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ANALYTICAL SOLUTION OF NONLINEAR KLEIN-GORDON EQUATIONS WITH CUBIC NONLINEARITY BY EXTENDED ADOMIAN DECOMPOSITION METHOD

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Abstract. In this paper, we are dedicated to acquire, for the first time, an analytically continuous result of the nonlinear Klein-Gordon equations (NKGE) with cubic nonlinearity via Adomian decomposition method (ADM) using multivariate Taylor's theorem. These class of equations are nonlinear partial differential equations with initial or boundary conditions being hyperbolic or trigonometric functions. Thus leading to large solution series on application of ADM during the invertible process in the integral equations. Which is often analysed at finite discrete points. To overcome this, we extend the series solution by traditional ADM with the multivariate Taylor's theorem. The process resulted to a simple solution series that was easier to understand and analysed continuously guaranteeing excellent convergence rate. We demonstrate our findings using four examples that were further depicted pictorially using Maple symbolic software.

Keywords: nonlinear Klein-Gordon equations; Adomian decomposition method; multi-variable Taylor's theorem; nonlinear partial differential equations.

2010 AMS Subject Classification: 81Q05, 65M25, 35G25, 42A10.

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1. Introduction

The NKGEs are partial differential equations that are of great applications in nonlinear optics, solid state physics, quantum field theory and many more other field of applications. Its general form is given as

$$u_{tt}(\vec{x},t)-\alpha\nabla^2u(\vec{x},t)+\beta u(\vec{x},t)+N(u(\vec{x},t))=f(\vec{x},t) \tag{1}$$

$$u(\vec{x},0)=b_0\vec{x}, \quad u_t(\vec{x},0)=b_1\vec{x},$$

where $u=u(\vec{x},t)$ is the wave displacement, t is the time, $\nabla^2u(\vec{x},t)=u(\vec{x},t)_{xx}+u(\vec{x},t)_{yy}+u(\vec{x},t)_{zz}$ and $\vec{x}=x+y+z$. α, $\beta\in\mathbb{R}$ and $N(u(\vec{x},t))$ is a given nonlinear force which comes in several nonlinear forms that specifies the actual name of NKGE.

Recently, [16] applied Laplace transform and ADM on inhomogeneous NKGE of at most quadratic nonlinearity to obtain analytically exact solutions; [9] adjusted the ADM to obtained exact solutions to NKGE with quadratic nonlinearity where the main stream ADM gave approximate results. [20] studied the Fuzzy fractional Klein-Gordon-Fock equation using variational iteration, Adomian decomposition and new iteration methods in fluid mechanics. [24] presented an auxiliary equation method on the nonlinear space-time fractional Klein-Gordon equation to obtain analytically exact solution. [21] followed suit by modifying the iteration method to obtain exact solutions to the linear and quadratic nonlinearity. [5] presented a multi-step modified reduced differential transform method with strictly quadratic nonlinearity and Adomian polynomials were applied on the examples considered to obtain approximate result. Similar studies were undertaken using iterative scheme and group preserving scheme with method of lines by [29] and [17] respectively.

Similar investigation has also been carried out in this class of equation. [26] presented a composite numerical method base on finite difference scheme and fixed point iteration method on coupled Klein-Gordon equations. Their findings resulted to approximate solutions with error analysed. [22] presented the variational homotopy perturbation method and compare his numerical result with those of the traditional ADM and variational iteration method. [27] adapted the Galerkin method, logarithmic Sobolev inequality and compactness theorem to the NKGE with logarithmic nonlinearity to obtain week solutions. [25] modified an existing iteration method to solve linear and NKGE of integer and fractional order. The study was based on manipulating the source term into integral representation of the nonlinear term which yielded interesting result. [7] presented a semi-analytical solution based on Jacobi-Gauss-Lobatto collocation method on NKGE other than cubic nonlinearities. Tanh method has also been used by [13] while [14] studied the recurrence and resonance on the NKGE with cubic nonlinearity. [8] investigated the reliability of fully implicit finite difference method over exponential finite difference on a single cubic nonlinearity problem; similar investigations was done on cubic and quadratic nonlinearity by [23]. [2] presented a numerical scheme to solve the time fractional NKGE using Sinc-Chebyshev collocation method.

Several other reported studies include [32]; where the author presented the modified exp-function method to seek a generalised solution on cubic nonlinearity problems. [4] studied theoretically the Dirac and the Klein-Gordon oscillators in non-commutative space, with similar studies using Hilbert space structures in conjunction with the Pseudo-Hermitian operators by [1]. Some other sophisticated numerical approximate methods has also been deplored to obtain results at discrete points from the defined domains. Some available in literature are ADM reported by [6] and [12]; Homotopy perturbation method studied by [11] and [19]. B-Spline Collocation method reported by [2] and [18]. Finite difference method studied by [28] and [30].

However, investigations on NKGE with cubic nonlinearity are quite scanty in existing literatures and the existing studies has mostly resulted to approximate results analysed at discrete point in the given domains. The goal of this paper is to extend the potentials of ADM by slight extension on the solution series using the multivariate Taylor's theorem. The solution procedure in the models considered is to apply the adjustment made after applying ADM. The remaining organisational structure of this paper will be by first stating the theoretical background of ADM and its adjustment on how it can be implemented on the generalised NKGE and its models of cubic nonlinearity. We then take examples to justify our findings and we conclude.

2. Theory of ADM and the Proposed Extension

2.1. Theory of ADM.

The ADM in [15] writes equation (1) using the linear operator L as

$$L(u(\vec{x},t)) = f(\vec{x},t) + \alpha \nabla^2 u(\vec{x},t) - \beta u(\vec{x},t) - N(u(\vec{x},t)) \tag{2}$$

Since the method is a constructive scheme, we assume that L^{-1} exist, resulting to the integral equation

$$u(\vec{x},t) = \phi + L^{-1}\left[f(\vec{x},t) + \alpha\nabla^2 u(\vec{x},t) - \beta u(\vec{x},t) - N(u(\vec{x},t))\right] \tag{3}$$

ϕ is a terms resulting from the initial or boundary conditions. Where the method dictate that we write the solution of equation (1) as

$$u(\vec{x},t) = \sum_{n=0}^{\infty} u_n(\vec{x},t) \tag{4}$$

with $N(u(\vec{x},t)) = \sum_{n=0}^{\infty} A_n u(\vec{x},t)$ and $A_n u(\vec{x},t)$ are the Adomian polynomials which can be generated easily with

$$A_n u(\vec{x},t) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[f\left(\sum_{k=0}^{\infty} u_k(\vec{x},t)\lambda^k\right)\right]_{\lambda=0} \qquad n \in \left[0 \cup \mathbb{Z}^+\right]$$

λ is as a stabilising parameter. The $A_n(\vec{x},t)$ are functionally defined explicitly as

$$\begin{aligned}
A_0 u(\vec{x},t) &= A(u_0(\vec{x},t)\\
A_1 u(\vec{x},t) &= A(u_0(\vec{x},t), u_1(\vec{x},t))\\
A_2 u(\vec{x},t) &= A(u_0(\vec{x},t), u_1(\vec{x},t), u_2(\vec{x},t))\\
A_3 u(\vec{x},t) &= A(u_0(\vec{x},t), u_1(\vec{x},t), u_2(\vec{x},t), u_3(\vec{x},t))\\
A_4 u(\vec{x},t) &= A(u_0(\vec{x},t), u_1(\vec{x},t), u_2(\vec{x},t), u_3(\vec{x},t), u_4(\vec{x},t))\\
A_5 u(\vec{x},t) &= A(u_0(\vec{x},t), u_1(\vec{x},t), u_2(\vec{x},t), u_3(\vec{x},t), u_4(\vec{x},t), u_5(\vec{x},t))\\
&\dots\\
A_n u(\vec{x},t) &= A(u_0(\vec{x},t), u_1(\vec{x},t), u_2(\vec{x},t), \dots, u_{n-2}(\vec{x},t), u_{n-1}(\vec{x},t), u_n(\vec{x},t))
\end{aligned}$$

See [10] and [9] for in-depth definition of hyperbolic $N(u)$ and trigonometric $N(u)$ respectively and more in [11]. The components in equation (4) are determined recursively as

$$\begin{aligned}
u_0(\vec{x},t) &= \phi + L^{-1}\left[f(\vec{x},t)\right]\\
u_{n+1}(\vec{x},t) &= L^{-1}\left[\alpha\nabla^2 u_n(\vec{x},t) - \beta u_n(\vec{x},t) - A_n(u_n(\vec{x},t))\right]
\end{aligned}$$

The term

$$\varphi_q(\vec{x},t) = \sum_{n=0}^{q} u_n(\vec{x},t) \tag{5}$$

converges to $u(\vec{x},t)$ as $q \to \infty$ and $q \in \mathbb{Z}^+$.

2.2. The Extension. From equation (5), $\varphi_q(\vec{x},t)$ is a series of partial sum. We assume that $\varphi_q(\vec{x},t)$ has a continuous partial derivative up through order $n+1$, for a self specified order in the neighbourhood of the point (x_0,y_0,z_0,t_0). We consider $\nabla\varphi_q(\vec{x},t) : \mathbb{R}^n \longrightarrow \mathbb{R}$. Where $\nabla\varphi_q(\vec{x},t)$ is the partial derivatives of $\varphi_q(\vec{x},t)$. This assumption is predicated on the existing literatures on ADM for a one space-variable function reported in [9], that the series solution of equation (1) is a Taylor series expansion about $(\vec{x}_0,t_0)$.

In accordance with Taylor's theorem, let $\varphi_q(\vec{x},t) = \varphi_q(\vec{w})$ be finitely differentiable around $(x_0,y_0,z_0,t_0) = \vec{w}_0$, then the theorem says that

$$\varphi_q(\vec{w}+\vec{w}_0) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n \varphi_q(\vec{w}) \tag{6}$$

Where $D = \vec{w}_0\nabla$ and now $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + \frac{\partial}{\partial t}$. Writing out equation (6) through quadratic terms approximation more elaborately gives a second order Taylor expansion of $\varphi_q(\vec{w})$ about $\vec{w}_0$ of the form

$$\varphi_q(\vec{w}) \approx \varphi_q(\vec{w}_0) + (\vec{w}-\vec{w}_0)\,.\,\nabla\varphi_q(\vec{w}_0) + \frac{1}{2!}(\vec{w}-\vec{w}_0)^T \mathbf{H}\varphi_q(\vec{w}_0)(\vec{w}-\vec{w}_0) \tag{7}$$

Where $(\vec{w}-\vec{w}_0)^T$ is a matrix transpose of 4-dimensional row vector with component $(\vec{w}-\vec{w}_0)^T = (x-x_0, y-y_0, z-z_0, t-t_0)$ and $\mathbf{H}\varphi_q(\vec{w}_0)$ is a symmetric Hessian matrix given as

$$\mathbf{H}\varphi_q(\vec{w}_0) = \begin{bmatrix} \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial x^2} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial x\partial y} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial x\partial z} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial x\partial t} \\ \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial y\partial x} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial y^2} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial y\partial z} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial y\partial t} \\ \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial z\partial x} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial z\partial y} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial z^2} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial z\partial t} \\ \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial t\partial x} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial t\partial y} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial t\partial z} & \frac{\partial^2\varphi_q(\vec{w}_0)}{\partial t^2} \end{bmatrix}$$

From the multivariate Taylor's theorem, the n-th order Taylor polynomial $T_n\varphi_q(\vec{w})$ of $\varphi_q(\vec{w})$ about $(\vec{w}_0)$ closely approximate $\varphi_q(\vec{w})$ to the n-th degree near $(\vec{w}_0)$ in the sense that

$$T_n\varphi_q(\vec{w}_0) \approx \varphi_q(\vec{w}_0) \tag{8}$$

and

$$\lim_{\vec{w}\to\vec{w}_0} \frac{\varphi_q(\vec{w}) - T_n\varphi_q(\vec{w})}{\left(\sqrt{\vec{w}^2}\right)^n} = 0$$

Also, the exact solution $u(\vec{w})$ has

(9) $$T_n u(\vec{w}_0) \approx u(\vec{w}_0)$$

and

$$\lim_{\vec{w}\to\vec{w}_0} \frac{u(\vec{w}) - T_n u(\vec{w})}{\left(\sqrt{\vec{u}^2}\right)^n} = 0$$

Where $\vec{w}^2 = x^2 + y^2 + z^2 + t^2 = \vec{u}^2$. Finally, we see that

$$T_n\varphi_q(\vec{w}) \approxeq T_n u(\vec{w})$$

Where the T_n of both functions are choices of approximating polynomials ranging from linear, quadratic, cubic, quartic, quintic, to any order as desired by the investigator.

Theorem 1. *If there exist $\vec{w} \in \mathbb{R}^n$ with positive components, then $T_n\varphi_q(\vec{w})$ is a unique analytical solution of* (1)

Proof. The proof can be found in [3] □

3. Practical Numerical Examples

Example 1. Consider the one-dimensional NKGE with cubic nonlinearity reported in [23] without a continuous closed form solution and also reported in [28].

(10) $$u_{tt} - u_{xx} + u + u^3 = 0$$

$u(x,0) = A\left[1+\cos\left(\frac{2\pi}{1.28}x\right)\right], u_t(x,0) = 0$. Where $x \in [0, 1.28], t > 0$ and $A \in [0.1, 100]$ is the amplitude. Following the exposition in equations (2) - (8) we have

$$\begin{aligned}\varphi_5(x,t) = A\theta - \frac{1}{2}\left(\beta^2\cos\beta x + A\theta + A^3\theta^3\right)t^2 - \frac{1}{4}(A^3\theta\beta^2\sin^2\beta x \\ - \frac{1}{2}\left(A^3\theta^2\beta^2\cos\beta x + A^3\theta^2(A\beta^2\cos\beta x + A\theta + A^3\theta^3)\right) \\ - \frac{1}{3}A\beta^2\cos\beta x - \frac{1}{6}\left(A\theta + A^3\theta^3 + A\beta^4\cos\beta x\right))t^4 - \cdots\end{aligned}$$

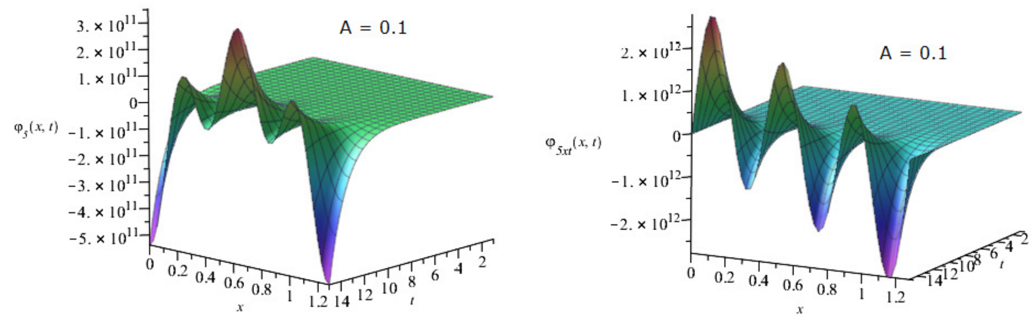

FIGURE 1. Space-time graph of Example 1: $A = 0.1$

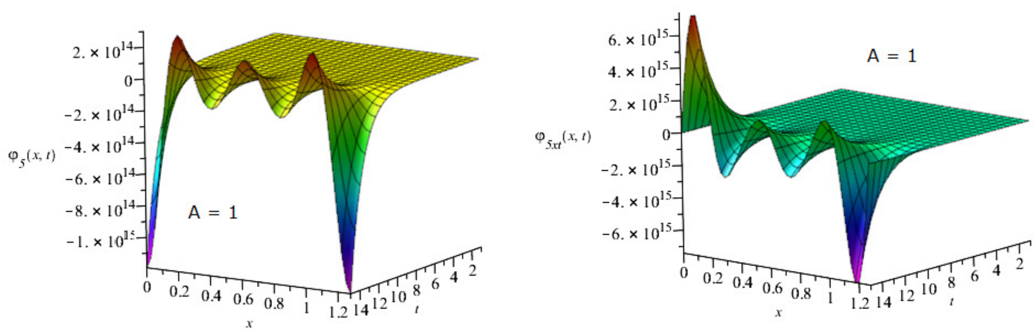

FIGURE 2. Space-time graph of Example 1: $A = 1$

with $\varphi_5(x,t)$ about $(0,0)$ of quintic approximation, $T_5\varphi_5(x,t)$ given as

$$\begin{aligned}T_5\varphi_5(x,t) &= 2A - \frac{1}{2}A\beta^2x^2 + \frac{1}{24}A\beta^4x^4 - \frac{1}{2}\left(8A^3 + A\beta^2 + 2A\right)t^2 \\ &+ \frac{1}{4}\left(12A^3\beta^2 + A\beta^4 + A\beta^2\right)x^2t^2 \\ &+ \frac{1}{24}\Big(96A^5 + 24A^3\beta^2 + A\beta^4 + 32A^3 + 2A\beta^2 + 2A\Big)t^4 + \cdots\end{aligned}$$

Where $\beta = \dfrac{2\pi}{1.28}$ and $\theta = 1 + \cos\beta x$. See more results in figures 1, 2 and 3 at varying amplitudes. Where $\varphi_{5xt}(x,t) = \dfrac{\partial^2}{\partial x\partial t}\varphi_5(x,t)$ and we noticed that higher amplitudes are replica of figure 3.

Example 2. Consider also the one-dimensional NKGE with cubic nonlinearity reported in [23] and [18].

$$u_{tt} - u_{xx} + u + u^3 = f(x,t) \tag{11}$$

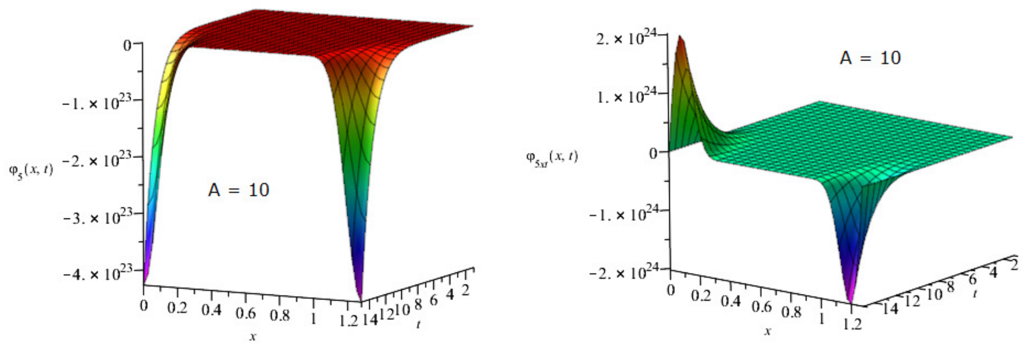

FIGURE 3. Space-time graph of Example 1: $A = 10$

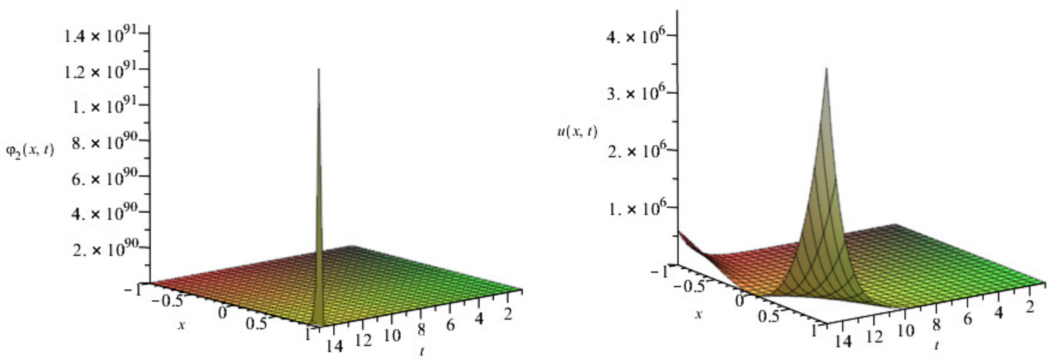

FIGURE 4. Space-time graph of Example 2

$u(x,0) = x^2 \cosh x, u_t(x,0) = x^2 \cosh x$. Where $f(x,t) = (x^2-2)\cosh(x+t) - 4x\sinh(x+t) + x^6\cosh^3(x+t), \quad x \in [-1,1]$ and $t > 0$. The exact solution is $u(x,t) = x^2\cosh(x+t)$. Nonetheless, following the exposition in equations (2) - (8) we have $\varphi_2(x,t)$ about $(0,0)$ of septic approximation $T_7\varphi_2(x,t)$, given as

$$T_7\varphi_2(x,t) = x^2 + \frac{1}{2}x^2t^2 + x^2t + \frac{1}{2}x^4 + \frac{1}{4}x^4t^2 + \cdots$$

and that of the exact solution $u(x,t)$ about $(0,0)$ of the same approximation is given as

$$T_7u(x,t) = x^2 + \frac{1}{2}x^2t^2 + x^3t + +\frac{1}{2}x^4 + \frac{1}{4}x^4t^2 + \cdots$$

More results are further depicted in figures 4 and 5

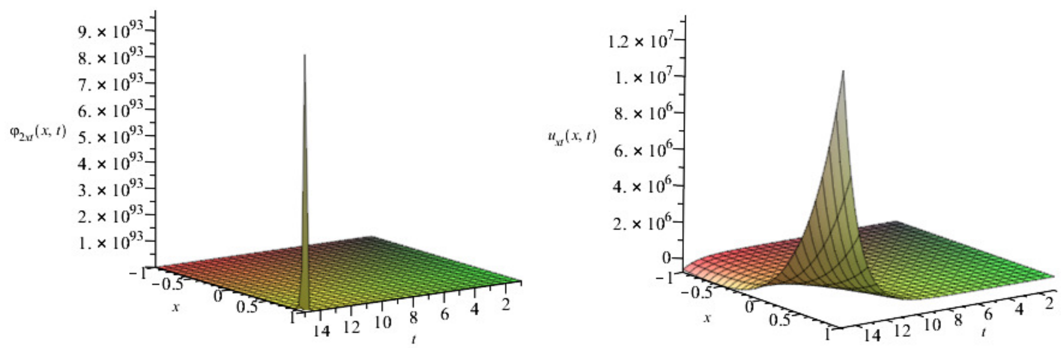

FIGURE 5. Space-time graph of Example 2

Example 3. Consider also the one-dimensional NKGE with cubic nonlinearity reported in [23] and [6].

$$u_{tt}-\frac{5}{2}u_{xx}+u+\frac{3}{2}u^3=f(x,t) \tag{12}$$

$u(x,0)=B\tan(kx), u_t(x,0)=Bck\sec^2(kx)$. Where $f(x,t)=0,\quad B=\frac{\sqrt{6}}{3},\quad k=\sqrt{-\frac{1}{2c^2-5}}$ $x\in[0,1]$ and $t>0$. The exact solution is $u(x,t)=B\tan(k(x+ct))$. We also consider $c=\frac{1}{2},\frac{1}{20}$ as studied by [23]. Nevertheless, following also the exposition in equations (2) - (8) we have $\varphi_4(x,t)$ about $(0,0)$ of septic approximation $T_7\varphi_4(x,t)$ given as

$$T_7\varphi_4(x,t)=tBck+Bkx+\frac{1}{3}Bk^3x^3+tBck^3x^2+\frac{2}{15}Bk^5x^5+\frac{2}{3}tBck^5x^4+tBck^3x^2+\cdots$$

and

$$T_7u(x,t)=tBck+Bkx+\frac{1}{3}Bk^3x^3+tBck^3x^2+\frac{2}{15}Bk^5x^5+\frac{2}{3}tBck^5x^4+tBck^3x^2+\cdots$$

The extended space-time graphs beyond what has already been studied in literature are as presented in figures 6, 7, 8 and 9 with $c=0.5$ and $c=0.05$.

Example 4. Consider also the one-dimensional NKGE with cubic nonlinearity reported in [12] and [31].

$$u_{tt}-u_{xx}+\frac{3}{4}u-\frac{3}{2}u^3=0 \tag{13}$$

$u(x,0)=-sechx, u_t(x,0)=\frac{1}{2}sechx\tanh x, t>0$. The exact solution is $u(x,t)=-sech(x+\frac{1}{2}t)$. Nonetheless, following the exposition in equations (2) - (8) we have $\varphi_5(x,t)$ about $(0,0)$ of

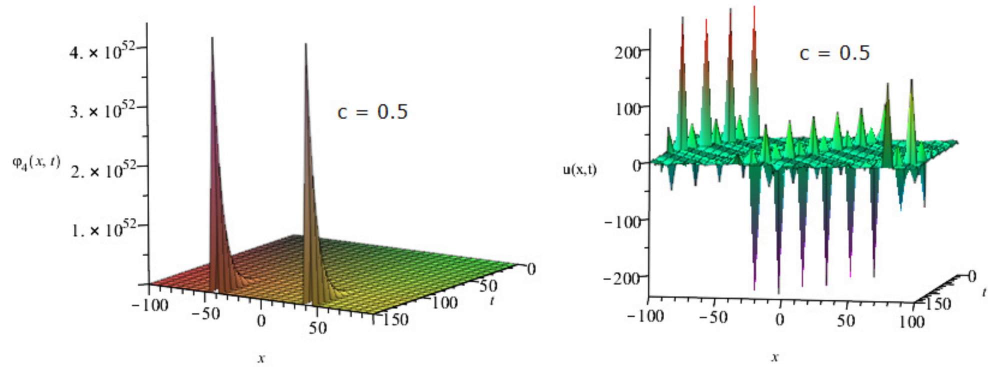

FIGURE 6. Space-time graph of Example 3

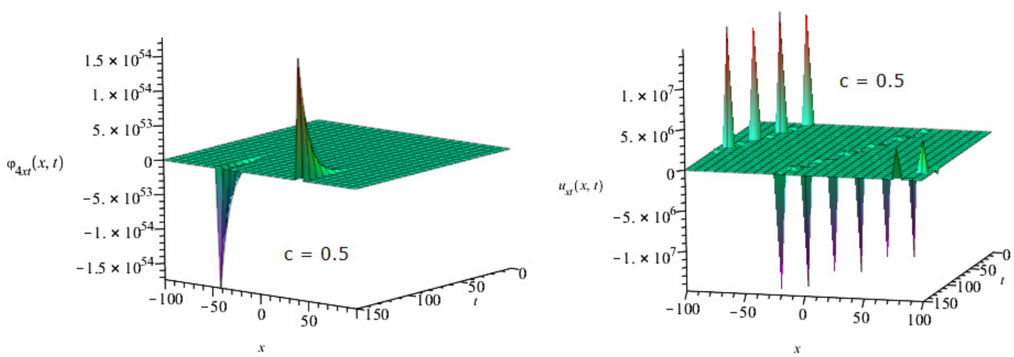

FIGURE 7. Space-time graph of Example 3

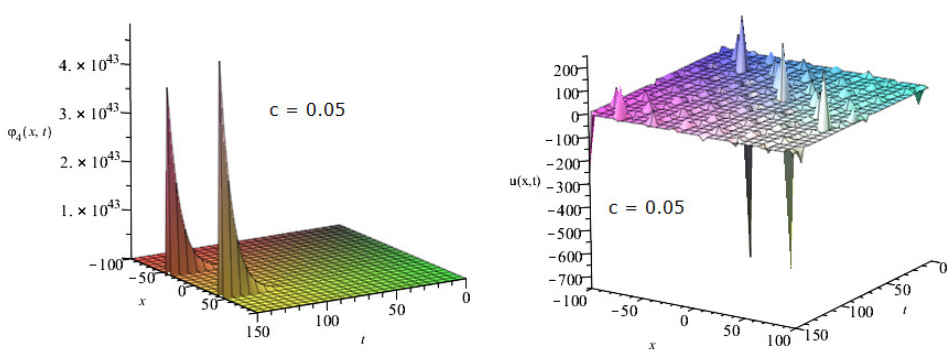

FIGURE 8. Space-time graph of Example 3

nonic approximation $T_9\varphi_5(x,t)$ given as

$$T_9\varphi_5(x,t) = -1 + \frac{1}{3}tx + \frac{1}{2}x^2 + \frac{1}{8}t^2 - \frac{5}{48}x^3t - \frac{5}{384}t^4 - \frac{5}{16}x^2t^2 - \frac{5}{24}x^4 - \frac{5}{12}tx^3 + \cdots$$

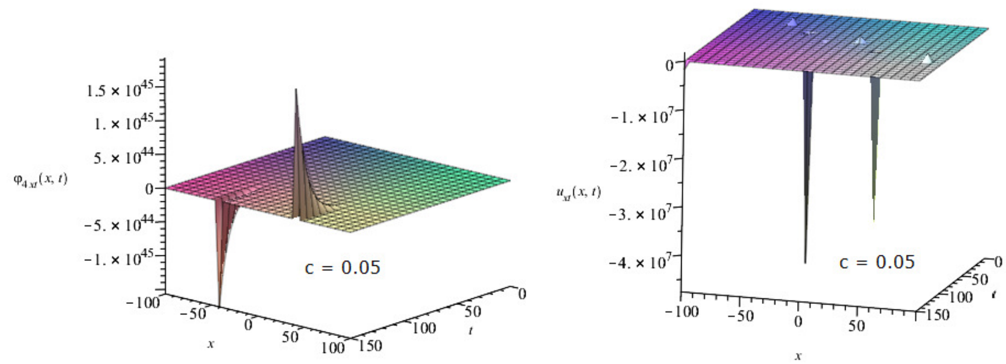

FIGURE 9. Space-time graph of Example 3

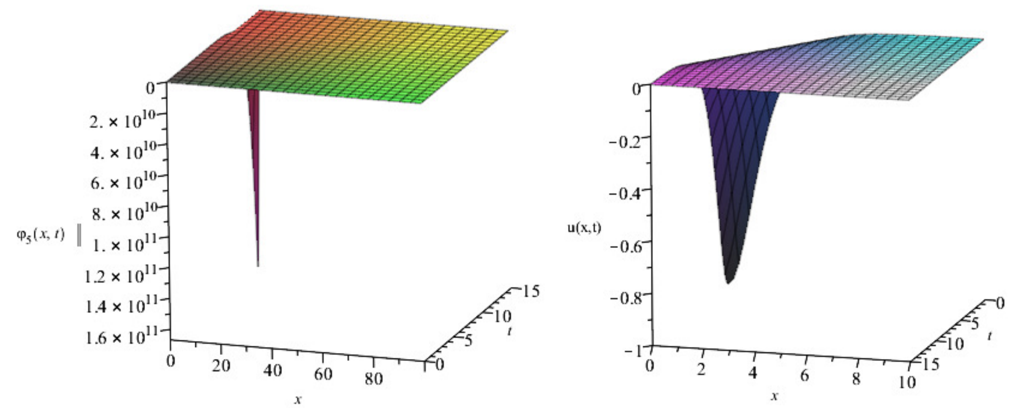

FIGURE 10. Space-time graph of Example 4

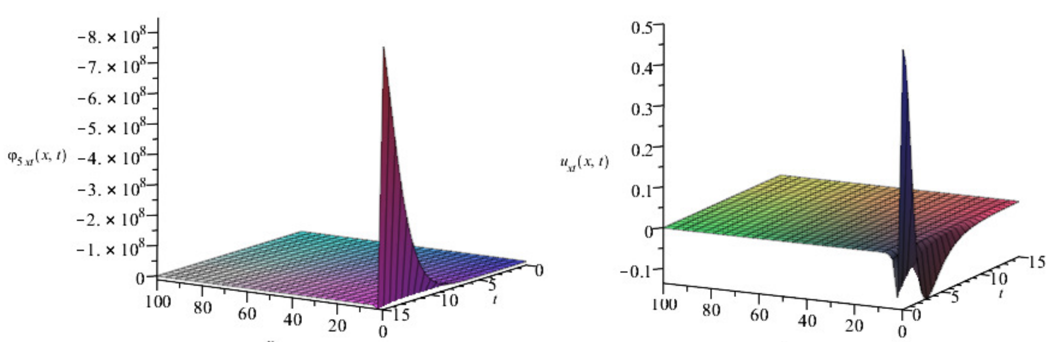

FIGURE 11. Space-time graph of Example 4

and that of the exact solution $u(x,t)$ about $(0,0)$ of the same approximation given as

$$T_9u(x,t) = -1 + \frac{1}{3}tx + \frac{1}{2}x^2 + \frac{1}{8}t^2 - \frac{5}{48}x^3t - \frac{5}{384}t^4 - \frac{5}{16}x^2t^2 - \frac{5}{24}x^4 - \frac{5}{12}tx^3 + \cdots$$

More results are further depicted in figures 10 and 11

CONCLUSION

We have, in this study, been able to extend the series solution by ADM on the NKGE using the multivariate Taylor's theorem to get simple analytically continuous result. The extension became necessary to obviate the troublesome proliferation of terms in an excessively large expression by the main stream ADM in the integral equation during the invertible process. Which in exiting literatures are only analysed at discrete and finite number of points. Although, the resulting outcome in this study did not come without alternating terms that cancelled out and/or fused up in pairs during the computational process.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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