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AN EFFICIENT APPROACH FOR INTEGER AND NON-INTEGER BARRIER OPTIONS MODEL IN A CAPUTO SENSE

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Abstract. This paper proposes an efficient approach for solving Fractional Barrier Option Model (FBOM). The approach of the Caputo Fractional Reduced Differential Transform Method (CFRDTM) which is the combination of the Caputo Fractional Derivative (CFD) and the Reduced Differential Transform Method (RDTM) is employed. The emphasis is laid on CFD which is more suitable for the study of differential equations of fractional order. It is assumed that the stock price pays no dividend and follows a marked point process. Based on CFRDTM, a series solution for FBOM has been obtained successfully. The valuation formula for the price of Barrier Option (BO) with fractional order is also obtained. Moreover, the approximate solution obtained via CFRDTM is expressed in the form of a convergent series with computed components. An illustrative example is presented to measure the performance of CFRDTM in terms of accuracy, efficiency and suitability. The results obtained via CFRDTM were compared with the other existing methods such as the Laplace Adomian Decomposition Method (LADM), Two Dimensional Differential Transform Method (TDDTM) and the Analytical Value (AV). Hence, CFRDTM is found to be accurate, efficient and a suitable approach for obtaining an approximate solution of FBOM.

Keywords: Barrier option; Caputo fractional derivative; fractional order; marked point process; analytical value.
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1. INTRODUCTION

A contingent claim is a versatile security that gives its holder the right, but not obligation, to buy (call option) or sell (put option) an underlying asset at an agreed upon price during a certain period of time. According to [1], “a barrier option is a type of derivative where the payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. A barrier option can be a knock-out, if the underlying exceeds a certain price, limiting profits for the holder and limiting losses for the writer. It can also be a knock-in, if it has no value until the underlying reaches a certain price”. Fractional differential and integral operators have been used extensively to describe practical dynamics phenomena arising from physical science, biological science social science, medical science and engineering; see [2, 3, 4, 5, 6, 7, 8], just to mention a few. Veeresha and Prakasha [9] studied the solution of fractional generalized Zakharov equations with Mittag-Leffler function. Fadugba [10] applied homotopy analysis method for the valuation of European call options with time-fractional Black-Scholes equation. Fadugba [11] studied the solution of fractional order equations in the domain of the Mellin transform. Sarwar et al. [12] carried out the free convective non-Newtonian fluid of Brinkman type flow near an upright plate moving with velocity $f(t)$. They also proposed a fractional order model for non-Newtonian fluid of Brinkman type flow. A numerical analysis for fractional model of the spread of pests in tea plants was studied by Kumar et al. [13]. They investigated the possibility for obtaining new chaotic behaviours with singular fractional operator and shows the chaotic behaviour at various values of arbitrary order. Cui et al. [14] considered the Riemann-Liouville-type general fractional derivatives of the non-singular kernel of the one-parametric Lorenzo-Hartley function. They also proposed a new general fractional-order-derivative Goldstein-Kac-type telegraph equation for the first time. In this paper, the solution of FBOM via CFRDTM is proposed. CFRDTM does not require linearization, perturbation or restrictive assumptions and offers solutions with easily computable components as convergent series. The emphasis is laid on the Caputo fractional derivative because of its suitability for the study of differential equations of fractional order and superiority over the Riemann-Liouville fractional derivative. The rest of the paper is structured as follows: Section Two presents the preliminaries which captures elements of FBOM, some definitions of terms and CFRDTM. In

Section Three, the solution of FBOM via CFRDTM is obtained. The valuation formula for the option is also obtained. Section Four captures the application of CFRDTM in the valuation of FBOM. The physical behaviour of the prices of FBOM obtained via CFRDTM is shown in terms of plots for different values of fractional order q is also presented. In Section Five, discussion of results and concluding remarks were presented.

2. PRELIMINARIES

This section presents the elements of FBOM, some definitions of concepts and CFRDTM.

2.1. Fractional Barrier Option Model (FBOM). Assume that the stock price follows geometric Brownian motion and pays no dividend, a Barrier Option Model (BOM) is given by [15]

$$\frac{\partial c^b}{\partial \tau} = \frac{\partial^2 c^b}{\partial x^2} + (\beta - 1)\frac{\partial c^b}{\partial x} - \beta c^b \tag{1}$$

subject to up-and-out barrier constraints, respectively

$$c^b(x,\tau) = 0, \exp(x) \geq \exp(B_u), \tau \in [0,T] \tag{2}$$

and

$$c^b(x,0) = (\exp(x) - K), 0 < \exp(x) < \exp(B_u) \tag{3}$$

with

$$S = e^x, t = T - \frac{2\tau}{\sigma^2}, B(S,t) = c^b(x,\tau), \beta = \frac{2r}{\sigma^2} \tag{4}$$

where $B(S,t)$ is the price of barrier option, S is the stock price, t is the current time, σ is the volatility, r is the risk-free interest rate, $(S,t)\ \varepsilon\ \mathbb{R}^+ \times (0,T)$, K is the strike price, T is the time to expiry and B_u is the barrier/ boundary. Let the stock price be driven by a marked point process as in [16]. Thus (1) turns to FBOM given by

$$\frac{\partial^q c^b}{\partial \tau^q} = \frac{\partial^2 c^b}{\partial x^2} + (\beta - 1)\frac{\partial c^b}{\partial x} - \beta c^b \tag{5}$$

subject to (2) and (3), where $q \in (0,1]$ is the fractional order.

2.2. Definition of Some Concepts. Definitions of some concepts are presented as follows:

Definition 1. A real valued function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $\rho > \mu$, such that $f(t) = t^\rho f_1(t)$, where $f_1(t) \in C[0,\infty]$ and is said to be in the space C_μ^n if and only if $f^n \in C_\mu, n \in N$

Definition 2. The Caputo fractional derivative of the function $f \in C_{-1}^n, n \in N$ is defined as

$$ {}_0^cD_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(\tau) d\tau \tag{6}$$

for $q \in (n-1, n]$, $t > 0$

Definition 3. The Caputo time-fractional derivative operator of order $q > 0$ is defined as

$$ {}_0^cD_t^q u = \begin{cases} \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} u^{(n)}(x,\tau) d\tau & q \in (0,1], \\ \\ \frac{\partial^n u(x,t)}{\partial t^n}, & q = n \end{cases} \tag{7}$$

where n is the smallest integer that exceeds q, $u = u(x,t), u^{(n)}(x,\tau) = \frac{\partial^n u(x,\tau)}{\partial \tau^n}$.

Definition 4. The Riemann-Liouville fractional integral operator of order $q \geq 0$ of a function $f \in C_\mu$, $\mu \geq 1$ is defined as

$$ J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau, \tau > 0 \tag{8}$$

where $\Gamma(q)$ is the gamma function of q.

Definition 5. The Riemann-Liouville fractional derivative operator of order $q > 0$ of a function $f(t)$ is defined as

$$ {}_0D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f(\tau) d\tau \tag{9}$$

for $q \in (n-1, n)$, $t > 0$ and $n \in N$

The relation between the Riemann-Liouville operator and Caputo fractional differential operator is given by

$$ J^q D_t^q f(t) = D_t^{-q} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \tag{10}$$

TABLE 1. The fundamental properties of CFRDTM

Functional form	Transformed form
$\psi(x,t) = \varphi(x,t) \pm \xi(x,t)$	$\Psi_k(x) = \Phi_k(x) \pm \Xi_k(x)$
$\psi(x,t) = c\varphi(x,t)$	$\Psi_k(x) = c\Phi_k(x)$, c is a constant.
$\psi(x,t) = \varphi(x,t)\xi(x,t)$	$\Psi_k(x) = \sum_{i=0}^{k} \Phi_i(x)\Xi_{k-i}(x)$
$\psi(x,t) = \frac{x^{mq}t^{nq}}{\Gamma(1+mq)\Gamma(1+nq)}$	$\Psi_k(x) = \frac{x^{mq}}{\Gamma(1+mq)} \frac{\delta_q(k-n)}{\Gamma(1+q)}$, $m,n \in N$
$\psi(x,t) = \frac{\partial^{nq}\varphi(x,t)}{\partial t^{nq}}$	$\Psi_k(x) = \frac{\Gamma(1+(k+n)q)}{\Gamma(1+kq)} \Phi_{k+n}(x), n \in N$
$\psi(x,t) = \frac{\partial^{mq}\varphi(x,t)}{\partial x^{mq}}$	$\Psi_k(x) = \frac{\partial^{mq}\Phi_k(x)}{\partial x^{mq}}$, $m \in N$
$\psi(x,t) = x_i^n t^r$	$\Psi_k(x) = x_i^n \delta_{(k-r)}, i = 1,...,m$
$\psi(x,t) = e^{\lambda t}$	$\Psi_k(x) = \frac{\lambda^k}{k!}$

Definition 6. The Mittag-Leffler function is defined as the series representation, valid in the whole complex plane [17]

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(nq+1)} \tag{11}$$

For more details about the properties and applications of fractional calculus; see [17] and [18].

2.3. Caputo Fractional Reduced Differential Transform Method (CFRDTM). The Caputo Fractional Reduced Differential Transform (CFRDT) of the function $\psi(x,t)$ is defined as

$$\Psi_k(x) = \frac{[{}_0^cD_t^{kq}\psi(x,t)]_{t=t_0}}{\Gamma(1+kq)}, q \in (0,1], \;\; k = 0,1,...,n, \tag{12}$$

where ${}_0^cD_t^{kq}\psi(x,t) = \frac{\partial^{kq}\psi(x,t)}{\partial t^{kq}}$. Conversely, the inverse CFRDT of $\Psi_k(x)$ is defined as

$$\psi(x,t) = \sum_{k=0}^{\infty} \Psi_k(x)(t-t_0)^{kq}, 0 < q \leq 1. \tag{13}$$

By means of (12) and (13), the fundamental properties of CFRDTM were summarized in Table 1. See Refs [19, 20, 21, 22] for more details on classical RDTM.

3. CFRDTM FOR THE SOLUTION OF FBOM

Applying CFRDT on both sides of (3) and (5) yields

$$C_0^b = \exp(x) - K \tag{14}$$

and

$$C_{k+1}^{b} = \frac{\Gamma(kq+1)}{\Gamma(q(k+1)+1)}\left[\frac{\partial^2 C_k^b}{\partial x^2} + (\beta-1)\frac{\partial C_k^b}{\partial x} - \beta C_k^b\right] \tag{15}$$

respectively. Using (14) and (15), we have the following

$$C_1^b = \frac{\beta K}{\Gamma(q+1)} \tag{16}$$

$$C_2^b = \frac{-\beta^2 K}{\Gamma(2q+1)} \tag{17}$$

$$C_3^b = \frac{\beta^3 K}{\Gamma(3q+1)} \tag{18}$$

$$C_4^b = \frac{-\beta^4 K}{\Gamma(4q+1)} \tag{19}$$

$$C_5^b = \frac{\beta^5 K}{\Gamma(5q+1)} \tag{20}$$

$$C_6^b = \frac{-\beta^6 K}{\Gamma(6q+1)} \tag{21}$$

$$\vdots$$

$$C_n^b = \frac{(-1)^{n+1}\beta^n K}{\Gamma(nq+1)} \tag{22}$$

$$\vdots$$

By means of inverse CFRDT, one obtains the solution as

$$c^b(x,\tau) = \sum_{k=0}^{\infty} C_k^b \tau^{qk} \tag{23}$$

Substituting (14), (16)-(22) into (23) and simplifying further, one gets

$$c^b(x,\tau) = \exp(x) - KE_q(-\beta\tau^q) \tag{24}$$

where

$$E_q(-\beta\tau^q) = \exp(-\beta\tau^q) = \sum_{m=0}^{\infty}\frac{(-\beta\tau^q)^m}{\Gamma(mq+1)}$$

TABLE 2. The parameters

Parameter	**Description**	**Values**
S	Stock price	\$80
K	Strike price	\$20, \$30, \$40, \$50, \$60
T	Time to expiry	2, 4, 6, 8, 10 (years)
r	Risk-free interest rate	0.1,0.2,0.3,0.4,0.5,0.6
σ	Volatility	0.25
q	Fractional order	0.25, 0.50, 0.75, 0.90, 0.95, 1
B_u	Barrier/ Boundary	\$85

is the Mittag-Leffler function. Using (4), the price of FBOM via CFRDTM is obtained as

$$B(S,t) = S - K\exp\left[-r(T-t)^q\left(\frac{\sigma^2}{2}\right)^{q-1}\right] \tag{25}$$

with $S < e^{B_u}$. For a special case $q = 1$, (25) becomes the result obtained by [23].

4. Application of CFRDTM in the Valuation of FBOM

Using (25) and the following parameters in Table 2. The results generated via CFRDTM were displayed in Figures 1-6. The comparative result analyzes of CFRDTM and Analytical Value (AV) [24] are presented in Figures 7-11. The physical behaviour of the FBOM prices is captured in terms of plots for different values of q in Figure 12. The comparative study of the FBOM prices generated via CFRDTM, Laplace Adomian Decomposition Method (LADM) [25], Two-Dimensional Differential Transform Method (TDDM) [15] and AV is captured in Figure 13.

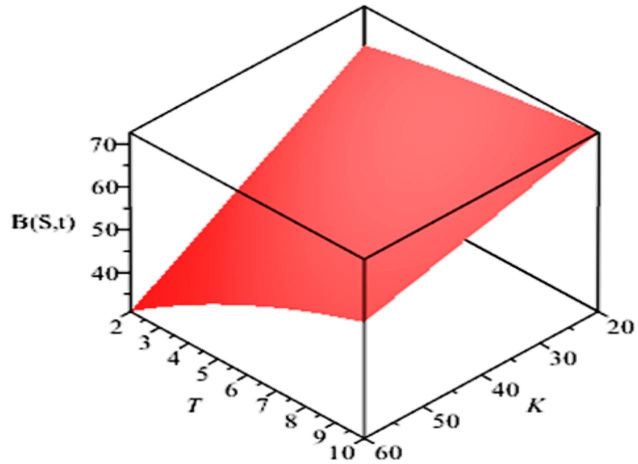

Figure 1: FBOM prices via CFRDTM with $r = 0.1$

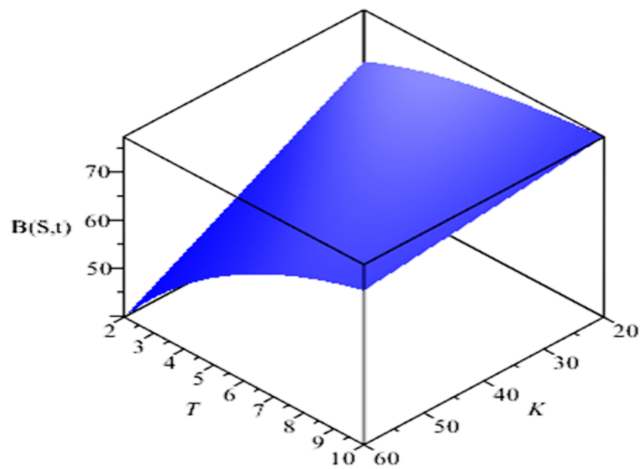

Figure 2: FBOM prices via CFRDTM with $r = 0.2$

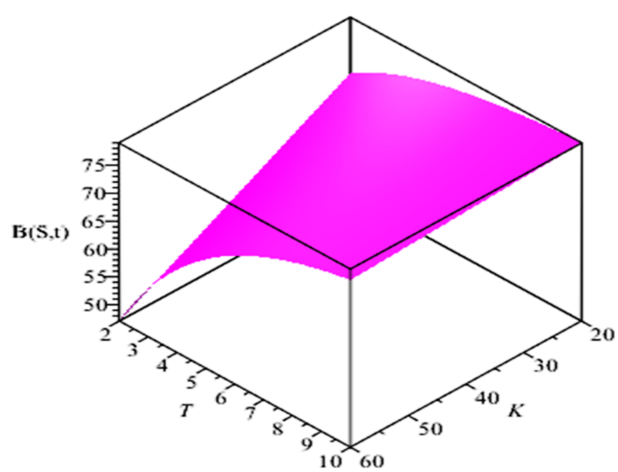

Figure 3: FBOM prices via CFRDTM with $r = 0.3$

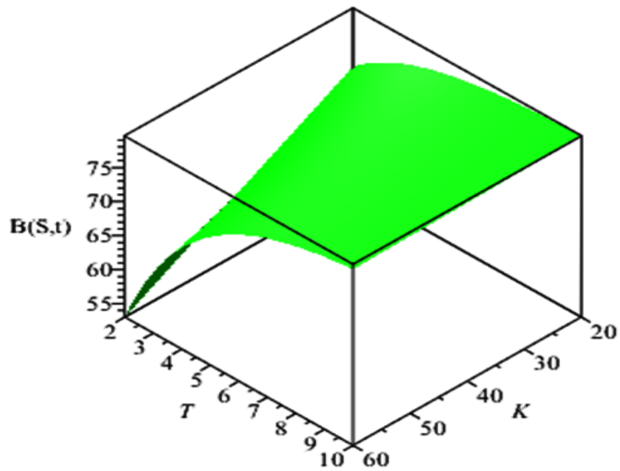

Figure 4: FBOM prices via CFRDTM with $r = 0.4$

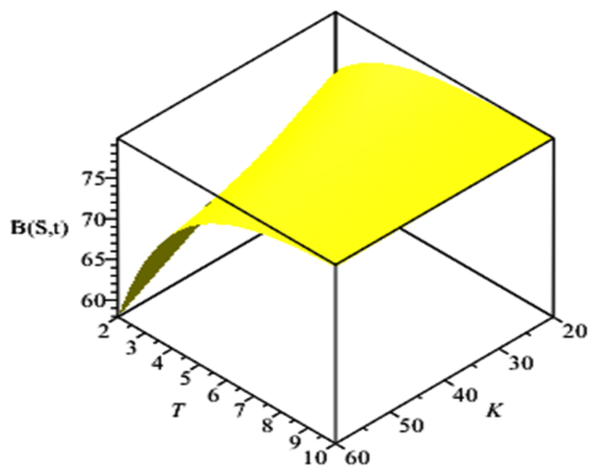

Figure 5: FBOM prices via CFRDTM with $r = 0.5$

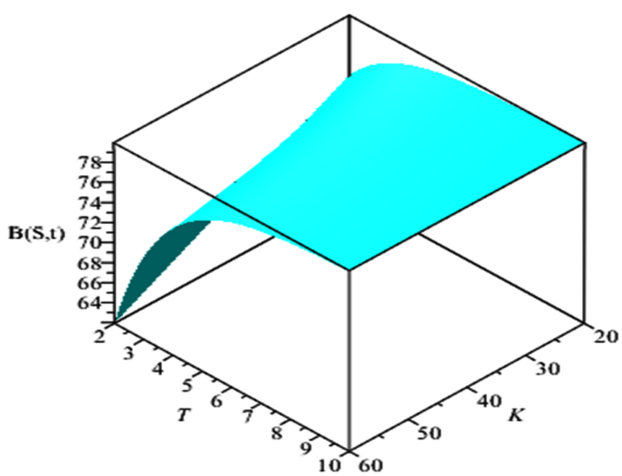

Figure 6: FBOM prices via CFRDTM with $r = 0.6$

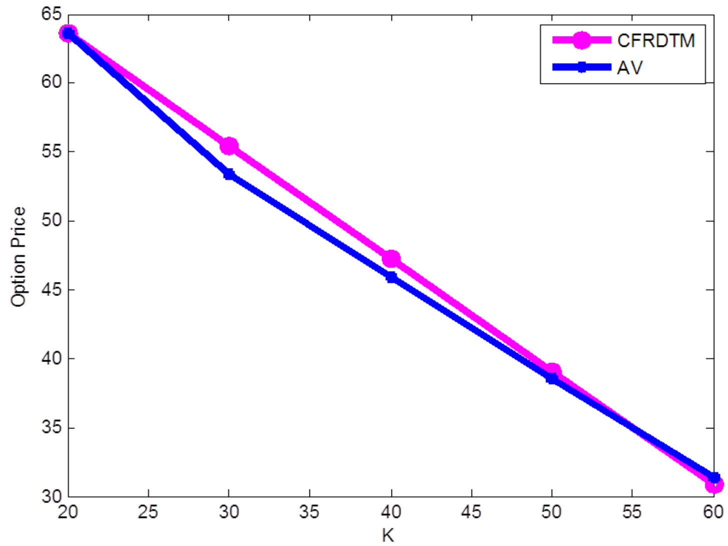

Figure 7: CFRDTM versus AV with $T = 2$ years

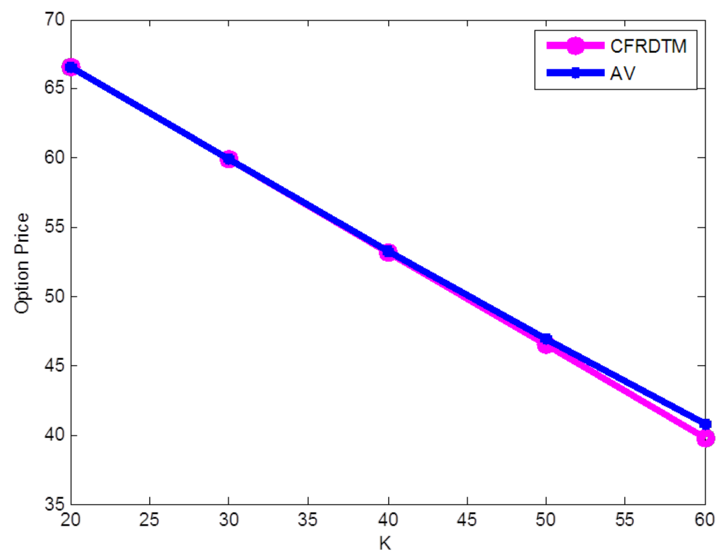

Figure 8: CFRDTM versus AV with $T = 4$ years

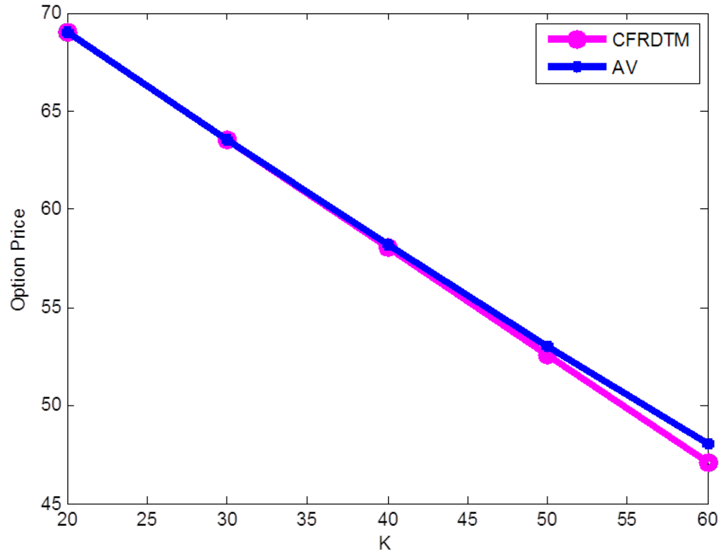

Figure 9: CFRDTM versus AV with $T = 6$ years

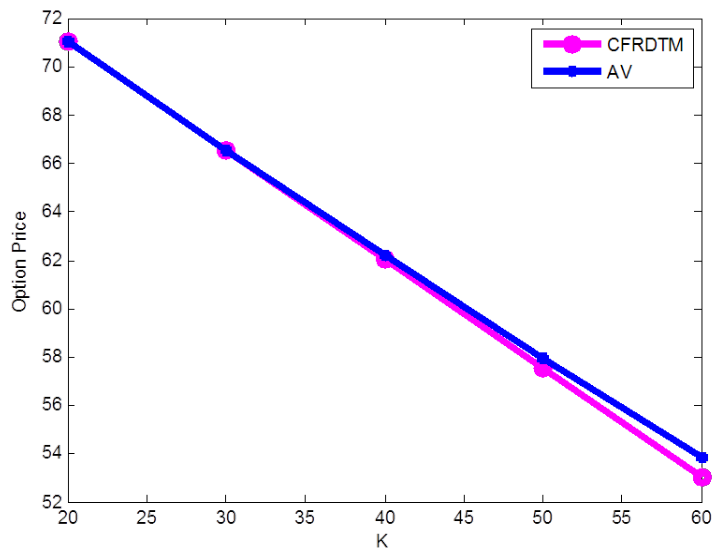

Figure 10: CFRDTM versus AV with $T = 8$ years

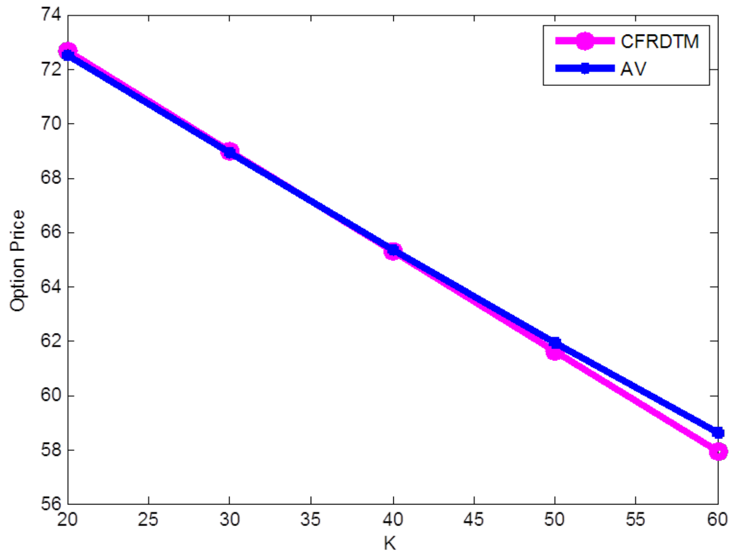

Figure 11: CFRDTM versus AV with $T = 10$ years

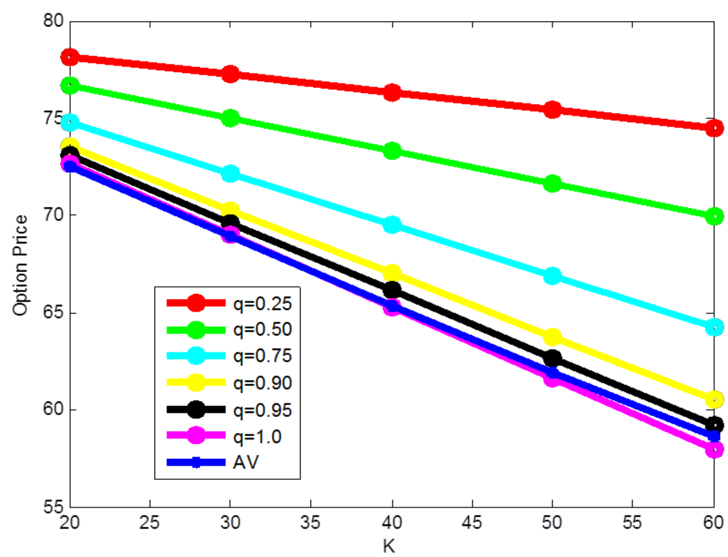

Figure 12: FBOM prices via CFRDTM versus AV with $\sigma = 0.25, r = 0.1, S = \$80, T = 10$ years for different values of q

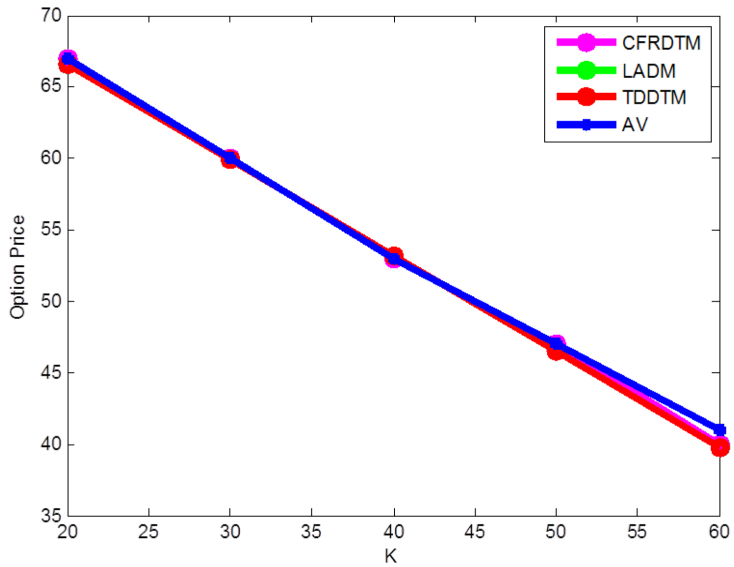

Figure 13: The comparative results analyses of CFRDTM, LADM, TDDTM and AV with $q = 1.0, \sigma = 0.25, r = 0.1, S = \$80, T = 4$ years

5. Discussion of Results and Concluding Remarks

5.1. Discussion of Results. It is observed from Figures 1-6 that the price of FBOM generated via CFRDTM increases as the risk-free interest rate increases. By varying the time to expiry, it is observed from Figures 7-11 that increase in time to expiry leads to increase in FBOM prices. It is also observed that CFRDTM curves follow that of AV more elegantly. Figure 12 shows the physical behaviour of the FBOM prices generated via CFRDTM for different values of fractional order q in the context of AV. It is observed from Figure 12 that CFRDTM compared favourably and agreed with AV for $q = 0.75$, $q = 0.90$, $q = 0.95$ and $q = 1.0$. It is also observed that when $q = 0.25$ and $q = 0.50$, FBOM is overpriced. In other words, when $q = 1$, FBOM has the lowest price in exercise time T. The payoff of FBOM increases as q decreases. It is observed from Figure 13 that the results obtained via CFRDTM agreed with that of LADM, TDDTM and AV.

5.2. Concluding Remarks. In this paper, a new efficient approach "CFRDTM" has been proposed for the solution of FBOM. The valuation formula for the price of Barrier Option (BO) with fractional order is also obtained. Furthermore, an illustrative example is presented to measure the performance of CFRDTM in terms of accuracy, efficiency and suitability. The results show that CFRDTM is in agreement with AV. It is also observed that CFRDTM compared favourably with other existing methods "LADM, TDDTM, AV" for the valuation of FBOM. Moreover, CFRDTM is found to be accurate, computationally efficient and suitable for obtaining an approximate solution of FBOM. In conclusion, CFRDTM is a good approach to be included in the class of methods for the valuation of FBOM. Results and figures are obtained by the help of MAPLE 18 and MATLAB R2014a, Version: 8.3.0.552, 32 bit (win 32) in double precision. The methodology can further be extended to solve barrier option pseudo differential equation with fractional order.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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