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THE SECOND HYPER-ZAGREB INDEX OF GRAPH OPERATIONS

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Abstract. A graph can be recognized by numeric number, polynomial or matrix which represent the whole graph. Topological index is a numerical descriptor of a molecule, based on a certain topological feature of the corresponding molecular graph, it is found that there is a strong correlation between the properties of chemical compounds and their molecular structure. Zagreb indices are numeric numbers related to graphs. In this study, the second Hyper-Zagreb index for some special graphs, and graph operations has been computed, that have been applied to compute the second Hyper-Zagreb index for Nano-tube and Nano-torus.

Keywords: Zagreb indices; the first Hyper-Zagreb index; the second Hyper-Zagreb index; graph operations.

2010 AMS Subject Classification: 11N45, 37C70.

1. Introduction

Topological indices are real numbers related to graphs, They have many applications as tools for modeling chemical and other properties of molecules. In practical applications, Zagreb Indices are among the best topological indices applications to recognize the physical properties, chemical reactions and biological activities. [4, 15]. Throughout this paper, we consider a finite

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connected graph G that has no loops or multiple edges. The vertex and the edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The degree of the vertex a is the number of edges joined with this vertex denoted by $\delta(a)$.

The first Zagreb index $M_1(G)$, and the second Zagreb index $M_2(G)$ were firstly considered by Gutman and Trinajstic in 1972 [1, 4]. They are defined as:

$$M_1(G) = \sum_{v \in V(G)} {\delta_G}^2(v) = \sum_{uv \in E(G)} \delta_G(u) + \delta_G(v)$$

$$M_2(G) = \sum_{uv \in E(G)} \delta_G(u)\, \delta_G(v)$$

In 2005, Li and Zheng [17] introduced the first general Zagreb index as:

$${M_1}^{\alpha+1}(G) = \sum_{v \in V(G)} {\delta_G}^{\alpha+1}(v) = \sum_{uv \in E(G)} {\delta_G}^{\alpha}(u) + {\delta_G}^{\alpha}(v)$$

In 2013, Shirdel et al [14]. introduced degree-based of Zagreb indices named Hyper-Zagreb index as:

$$HM(G) = \sum_{uv \in E(G)} (\delta_G(u) + \delta_G(v))^2$$

In 2013, Ranjini et al [13, 18]. re-defined the Zagreb indices, the third Zagreb indices for a graph G as:

$$ReZG_3(G) = \sum_{uv \in E(G)} \delta_G(u)\, \delta_G(v) [\delta_G(u) + \delta_G(v)]$$

Furtula and Gutman in 2015 introduced forgotten index (F-index) [7, 9] which defined as:

$$F(G) = \sum_{v \in V(G)} {\delta_G}^3(v) = \sum_{uv \in E(G)} \left({\delta_G}^2(u) + {\delta_G}^2(v) \right)$$

In 2016, computed exact formulas for the Zagreb and Hyper-Zagreb indices of Carbon Nanocones $CNC_k[n]$ by Gao et al. They defined a new degree-based of Zagreb indices named second Hyper-Zagreb index [16].

Definition 1.1: The second Hyper-Zagreb index of a graph G defined as:

$$HM_2(G) = \sum_{uv\in E(G)} [\delta_G(u)\,\delta_G(v)]^2 = \sum_{uv\in E(G)} {\delta_G}^2(u){\delta_G}^2(v)$$

In this paper, we present some exact formulaes of the second Hyper-Zagreb index for some special graphs and some graph binary operations such as tensor product $G_1 \otimes G_2$, Cartesian product $G_1 \times G_2$,, composition $G_1 \circ G_2$,, strong product $G_1 * G_2$,, disjunction $G_1 \vee G_2$ and symmetric difference $G_1 \oplus G_2$, of graphs. We apply some results to compute the second Hyper-Zagreb index for some important classes of nano-structures such as nano-tube and nano-torus. In order to calculate the second Hyper-Zagreb index of graph operations, we need one modern versions of Zagreb index is given by [7].

$$Y(G) = \sum_{u\in V(G)} \delta_G^4(u) = \sum_{uv\in E(G)} [\delta_G^3(u) + \delta_G^3(v)]$$

We named this index ”Y-index”, which studied of some graph operations [2].

Definition 1.2: Product binary operations create a new graph G from two initial graphs G_1, G_2, the resulting graph has the same set of vertices $V(G_1), V(G_2)$ and $|V(G)| = |V(G_1)||V(G_2)|$ but its set of edges depends of the considered operation, i.e., if $|V(G_1)| = p_1, |V(G_2)| = p_2, |E(G_1)| = q_1$ and $|E(G_2)| = q_2$. Then [11].

$$E(G_1 \otimes G_2) = \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E_1, v_1v_2 \in E_2\}.$$

$$E(G_1 \times G_2) = \{(u_1,v_1)(u_2,v_2)\} : \begin{cases} u_1u_2 \in E_1, v_1 = v_2 \\ v_1v_2 \in E_2, u_1 = u_2 \end{cases}$$

$$E(G_1 \circ G_2) = \{(u_1,v_1)(u_2,v_2)\} : \begin{cases} u_1u_2 \in E_1 \\ v_1v_2 \in E_2, u_1 = u_2 \end{cases}$$

$$E(G_1 * G_2) = \{(u_1,v_1)(u_2,v_2)\} : \begin{cases} u_1u_2 \in E_1, v_1 = v_2 \\ v_1v_2 \in E_2, u_1 = u_2 \\ u_1u_2 \in E_1, v_1v_2 \in E_2 \end{cases}$$

$$E(G_1 \oplus G_2) = \{(u_1,v_1)(u_2,v_2)\} : \begin{cases} u_1u_2 \in E_1, v_1v_2 \notin E_2 \\ v_1v_2 \in E_2, u_1u_2 \notin E_1 \end{cases}$$

$$E(G_1 \vee G_2) = \{(u_1,v_1)(u_2,v_2)\} : \begin{cases} u_1u_2 \in E_1 \\ v_1v_2 \in E_2 \end{cases}$$

Lemma 1.3: Let G_1 and G_2 be graphs with $|V(G_1)| = p_1, |V(G_2)| = p_2$, $|E(G_1)| = q_1$ and $|E(G_2)| = q_2$. Then [8, 11]

(a)

1. $|E(G_1 \otimes G_2)| = 2q_1q_2$
2. $|E(G_1 \times G_2)| = p_1q_2 + p_2q_1$
3. $|E(G_1 \circ G_2)| = p_2^2q_1 + p_1q_2$
4. $|E(G_1 * G_2)| = p_1q_2 + p_2q_1 + 2q_1q_2$
5. $|E(G_1 \vee G_2)| = p_1^2q_2 + p_2^2q_1 - 2q_1q_2$
6. $|E(G_1 \oplus G_2)| = p_1^2q_2 + p_2^2q_1 - 4q_1q_2$

(b)

1. $\delta_{G_1 \otimes G_2}(u,v) = \delta_{G_1}(u)\delta_{G_2}(v)$
2. $\delta_{G_1 \times G_2}(u,v) = \delta_{G_1}(u) + \delta_{G_2}(v)$
3. $\delta_{G_1 \circ G_2}(u,v) = p_2\delta_{G_1}(u) + \delta_{G_2}(v)$
4. $\delta_{G_1 * G_2}(u,v) = \delta_{G_1}(u) + \delta_{G_2}(v) + \delta_{G_1}(u)\delta_{G_2}(v)$
5. $\delta_{G_1 \vee G_2}(u,v) = p_2\delta_{G_1}(u) + p_1\delta_{G_2}(v) - \delta_{G_1}(u)\delta_{G_2}(v)$
6. $\delta_{G_1 \oplus G_2}(u,v) = p_2\delta_{G_1}(u) + p_1\delta_{G_2}(v) - 2\delta_{G_1}(u)\delta_{G_2}(v)$

Any unexplained terminology is standard, typically as in [3, 6, 10].

2. Preliminaries

In this part, we give the second Hyper-Zagreb index of some special graphs as: complete graph K_n, cycle C_n, path P_n, complete bipartite graph $K_{m,n}$ and conical graph $C_{m,n}$ (cf. Fig. 1) [5].

(1) $HM_2(K_n) = \frac{1}{2}n(n-1)^5$

(2) $HM_2(C_n) = 16n$

(3) $HM_2(P_n) = 8(2n-5)$

(4) $HM_2(K_{m,n}) = m^3n^3$

(5) $HM_2(C_{m,n}) = n(9n^2+225n+512m-768)$

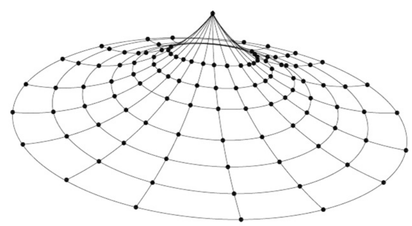

FIGURE 1. $C_{7,19}$

3. MAIN RESULTS

In the following section, we study the second Hyper-Zagreb index of some graph operations.

Theorem 3.1: The second Hyper-Zagreb index of $(G_1 \otimes G_2)$ is given by.

$$HM_2(G_1 \otimes G_2) = 2HM_2(G_1)HM_2(G_2)$$

Proof. By Definition 1.1 and Lemma 1.3, we have

$$\begin{aligned}
HM_2(G_1 \otimes G_2) &= \sum_{(a,c)(b,d)\in E(G_1\otimes G_2)} [\delta_{(G_1\otimes G_2)}(a,c)\,\delta_{(G_1\otimes G_2)}(b,d)]^2 \\
&= \sum_{(a,c)(b,d)\in E(G_1\otimes G_2)} [\delta^2_{(G_1\otimes G_2)}(a,c)][\delta^2_{(G_1\otimes G_2)}(b,d)] \\
&= 2\sum_{ab\in E(G_1)}\sum_{cd\in E(G_2)} [\delta^2_{G_1}(a)\delta^2_{G_2}(c)][\delta^2_{G_1}(b)\delta^2_{G_2}(d)] \\
&= 2\sum_{ab\in E(G_1)} [\delta^2_{G_1}(a)\delta^2_{G_1}(b)] \sum_{cd\in E(G_2)} [\delta^2_{G_2}(c)\delta^2_{G_2}(d)] \\
&= 2HM_2(G_1)HM_2(G_2). \quad \square
\end{aligned}$$

Theorem 3.2: The second Hyper-Zagreb index of $(G_1 \times G_2)$ is given by.

$$\begin{aligned} HM_2(G_1 \times G_2) &= p_1 HM_2(G_2) + p_2 HM_2(G_1) + 3F(G_1)M_1(G_2) + 3F(G_2)M_1(G_1) \\ &+ q_1[Y(G_2) + 4ReZG_3(G_2)] + q_2[Y(G_1) + 4ReZG_3(G_1)] \\ &+ 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1). \quad \square \end{aligned}$$

Proof. By Definition 1.1 and Lemma 1.3, we have $HM_2(G_1 \times G_2)$

$$\begin{aligned} &= \sum_{(a,c)(b,d)\in E(G_1\times G_2)} [\delta_{(G_1\times G_2)}(a,c)\,\delta_{(G_1\times G_2)}(b,d)]^2 \\ &= \sum_{(a,c)(b,d)\in E(G_1\times G_2)} (\delta_{G_1}(a) + \delta_{G_2}(c))^2(\delta_{G_1}(b) + \delta_{G_2}(d))^2 \\ &= \sum_{a=b\in V(G_1)} \sum_{cd\in E(G_2)} [\delta_{G_1}(a)\delta_{G_1}(b) + \delta_{G_1}(a)\delta_{G_2}(d) + \delta_{G_1}(b)\delta_{G_2}(c) + \delta_{G_2}(c)\,\delta_{G_2}(d)]^2 \\ &+ \sum_{ab\in E(G_1)} \sum_{c=d\in V(G_2)} [\delta_{G_1}(a)\delta_{G_1}(b) + \delta_{G_1}(a)\delta_{G_2}(d) + \delta_{G_1}(b)\delta_{G_2}(c) + \delta_{G_2}(c)\,\delta_{G_2}(d)]^2 \end{aligned}$$

Step1.

$$\sum_{a=b\in V(G_1)} \sum_{cd\in E(G_2)} (\delta_{G_1}(a)\delta_{G_1}(b) + \delta_{G_1}(a)\delta_{G_2}(d) + \delta_{G_1}(b)\delta_{G_2}(c) + \delta_{G_2}(c)\,\delta_{G_2}(d))^2$$

$$\begin{aligned} &= \sum_{a=b\in V(G_1)} \sum_{cd\in E(G_2)} [\delta^2_{G_1}(a)\delta^2_{G_1}(b) + \delta^2_{G_1}(a)\delta^2_{G_2}d + \delta^2_{G_1}(b)\delta^2_{G_2}(c) + \delta^2_{G_2}(c)\,\delta^2_{G_2}(d) \\ &+ 2\delta^2_{G_1}(a)\delta_{G_1}(b)\delta_{G_2}(d) + 2\delta_{G_1}(a)\delta^2_{G_1}(b)\delta_{G_2}(c) + 2\delta_{G_1}(a)\delta_{G_2}(c)\delta^2_{G_2}(d) \\ &+ 2\delta_{G_1}(b)\delta^2_{G_2}(c)\delta_{G_2}(d) + 4\delta_{G_1}(a)\delta_{G_1}(b)\delta_{G_2}(c)\,\delta_{G_2}(d)] \\ &= \sum_{a=b\in V(G_1)} \delta^2_{G_1}(a)\delta^2_{G_1}(b) \sum_{cd\in E(G_2)} 1 + \sum_{a=b\in V(G_1)} \delta^2_{G_1}(a) \sum_{cd\in E(G_2)} \delta^2_{G_2}(d) \\ &+ \sum_{a=b\in V(G_1)} \delta^2_{G_1}(b) \sum_{cd\in E(G_2)} \delta^2_{G_2}(c) + \sum_{a=b\in V(G_1)} 1 \sum_{cd\in E(G_2)} \delta^2_{G_2}(c)\delta^2_{G_2}(d) \\ &+ 2\sum_{a=b\in V(G_1)} \delta^2_{G_1}(a)\delta_{G_1}(b) \sum_{cd\in E(G_2)} \delta_{G_2}(d) + 2\sum_{a=b\in V(G_1)} \delta_{G_1}(a)\delta^2_{G_1}(b) \sum_{cd\in E(G_2)} \delta_{G_2}(c) \\ &+ 2\sum_{a=b\in V(G_1)} \delta_{G_1}(a) \sum_{cd\in E(G_2)} \delta_{G_2}(c)\delta^2_{G_2}(d) + 2\sum_{a=b\in V(G_1)} \delta_{G_1}(b) \sum_{cd\in E(G_2)} \delta^2_{G_2}(c)\delta_{G_2}(d) \\ &+ 4\sum_{a=b\in V(G_1)} \delta_{G_1}(a)\delta_{G_1}(b) \sum_{cd\in E(G_2)} \delta_{G_2}(c)\delta_{G_2}(d) \end{aligned}$$

$$
\begin{aligned}
&= [\sum_{a\in V(G_1)} \delta^4_{G_1}(a)][q_2] + [M_1(G_1)][\sum_{cd\in E(G_2)} \delta^2_{G_2}(c)+\delta^2_{G_2}(d)] + [p_1][HM_2(G_2)] \\
&+ 2[F(G_1)][\sum_{cd\in E(G_2)} \delta_{G_2}(c)+\delta_{G_2}(d)] + 4[M_1(G_1)][M_2(G_2)] \\
&+ 2[2q_1][\sum_{cd\in E(G_2)} \delta^2_{G_2}(c)\delta_{G_2}(d)(\delta_{G_2}(c)+\delta_{G_2}(d)] \\
&= [Y(G_1)][q_2] + [M_1(G_1)][F(G_2)] + p_1 HM_2(G_2) + 2[F(G_1][M_1(G_2)] \\
&+ 4[M_1(G_1)][M_2(G_2)] + 4[q_1][ReZG_3(G_2)]
\end{aligned}
$$

Step2. Similar step1. we have

$$
\sum_{ab\in E(G_1)} \sum_{c=d\in V(G_2)} [\delta_{G_1}(a)\delta_{G_1}(b)+\delta_{G_1}(a)\delta_{G_2}(d)+\delta_{G_1}(b)\delta_{G_2}(c)+\delta_{G_2}(c)\,\delta_{G_2}(d)]^2
$$

$$
\begin{aligned}
&= [Y(G_2)][q_1] + [M_1(G_2)][F(G_1)] + p_2 HM_2(G_1) + 2[F(G_2][M_1(G_1)] \\
&+ 4[M_1(G_2)][M_2(G_1)] + 4[q_2][ReZG_3(G_1)]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
HM_2(G_1\times G_2) &= p_1 HM_2(G_2) + p_2 HM_2(G_1) + 3F(G_1)M_1(G_2) + 3F(G_2)M_1(G_1) \\
&+ q_1[Y(G_2)+4ReZG_3(G_2)] + q_2[Y(G_1)+4ReZG_3(G_1)] \\
&+ 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1). \quad \square
\end{aligned}
$$

Theorem 3.3: The second Hyper-Zagreb index of $(G_1 \circ G_2)$ is given by.

$$
\begin{aligned}
HM_2 G_1 \circ G_2) &= p_2^6 HM_2(G_1) + p_1 HM_2(G_2) + p_2^4 q_2[4ReZG_3(G_1)+Y(G_1)] + 4p_2 q_1 ReZG_3(G_2) \\
&+ 3p_2^3 M_1(G_2)F(G_1) + p_2^2 M_1(G_1)[F(G_2)+4M_2(G_2)] + q_1 {M_1}^2(G_2) \\
&+ 4p_2 q_2[4p_2 q_2 M_2(G_1) + M_1(G_1)M_1(G_2)]
\end{aligned}
$$

Proof. By Definition 1.1 and Lemma 1.3, we have

$$
\begin{aligned}
HM_2(G_1\circ G_2) &= \sum_{(a,c)(b,d)\in E(G_1\circ G_2)} [\delta_{(G_1\circ G_2)}(a,c)\,\delta_{(G_1\circ G_2)}(b,d)]^2\\
&= \sum_{(a,c)(b,d)\in E(G_1[G_2])} [p_2\delta_{G_1}(a)+\delta_{G_2}(c)]^2[p_2\delta_{G_1}(b)+\delta_{G_2}(d)]^2\\
&= \sum_{c\in V(G_2)}\sum_{d\in V(G_2)}\sum_{ab\in E(G_1)} [p_2^2\delta_{G_1}(a)\delta_{G_1}(b)+p_2\delta_{G_1}(a)\delta_{G_2}(d)\\
&+ p_2\delta_{G_1}(b)\delta_{G_2}(c)+\delta_{G_2}(c)\delta_{G_2}(d)]^2\\
&+ \sum_{a=b\in V(G_1)}\sum_{cd\in E(G_2)} [p_2^2\delta_{G_1}(a)\delta_{G_1}(b)+p_2\delta_{G_1}(a)\delta_{G_2}(d)\\
&+ p_2\delta_{G_1}(b)\delta_{G_2}(c)+\delta_{G_2}(c)\delta_{G_2}(d)]^2
\end{aligned}
$$

Step1.

$$
\begin{aligned}
&\sum_{c\in V(G_2)}\sum_{d\in V(G_2)}\sum_{ab\in E(G_1)} [p_2^2\delta_{G_1}(a)\delta_{G_1}(b)+p_2\delta_{G_1}(a)\delta_{G_2}(d)+p_2\delta_{G_1}(b)\delta_{G_2}(c)+\delta_{G_2}(c)\,\delta_{G_2}(d)]^2\\
&= \sum_{c\in V(G_2)}\sum_{d\in V(G_2)}\sum_{ab\in E(G_1)} [p_2^4\delta_{G_1}^2(a)\delta_{G_1}^2(b)+p_2^2\delta_{G_1}^2(a)\delta_{G_2}^2 d+p_2^2\delta_{G_1}^2(b)\delta_{G_2}^2(c)+\delta_{G_2}^2(c)\delta_{G_2}^2(d)\\
&+ 2p_2^3\delta_{G_1}^2(a)\delta_{G_1}(b)\delta_{G_2}(d)+2p_2^3\delta_{G_1}(a)\delta_{G_1}^2(b)\delta_{G_2}(c)+2p_2\delta_{G_1}(a)\delta_{G_2}(c)\delta_{G_2}^2(d)\\
&+ 2p_2\delta_{G_1}(b)\delta_{G_2}^2(c)\delta_{G_2}(d)+4\delta_{G_1}(a)\delta_{G_1}(b)\delta_{G_2}(c)\delta_{G_2}(d)]\\
&= p_2^6 HM_2(G_1)+p_2^3M_1(G_2)F(G_1)+q_1M_1^2(G_2)+16p_2^2q_2M_2(G_1)\\
&+ 4p_2^4q_2ReZG_3(G_1)+4p_2q_2M_1(G_1)M_1(G_2)
\end{aligned}
$$

Step2.

$$
\begin{aligned}
&\sum_{a=b\in V(G_1)}\sum_{cd\in E(G_2)} [p_2^2\delta_{G_1}(a)\delta_{G_1}(b)+p_2\delta_{G_1}(a)\delta_{G_2}(d)+p_2\delta_{G_1}(b)\delta_{G_2}(c)+\delta_{G_2}(c)\,\delta_{G_2}(d)]^2\\
&= \sum_{a=b\in V(G_1)}\sum_{cd\in E(G_2)} [p_2^4\delta_{G_1}^2(a)\delta_{G_1}^2(b)+p_2^2\delta_{G_1}^2(a)\delta_{G_2}^2 d+p_2^2\delta_{G_1}^2(b)\delta_{G_2}^2(c)+\delta_{G_2}^2(c)\delta_{G_2}^2(d)\\
&+ 2p_2^3\delta_{G_1}^2(a)\delta_{G_1}(b)\delta_{G_2}(d)+2p_2^3\delta_{G_1}(a)\delta_{G_1}^2(b)\delta_{G_2}(c)+2p_2\delta_{G_1}(a)\delta_{G_2}(c)\delta_{G_2}^2(d)\\
&+ 2p_2\delta_{G_1}(b)\delta_{G_2}^2(c)\delta_{G_2}(d)+4\delta_{G_1}(a)\delta_{G_1}(b)\delta_{G_2}(c)\delta_{G_2}(d)]\\
&= p_2^4q_2Y(G_1)+p_2^2M_1(G_1)F(G_2)+p_1HM_2(G_2)+4p_2^2M_1(G_1)M_2(G_2)\\
&+ 2p_2^3M_1(G_2)F(G_1)+4p_2q_1ReZG_3(G_2)
\end{aligned}
$$

It is easy to see that the summation of step 1, step 2 complete the proof. □

Theorem 3.4: The second Hyper-Zagreb index of $(G_1 * G_2)$ is given by.

$$HM_2(G_1 * G_2)$$

$$\begin{aligned}
&= HM_2(G_2)[p_1 + 10q_1 + 10M_1(G_1) + 8M_2(G_1) + 6F(G_1) + 4ReZG_3(G_1) + Y(G_1)] \\
&+ HM_2(G_1)[p_2 + 10q_2 + 10M_1(G_2) + 8M_2(G_2) + 6F(G_2) + 4ReZG_3(G_2) + Y(G_2)] \\
&+ Y(G_2)[q_1 + 2M_1(G_1) + 4M_2(G_1) + F(G_1) + 2ReZG_3(G_1)] \\
&+ Y(G_1)[q_2 + 2M_1(G_2) + 4M_2(G_2) + F(G_2) + 2ReZG_3(G_2)] \\
&+ 4ReZG_3(G_2)[q_1 + 2M_1(G_1) + 2M_2(G_1) + 2F(G_1)] \\
&+ 4ReZG_3(G_1)[q_2 + 2M_1(G_2) + 2M_2(G_2) + 2F(G_2)] \\
&+ F(G_2)[3M_1(G_1) + 8M_2(G_1)] + F(G_1)[3M_1(G_2) + 8M_2(G_2)] \\
&+ 8M_2(G_1)M_2(G_2) + 4M_1(G_1)M_2(G_2) + 4M_1(G_2)M_2(G_1) \\
&+ 2HM_2(G_1)HM_2(G_2) + 5F(G_1)F(G_2) + 6ReZG_3(G_1)ReZG_3(G_2)
\end{aligned}$$

Proof. By Definition 1.1 and Lemma 1.3, we have

$$HM_2(G_1 * G_2) = \sum_{(a,c)(b,d)\in E(G_1*G_2)} \delta^2_{G_1*G_2}(a,c)\delta^2_{G_1*G_2}(b,d)$$

$$\begin{aligned}
&= \sum_{(a,c)(b,d)\in E(G_1*G_2)} [(\delta_{G_2}(c) + \delta_{G_1}(a) + \delta_{G_1}(a)\delta_{G_2}(c))(\delta_{G_2}(d) + \delta_{G_1}(b) + \delta_{G_1}(b)\delta_{G_2}(d))]^2 \\
&= \sum_{(a,c)(b,d)\in E(G_1*G_2)} [\delta_{G_1}(a)\delta_{G_1}(b) + \delta_{G_1}(a)\delta_{G_2}(d) + \delta_{G_1}(b)\delta_{G_2}(c) + \delta_{G_2}(c)\delta_{G_2}(d) \\
&+ \delta_{G_1}(a)\delta_{G_1}(b)\delta_{G_2}(c)\delta_{G_2}(d) + \delta_{G_1}(a)\delta_{G_1}(b)(\delta_{G_2}(c) + \delta_{G_2}(d)) + \delta_{G_2}(c)\delta_{G_2}(d)(\delta_{G_1}(a) + \delta_{G_1}(b))]^2 \\
&= \sum_{a=b\in V(G_1)} \sum_{cd\in E(G_2)} [\cdots] + \sum_{c=d\in V(G_2)} \sum_{ab\in E(G_1)} [\cdots] + 2\sum_{ab\in E(G_1)} \sum_{cd\in E(G_2)} [\cdots]
\end{aligned}$$

Step1.

$$\sum_{a=b\in V(G_1)} \sum_{cd\in E(G_2)} [\cdots\cdots]$$

$$
\begin{aligned}
&= HM_2(G_2)[p_1+8q_1+6M_1(G_1)+4F(G_1)+Y(G_1)]+4q_1ReZG_3(G_2)\\
&+ Y(G_1)[q_2+2M_1(G_2)+2M_2(G_2)+HM_1(G_2)+2ReZG_3(G_2)]\\
&+ F(G_1)[2M_1(G_2)+4M_2(G_2)+2HM_1(G_2)+6ReZG_3(G_2)]\\
&+ M_1(G_1)[F(G_2)+4M_2(G_2)+6ReZG_3(G_2)]
\end{aligned}
$$

Step2. $\sum\limits_{c=d\in V(G_2)}\sum\limits_{ab\in E(G_1)}[\cdots\cdots]$

$$
\begin{aligned}
&= HM_2(G_1)[p_2+8q_2+6M_1(G_2)+4F(G_2)+Y(G_2)]+4q_2ReZG_3(G_1)\\
&+ Y(G_2)[q_1+2M_1(G_1)+2M_2(G_1)+HM_1(G_1)+2ReZG_3(G_1)]\\
&+ F(G_2)[2M_1(G_1)+4M_2(G_1)+2HM_1(G_1)+6ReZG_3(G_1)]\\
&+ M_1(G_2)[F(G_1)+4M_2(G_1)+6ReZG_3(G_1)]
\end{aligned}
$$

Step3. $2\sum\limits_{ab\in E(G_1)}\sum\limits_{cd\in E(G_2)}[\cdots\cdots]$

$$
\begin{aligned}
&= HM_2(G_2)[2q_1+4M_1(G_1)+4M_2(G_1)+2HM_1(G_1)+4ReZG_3(G_1)]+4M_2(G_2)ReZG_3(G_1)\\
&+ HM_2(G_1)[2q_2+4M_1(G_2)+4M_2(G_2)+2HM_1(G_2)+4ReZG_3(G_2)]+4M_2(G_1)ReZG_3(G_2)\\
&+ 2HM_2(G_1)HM_2(G_2)+8M_2(G_1)M_2(G_2)+F(G_1)F(G_2)+6ReZG_3(G_1)ReZG_3(G_2)\\
&+ 2ReZG_3(G_2)[M_1(G_1)+HM_1(G_1)]+2ReZG_3(G_1)[M_1(G_2)+HM_1(G_2)]
\end{aligned}
$$

Combining these three Steps (1, 2, 3) will complete the proof. □

Theorem 3.5: The second Hyper-Zagreb index of $(G_1 \vee G_2)$ is given by.

$$
\begin{aligned}
HM_2(G_1\vee G_2) &= HM_2(G_2)[p_1(p_1^5+16p_1q_1^2-10p_1^3q_1-8p_1M_2(G_1)-2p_1F(G_1)+4ReZG_3(G_1))\\
&+ M_1(G_1)(M_1(G_1)+6p_1^3-8p_1q_1)]+p_2^2F(G_2)M_1(G_1)[p_1^3-4p_1q_1+M_1(G_1)]\\
&+ HM_2(G_1)[p_2(p_2^5+16p_2q_2^2-10p_2^3q_2-8p_2M_2(G_2)-2p_2F(G_2)+4ReZG_3(G_2))\\
&+ M_1(G_2)(M_1(G_2)+6p_2^3-8p_2q_2)]+p_1^2F(G_1)M_1(G_2)[p_2^3-4p_2q_2+M_1(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2p_2ReZG_3(G_2)[p_1^2(2p_1^2q_1+F(G_1)-8q_1^2+4M_2(G_1))-M_1(G_1)(M_1(G_1)\\
&+ \quad 2p_1^3-6p_1q_1)]+4p_2^2M_2(G_2)[4p_1^2q_1^2+M_1^2(G_1)-4p_1q_1M_1(G_1)]\\
&+ \quad 2p_1ReZG_3(G_1)[p_2^2(2p_2^2q_2+F(G_2)-8q_2^2+4M_2(G_2))-M_1(G_2)(M_1(G_2)\\
&+ \quad 2p_2^3-6p_2q_2)]+4p_1^2M_2(G_1)[4p_2^2q_2^2+M_1^2(G_2)-4p_2q_2M_1(G_2)]\\
&+ \quad 2M_1(G_1)M_1(G_2)[p_1^3(2p_2q_2-M_1(G_2))+p_2^3(2p_1q_1-M_1(G_1))]\\
&+ \quad p_1^4q_1M_1^2(G_2)+p_2^4q_2M_1^2(G_1)-6p_1p_2ReZG_3(G_1)ReZG_3(G_2)\\
&- \quad p_1^2p_2^2[F(G_1)F(G_2)+8M_2(G_1)M_2(G_2)]-2HM_2(G_1)HM_2(G_2)
\end{aligned}
$$

Proof. By Definition 1.1, we have

$$HM_2(G_1 \vee G_2) = \sum_{(a,c)(b,d)\in E(G_1\vee G_2)} \delta^2_{G_1\vee G_2}(a,c)\delta^2_{G_1\vee G_2}(b,d)$$

$$= \sum_{a\in V(G_1)}\sum_{b\in V(G_1)}\sum_{cd\in E(G_2)}[\cdots] + \sum_{c\in V(G_2)}\sum_{d\in V(G_2)}\sum_{ab\in E(G_1)}[\cdots] - 2\sum_{ab\in E(G_1)}\sum_{cd\in E(G_2)}[\cdots]$$

But

$$|E(G_1\vee G_2)| = p_1^2q_2+p_2^2q_1-2q_1q_2$$

Therefore,

$$HM_2(G_1 \vee G_2) = \sum_{a\in V(G_1)}\sum_{b\in V(G_1)}\sum_{cd\in E(G_2)}[\cdots] + \sum_{c\in V(G_2)}\sum_{d\in V(G_2)}\sum_{ab\in E(G_1)}[\cdots] - 2\sum_{ab\in E(G_1)}\sum_{cd\in E(G_2)}[\cdots]$$

Where $\delta_{G_1\vee G_2}(a,c) = p_1\delta_{G_2}(c)+p_2\delta_{G_1}(a)-\delta_{G_1}(a)\delta_{G_2}(c)$

Then $\delta^2_{G_1\vee G_2}(a,c) = (p_1\delta_{G_2}(c)+p_2\delta_{G_1}(a)-\delta_{G_1}(a)\delta_{G_2}(c))^2$

$$= p_1^2\delta^2_{G_2}(c)+p_2^2\delta^2_{G_1}(a)+\delta^2_{G_1}(a)\delta^2_{G_2}(c)+2p_1p_2\delta^2_{G_1}(a)\delta^2_{G_2}(c)-2p_1\delta_{G_1}(a)\delta^2_{G_2}(c)-2p_2\delta^2_{G_1}(a)\delta_{G_2}(c)$$

Similar $\delta_{G_1\vee G_2}(b,d) = p_1\delta_{G_2}(d)+p_2\delta_{G_1}(b)-\delta_{G_1}(b)\delta_{G_2}(d)$

Then $\delta^2_{G_1\vee G_2}(b,d) = (p_1\delta_{G_2}(d)+p_2\delta_{G_1}(b)-\delta_{G_1}(b)\delta_{G_2}(d))^2$

$$= p_1^2\delta^2_{G_2}(d)+p_2^2\delta^2_{G_1}(a)+\delta^2_{G_1}(b)\delta^2_{G_2}(d)+2p_1p_2\delta^2_{G_1}(b)\delta^2_{G_2}(d)-2p_1\delta_{G_1}(b)\delta^2_{G_2}(d)-2p_2\delta^2_{G_1}(b)\delta_{G_2}(d)$$

Step1.

$$\sum_{a\in V(G_1)}\sum_{b\in V(G_1)}\sum_{cd\in E(G_2)}[\cdots\cdots]$$

$$
\begin{aligned}
= & \quad HM_2(G_2)[p_1^6+{M_1}^2(G_1)+16{p_1}^2q_1^2+2p_1^3M_1(G_1)-8p_1^4q_1-8p_1q_1M_1(G_1)]+p_2^4q_2M_1^2(G_1)\\
+ & \quad p_2ReZG_3(G_2)[4p_1^4q_1-2p_1^3M_1(G_1)+4p_1q_1M_1(G_1)-2M_1^2(G_1)-16{p_1}^2q_1^2+8p_1q_1M_1(G_1)]\\
+ & \quad 4p_2^2M_2(G_2)[4{p_1}^2q_1^2+M_1^2(G_1)-4p_1q_1M_1(G_1)]+2p_2^3M_1(G_1)M_1(G_2)[2p_1q_1-M_1(G_1)]\\
+ & \quad p_2^2F(G_2)M_1(G_1)[p_1^3-4p_1q_1+M_1(G_1)]
\end{aligned}
$$

Steps 2,3

$$
\sum_{c\in V(G_2)}\sum_{d\in V(G_2)}\sum_{ab\in E(G_1)}[\cdots\cdots]-2\sum_{ab\in E(G_1)}\sum_{cd\in E(G_2)}[\cdots\cdots]
$$

$$
\begin{aligned}
= & \quad HM_2(G_1)[p_2^6+{M_1}^2(G_2)+16{p_2}^2q_2^2+2p_2^3M_1(G_2)-8p_2^4q_2-8p_2q_2M_1(G_2)]+p_1^4q_1M_1^2(G_2)\\
+ & \quad p_1ReZG_3(G_1)[4p_2^4q_2-2p_2^3M_1(G_2)+4p_2q_2M_1(G_2)-2M_1^2(G_2)-16{p_2}^2q_2^2+8p_2q_2M_1(G_2)]\\
+ & \quad 4p_1^2M_2(G_1)[4{p_2}^2q_2^2+M_1^2(G_2)-4p_2q_2M_1(G_2)]+2p_1^3M_1(G_1)M_1(G_2)[2p_2q_2-M_1(G_2)]\\
+ & \quad p_1^2F(G_1)M_1(G_2)[p_2^3-4p_2q_2+M_1(G_2)]\\
= & \quad -2p_1HM_2(G_2)[p_1^3q_1-2p_1^2M_1(G_1)+4p_1M_2(G_1)+p_1F(G_1)-2ReZG_3(G_1)]\\
- & \quad 2p_2HM_2(G_1)[p_2^3q_2-2p_2^2M_1(G_2)+4p_2M_2(G_2)+p_2F(G_2)-2ReZG_3(G_2)]\\
- & \quad 2p_1^2p_2ReZG_3(G_2)[p_1M_1(G_1)-F(G_1)-4M_2(G_1)]-8p_1^2p_2^2M_2(G_1)M_2(G_2)\\
- & \quad 2p_2^2p_1ReZG_3(G_1)[p_2M_1(G_2)-F(G_2)-4M_2(G_2)]-p_1^2p_2^2F(G_1)F(G_2)\\
- & \quad 2HM_2(G_1)HM_2(G_2)-6p_1p_2ReZG_3(G_1)ReZG_3(G_1)
\end{aligned}
$$

Buy summation of step 1, step 2 step 3 complete the proof. □

Example 3.6: Khalifeh, M. H., et al. computed the *PI* index of *C*4 nanotubes and nanotori in [16]. They also computed the Hyper-Wiener index of these molecular graphs in [12]. In this example, we compute the second Hyper-Zagreb index of these molecular graphs. Suppose R and S denote a C_4 nanotube and nanotorus, respectively. Then,

$$R=P_n\times C_m, S=C_n\times C_m$$

By (Theorem 4.2) we have

$$HM_2(R)=HM_2(P_n\times C_m)=2m(256n-391)$$

Where

$$G_1 \equiv P_n, \quad p_1 = n, q_1 = n-1, M_1(P_n) = 4n-6, M_2(P_n) = 4n-8,$$

$$F(P_n) = 8n-14, Y(P_n) = 16n-30, ReZG_3(P_n) = 16n-24, HM_2(P_n) = 16n-40,$$

$$G_1 \equiv C_m, \quad p_2 = q_2 = m, M_1(C_m) = M_2(C_m) = 4m,$$

$$F(C_m) = 8m, Y(C_m) = ReZG_3(C_m) = HM_2(C_m) = 16m$$

Similar we have $\quad HM_2(S) = HM_2(C_n \times C_m) = 512nm$

Finally, using a similar method, one can prove the following exact formula for the second Hyper-Zagreb index of symmetric difference of graphs.

Theorem 3.7: The second Hyper-Zagreb index of $(G_1 \oplus G_2)$ is given by.

$$\begin{aligned}
& HM_2(G_1 \oplus G_2) \\
= \; & HM_2(G_2)[p_1(p_1^5 + 64p_1q_1^2 - 20p_1^3q_1 - 64p_1M_2(G_1) - 16p_1F(G_1) + 64ReZG_3(G_1)) \\
+ \; & M_1(G_1)(16M_1(G_1) + 24p_1^3 - 64p_1q_1)] + p_2^2F(G_2)M_1(G_1)[p_1^3 - 8p_1q_1 + 4M_1(G_1)] \\
+ \; & HM_2(G_1)[p_2(p_2^5 + 64p_2q_2^2 - 20p_2^3q_2 - 64p_2M_2(G_2) - 16p_2F(G_2) + 64ReZG_3(G_2)) \\
+ \; & M_1(G_2)(16M_1(G_2) + 24p_2^3 - 64p_2q_2)] + p_1^2F(G_1)M_1(G_2)[p_2^3 - 8p_2q_2 + 4M_1(G_2)] \\
+ \; & 4p_2ReZG_3(G_2)[p_1^2(p_1^2q_1 + 2F(G_1) - 8q_1^2 + 8M_2(G_1)) - M_1(G_1)(4M_1(G_1) \\
+ \; & 2p_1^3 - 12p_1q_1)] + 16p_2^2M_2(G_2)[p_1^2q_1^2 + M_1^2(G_1) - 2p_1q_1M_1(G_1)] \\
+ \; & 4p_1ReZG_3(G_1)[p_2^2(p_2^2q_2 + 2F(G_2) - 8q_2^2 + 8M_2(G_2)) - M_1(G_2)(4M_1(G_2) \\
+ \; & 2p_2^3 - 12p_2q_2)] + 16p_1^2M_2(G_1)[p_2^2q_2^2 + M_1^2(G_2) - 2p_2q_2M_1(G_2)] \\
+ \; & 4M_1(G_1)M_1(G_2)[p_1^3(p_2q_2 - M_1(G_2)) + p_2^3(p_1q_1 - M_1(G_1))] \\
+ \; & p_1^4q_1M_1^2(G_2) + p_2^4q_2M_1^2(G_1) - 48p_1p_2ReZG_3(G_1)ReZG_3(G_2) \\
- \; & 2p_1^2p_2^2[F(G_1)F(G_2) + 8M_2(G_1)M_2(G_2)] - 64HM_2(G_1)HM_2(G_2)
\end{aligned}$$

Example 3.8: Let P_2, P_3 denote paths with 2 and 3 vertices, respectively. By theorem 4.1-7. we have The second Hyper-Zagreb index of some graph operations of P_2, P_3 in Table 4.

TABLE 1. The second Hyper-Zagreb index of some graph operations

G	$P_2 \otimes P_3$	$P_2 \times P_3$	$P_2 \circ P_3$	$P_2 * P_3$	$P_2 \oplus P_3$	$P_2 \vee P_3$
$HM_2(G)$	16	257	4849	2587	729	4849

4. CONCLUDING REMARKS

Hyper-Zagreb indices are a pair of recently introduced graph invariants that generalize much used Zagreb indices. In this paper, it has investigated the basic mathematical properties of the second Hyper-Zagreb index and obtained explicit formula for their values under several graph operations, where some of the above results have been applied to compute the second Hyper-Zagreb index for nano-tube and nano-torus. Much work still needs to be done, and here we mention some directions for future research as multiplicative second Hyper-Zagreb indices.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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