\} \rightarrow p$ , so there exists  $z \in Y$  such that  $gz = p$ .

Put  $x = x_n, y = z$  in (ii), we have

$$M(fx_n, fz, t) \geq c(M(gx_n, gz, t)) \text{ and } N(fx_n, fz, t) \leq c'(N(gx_n, gz, t)),$$

As  $n \rightarrow \infty$ , we get

$$M(p, fz, t) \geq c(M(p, p, t)) = 1 \text{ and } N(p, fz, t) \leq c'(N(p, p, t)) = 0,$$

That is,  $p = fz = gz$

Therefore  $f$  and  $g$  have coincident point.

**Theorem 3.2:** Let  $(X, M, N, *, \diamond)$  be an intuitionistic fuzzy metric space and  $f, g: X \rightarrow X$  are mappings satisfying the following conditions:

(i)  $f(X) \subset g(X)$  and either  $f(X)$  or  $g(X)$  is complete

(ii)  $M(fx, fy, t) \geq c(M(gx, gy, t)), N(fx, fy, t) \leq c'(N(gx, gy, t)),$

where  $c: [0,1] \rightarrow [0,1]$  and  $c': [0,1] \rightarrow [0,1]$  are continuous functions of  $t$  such that  $c(t) > t$  and  $c'(t) < t \forall t \in (0,1)$  and  $c(1) = 1, c(0) = 0$ .

(iii)  $f$  and  $g$  are coincidentally commuting.

Then  $f$  and  $g$  have unique common fixed point.

Proof: If we take  $Y = X$  in theorem 3.1 then we get  $z$  as the coincident point of  $f$  and  $g$  and  $fz = gz = p$ . Since  $f$  and  $g$  are coincidentally commuting so  $fgz = gfp = gp = fp$ .

Putting  $x = z$  and  $y = fz$  in (ii), we have the inequalities

$$M(fz, ffz, t) \geq c(M(gz, gfp, t)) \text{ and } N(fz, ffz, t) \leq c'(N(gz, gfp, t))$$

$$\text{i.e. } M(p, fp, t) \geq c(M(p, fp, t)) \text{ and } N(p, fp, t) \leq c'(N(p, fp, t))$$

If  $p \neq fp$ , then

$$M(p, fp, t) > M(p, fp, t) \text{ and } N(p, fp, t) < N(p, fp, t)$$

But, this is not possible. Hence we must have  $p = fp = gp$ . Hence  $p$  is common fixed point of  $f$  and  $g$ .

For uniqueness, suppose  $p, q$  are common fixed points of  $f$  and  $g$ .

Put  $x = p$  and  $y = q$  in (ii), we get,

$$M(fp, fq, t) > M(gp, gq, t) \text{ and } N(fp, fq, t) < N(gp, gq, t), \text{ which is a contradiction.}$$

Hence  $p = q$ . This proves the uniqueness.

#### 4. APPLICATION

Fuzzy metrics is advantageous over classical metrics. Gregori et al. [8] used fuzzy metric by taking into account two distance criteria (namely, the distance between two colour vectors defining the similarity of colour components and the one which measures the spatial closeness between the pixels) between pixels of colour image (RGB) with method of impulse noise filtering and showed the improved results over classical metric.

#### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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