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ON GENERALIZED SINGULAR FRACTIONAL DIFFERENTIAL INITIAL VALUE PROBLEMS

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Abstract. The main object of the present article is the study of an abstract singular fractional differential equation subject to some initial conditions. Under appropriate assumptions on the initial conditions we obtained the explicit solution of the given singular fractional differential equation after decoupling it into two separate equations of different nature. Finally, two concrete examples of singular fractional differential initial value problems are given at the end of this work.

Keywords: singular; Drazin inverse; Caputo's fractional derivative; existence and uniqueness.

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1. INTRODUCTION

Systems of differential algebraic equations (DAEs) have been used to model a large variety of areas in science and technology such as in multi-body mechanics, chemical engineering, control theory as well as incompressible fluids, see [3, 4, 10, 13]. Thus, the theory of differential algebraic equations has known a remarkable development in the last four decades, and despite the fact that the DAEs are considered somehow as a generalization of Ordinary Differential

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Equations (ODEs), the study of this kind of systems is quite different from the standard analysis of the classical ODEs. Another field that has enriched nowadays the classical concepts of ODEs is Fractional Calculus whose genesis began by an innocent question asked by Marquise De l'Hopital to the mathematician Gottfried W. Leibniz in 1695 regarding the new notation of the n^{th} derivative introduced by the latter, namely $\frac{d^n}{dx^n}$. He asked him whether it is propitious to use the noninteger value $n = \frac{1}{2}$. As it is too hard to give a rigorous mathematical answer, he just answered him in a prescient manner that "*This is an apparent paradox from which, one day, useful consequences will be drawn...*" Effectively, that naive question is now an unveiled reality that a great deal of investigators have deeply explored and used in the framework of Fractional Calculus [1, 7]. So, due to the nonlocal character of fractional derivative several real world processes and natural phenomena are mathematically modeled by different types of fractional derivatives among them Caputo's and Riemann-Liouville's, for more details see [5, 9, 14]. In regard to singular fractional differential initial value problems, we address the reader to the recent investigations by the authors E. Shishkina & S. Sitnik [15], Y. Zhao [17], S. Bu & G. Cai [2], M. Plekhanova [12], and the references therein.

Let $\mathbb{R}^+ := [0,\infty)$ be the set of nonnegative real numbers, α a positive non integer number and let $N = [\alpha]+1$, where $[\alpha]$ is the integral part of α. We are interested in solving explicitly the following singular fractional differential initial value problem with respect to Caputo's fractional derivative in the unknown vector function $x(t) : \mathbb{R}^+ \to X$, namely

$$E\mathscr{D}^{\alpha}_{0_+}x(t) = Ax(t) + f(t), t > 0, \tag{1}$$

subject to the initial conditions

$$x^{(k)}(0) = v_k,\ k = 0,1,\dots,N-1, \tag{2}$$

where $E,A \in \mathscr{B}(X)$, so that $\ker E \neq \{0\}$ (and possibly $\ker A \neq \{0\}$), and $\mathscr{D}^{\alpha}_{0_+}$ denotes the (left sided) Caputo's fractional derivative of order $\alpha > 0$ initiated at 0, $v_0,\ v_1,...,v_N$ are known vectors in X and f is a given absolutely continuous function defined on $\mathbb{R}^+$. Unlike the projection operators approach used in the works [6, 12], we shall express the solution to problem (1)-(2) in terms of Mittag-Leffler functions and Drazin inverses [8, 16] of the operators A and E, when

$AE = EA$. In particular, if these operators are non singular, we obtain the explicit solution of a regular fractional initial value problem. The technique used in our investigation consists in decoupling the operator E into the sum of two operators, one of them is nilpotent, so that the given problem (1)-(2) is equivalent to a certain couple of manageable subproblems. In order to investigate general singular fractional initial value problems when the operators A and E don't necessarily commute we introduce a new notion of regularity that allows solving this type of problems.

The present article is organized as follows: we first start by stating some basic definitions and properties from Fractional Calculus as well as the notion of Drazin inverse of non invertible linear operators in a Banach space. Next, we investigate a certain singular fractional differential initial value problem as we establish the existence and uniqueness of the solution; moreover, we derive the explicit representation of the solution besides some illustrating examples. Finally, we finish our investigation with some concluding remarks.

2. PRELIMINARIES

Let $(X, \|.\|)$ be a complex Banach space. We denote by $\mathscr{B}(X)$ the Banach space of linear bounded operators from X into itself endowed with the norm $\|A\|_{op} = \sup\{\|Ax\| : \|x\| = 1\}$, for every $A \in \mathscr{B}(X)$. First of all we give some background regarding Fractional Calculus chiefly the notion of (left-sided) fractional Riemann-Liouville integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to X$ as well as its (left-sided) fractional derivative of order α in the sense of Caputo [5, 9, 11].

First, we recall the definition of vector-valued absolutely continuous functions over the non-negative real line $\mathbb{R}^+$ and taking their values in X; we have

Definition 1. *A function $f : \mathbb{R}^+ \to X$ is said to be absolutely continuous, if for any compact interval $J \subset \mathbb{R}^+$ and, for any $\varepsilon > 0$, there exists a positive real number $\delta > 0$, such that*

$$\sum_{k=1}^{n} \|f(b_k) - f(a_k)\| < \varepsilon,$$

for any finite set of mutually disjoint intervals $[a_k, b_k] \subset J$, $k = 1, 2, \ldots, n$, such that

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

Remark 1. *We notice that a function $f:\mathbb{R}^+\to X$ is absolutely continuous if and only if there are $\varphi\in L^1(\mathbb{R}^+;X)$ and a constant vector $c\in X$ such that $f(x)=c+\int_0^x\varphi(t)dt,\ x\geq 0$, and from which we get $f'(x)=\varphi(x)$, a.e. $x\geq 0$.*

Notation 1. *The vector space of all absolutely continuous functions on $\mathbb{R}^+$ taking their values in X is denoted by $AC(\mathbb{R}^+;X)$. Moreover, we shall use the following generalization: If $n\in\mathbb{N}^*:=\{1,2,3,...\}$, then*

$$AC^n\left(\mathbb{R}^+;X\right)=\left\{f:\mathbb{R}^+\to X: f\in C^{n-1}\left(\mathbb{R}^+;X\right)\text{ and } f^{(n-1)}\in AC\left(\mathbb{R}^+;X\right)\right\}.$$

In particular, we have $AC^1(\mathbb{R}^+;X):=AC(\mathbb{R}^+;X)$.

Let α be a positive real constant, we define the left-sided Riemann-Liouville fractional integral of order α of an integrable function $f:\mathbb{R}^+\to X$ as follows

$$J^{\alpha}_{0^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad t>0,$$

where $\Gamma(\alpha)$ is the Gamma function, given by $\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt$, which is a generalization of the factorial of an integer number. Let $N=\alpha$, if α is an integer number, and $N=[\alpha]+1$, if α is not. Next, we define the left-sided Caputo's fractional derivative of order α of $f:\mathbb{R}^+\to X$ by

$$\begin{aligned}
{}^C\mathscr{D}^{\alpha}_{0^+}f(t) &= f^{(N)}(t),\ t>0,\text{ if } \alpha=N \text{ is an integer,}\\
{}^C\mathscr{D}^{\alpha}_{0^+}f(t) &= J^{N-\alpha}_{0^+}f^{(N)}(t)=\frac{1}{\Gamma(N-\alpha)}\int_0^t(t-s)^{N-\alpha-1}f^{(N)}(s)ds,\quad t>0,\\
&\quad\text{if } \alpha \text{ is not an integer.}
\end{aligned}$$

We point out that the Caputo's fractional derivative of order α of $f:\mathbb{R}^+\to X$ is well defined whenever $f\in AC^N(\mathbb{R}^+;X)$. If we define the convolution product $\varphi*\psi$ of two functions φ and ψ by

$$\varphi*\psi(t)=\int_0^t\varphi(s)\,\psi(t-s)\,ds,\ t>0,$$

and we set $h_\gamma(s)=\begin{cases}\frac{1}{\Gamma(\gamma)}s^{\gamma-1},\ s>0\\ 0,\ s\leq 0\end{cases},\ (\gamma\geq 0),$

then

$$J_{0^+}^{\alpha} f(t) = (h_\alpha * f)(t), \quad t > 0,$$

and

$${}^{C}\mathscr{D}_{0^+}^{\alpha} f(t) = \left(h_{N-\alpha} * f^{(N)}\right)(t), \quad t > 0, \text{ if } \alpha \text{ is not an integer.}$$

We have these two useful relations:

i) ${}^{C}\mathscr{D}_{0^+}^{\alpha} J_{0^+}^{\alpha} f = f$, whenever $J_{0^+}^{\alpha} f \in AC^N(\mathbb{R}^+; X)$,

ii) $J_{0^+}^{\alpha C}\mathscr{D}_{0^+}^{\alpha} f(t) = f(t) - \sum_{k=0}^{N-1} \frac{f^{(k)}(0^+)}{k!} t^k$, for $f \in AC^N(\mathbb{R}^+; X)$.

We shall denote in the remaining of the present article the left-sided Caputo's fractional derivative of order α of $f : \mathbb{R}^+ \to X$, initiated at 0, by $\mathscr{D}_{0^+}^{\alpha} f(t)$ instead of ${}^{C}\mathscr{D}_{0^+}^{\alpha} f(t)$.

We recall the following definition,

Definition 2. *The index of an operator $E \in \mathscr{B}(X)$, denoted indE, is the least nonnegative integer m such that* $\ker E^m = \ker E^{m+1}$ *and* $\mathscr{R}(E^m) = \mathscr{R}\left(E^{m+1}\right)$. *In particular, if E is invertible, then* $indE = 0$*; we set* $indE = 1$*, if* $E = 0$.

Moreover, if $indE = m < \infty$*, and* $\mathscr{R}(E^m)$ *is closed, then the unique operator* $E^D \in \mathscr{B}(X)$ *satisfying*

$$E^D E E^D = E^D, E^D E = E E^D, E^D E^{m+1} = E^m,$$

is called the Drazin inverse of E.

We have the following Proposition,

Proposition 1. *Let* $A, L \in \mathscr{B}(X)$ *such that L is bijective and* $LA = AL$*. Then*

$$\ker(LA) = \ker A \text{ and } \mathscr{R}(LA) = \mathscr{R}(A).$$

Moreover, if $indA = m < \infty$*, then*

$$ind(LA) = indA.$$

Proof. Let $x \in \ker(LA)$, then $L(Ax) = 0$; hence $Ax = 0$, that is $x \in \ker A$. Conversely, if $x \in \ker A$, then $Ax = 0$ implying that $LAx = 0$; hence $x \in \ker(LA)$. Therefore, we obtain $\ker(LA) = \ker A$.

To prove the other relation, let $z \in \mathscr{R}(LA)$, there is $x \in X : z = LAx = A(Lx)$. It follows that $z \in \mathscr{R}(A)$. Conversely, if $z \in \mathscr{R}(A)$, then there exists $x \in X$: $z = Ax = LA\left(L^{-1}x\right)$, and so, $z \in \mathscr{R}(LA)$. Therefore, $\mathscr{R}(LA) = \mathscr{R}(A)$.

Suppose that $indA = m$, then m is the least integer number for which we have $\ker A^m = \ker A^{m+1}$ and $\mathscr{R}(A^m) = \mathscr{R}\left(A^{m+1}\right)$. Since L^m and L^{m+1} are bijective we can apply the previous assertion of this Proposition to $\{L^m, A^m\}$ and $\left\{L^{m+1}, A^{m+1}\right\}$ to get

$$\begin{aligned} \ker(LA)^m &= \ker A^m = \ker A^{m+1} = \ker(LA)^{m+1}, \\ \mathscr{R}(LA)^m &= \mathscr{R}(A^m) = \mathscr{R}\left(A^{m+1}\right) = \mathscr{R}(LA)^{m+1}. \end{aligned}$$

We conclude that

$$ind(LA) = indA = m.$$

□

We need the following decomposition's Theorem of a bounded linear operator, see for instance [16]:

Theorem 1. *Let $E \in \mathscr{B}(X)$ with $indE = m < \infty$ and $\mathscr{R}(E^m)$ is closed. Then*

$$E = \mathbf{C} + \mathbf{N}, \tag{3}$$

where $\mathbf{C} = EE^DE$ and $\mathbf{N} = E - \mathbf{C} = E\left(I - E^DE\right)$.

Moreover, we have

$$\mathbf{CN} = \mathbf{NC} = 0, \mathbf{N}^m = 0, \mathbf{N}^k = E^k\left(I - E^DE\right) \neq 0, \textit{ for } k < m; \tag{4}$$

$$\mathbf{N}E^D = E^D\mathbf{N} = 0; \tag{5}$$

$$\mathbf{N}^D\mathbf{C} = 0,\ \mathbf{C}^D\mathbf{N} = 0; \tag{6}$$

$$E^D = \mathbf{C}^D; \tag{7}$$

$$\mathbf{C}^D\mathbf{C} = E^DE; \tag{8}$$

$$\left(E^D\right)^D = \mathbf{C}; \tag{9}$$

$$\mathbf{C}EE^D = EE^D\mathbf{C} = \mathbf{C}; \tag{10}$$

$$\left(E^D\right)^* = (E^*)^D; \tag{11}$$

$$\left(E^D\right)^p = (E^p)^D,\ p \in \mathbb{N}. \tag{12}$$

Here are some other interesting properties regarding the Drazin inverse of linear bounded operators,

Lemma 1. *Let $E,\ A \in \mathscr{B}(X)$ such that E^D and A^D exist. If $EA = AE$, then, we have*

$$\begin{aligned} EA^D &= A^D E, \\ E^D A &= AE^D, \\ (EA)^D &= A^D E^D = E^D A^D. \end{aligned}$$

Proof. It suffices to follow the steps of the Proof of Lemma 2.21 [10] which is still valid for bounded linear operators. □

Remark 2. *We note that if $A,\ B \in \mathscr{B}(X)$ (not necessarily commuting), then AB is Drazin invertible if and only if BA is Drazin invertible, see* [8]. *If it is the case, then*

$$(AB)^D = A\left[(BA)^D\right]^2 B.$$

Furthermore, if $AB = BA$, then

$$(AB)^D = B^D A^D = (BA)^D = A^D B^D.$$

Let us state without proof some facts about the application of Laplace transform to Caputo's fractional derivative initiated at the origin. We have

Definition 3. *Let $f : \mathbb{R}^+ \to X$ be a piecewise continuous on every finite interval $[0,T]$, $T > 0$, and there exist positive constants M and a such that $\|f(t)\| \leq Me^{at}$, $t \geq 0$. Then the Laplace transform of $f(t)$ is defined by*

$$F(p) = \mathscr{L}(f)(p) = \int_0^{+\infty} e^{-ps} f(s)ds, Re(p) > a.$$

The inverse Laplace transform is formally given by

$$\frac{f(t^+)+f(t^-)}{2} = \mathscr{L}^{-1}(F)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tp}F(p)dp,\ t>0,$$

where the integral is carried out along the line $c+iy, -\infty < y < +\infty$*, with* $c > a$*.*

Let $\alpha > 0$ and $N = [\alpha]+1$, if α is non integer and $N = \alpha$, if α is integer, then the Laplace transform of the Caputo's fractional derivative $\mathscr{D}^{\alpha}_{0^+}g$ of a vector-valued function $g \in C^N(\mathbb{R}^+;X)$ such that $g^{(N)} \in L^1(0,T;X)$, for every $T > 0$ and

$$\left|g^{(N)}(x)\right| \leq Me^{ax}, \text{ for every } x > T > 0,$$

for some constants $M > 0$ and $a > 0$, is given by

$$(\mathscr{L}\mathscr{D}^{\alpha}_{0^+}g)(p) = p^{\alpha}(\mathscr{L}g)(p) - \sum_{k=0}^{N-1} g^{(k)}(0)\,p^{\alpha-k-1}, Re(p) > a). \tag{13}$$

3. Main Results

Let us first state and solve explicitly some fractional differential equation with a nilpotent operator coefficient. The obtained solution is unique and there is no initial value imposed. We have

Lemma 2. *Let* B*,* $\mathbf{N}$*,* $L \in \mathscr{B}(X)$ *such that* B *is invertible,* $\mathbf{N}$ *a nilpotent operator of index (of nilpotency)* $m \in \mathbb{N}^*$ *so that* $BL\mathbf{N} = L\mathbf{N}B$*. Then, for any function* $f : \mathbb{R}^+ \to X$ *such that*

$$\left(B^{-1}\right)^{k+1}(L\mathbf{N})^k\left(\mathscr{D}^{\alpha}_{0^+}\right)^k f \in AC\left(\mathbb{R}^+;X\right),\ \textit{for}\ k = 0,1,\ldots,m-1,$$

the fractional differential equation

$$L\mathbf{N}\mathscr{D}^{\alpha}_{0^+}\xi(t) = B\xi(t) + f(t),\ t>0, \tag{14}$$

has a unique solution given by

$$\xi(t) = -\sum_{k=0}^{m-1}\left(B^{-1}\right)^{k+1}(L\mathbf{N})^k\left(\mathscr{D}^{\alpha}_{0^+}\right)^k f(t),\ t>0. \tag{15}$$

Proof. Applying B^{-1} to the both sides of the first equation of (14) we find

$$B^{-1}L\mathbf{N}\mathscr{D}^{\alpha}_{0^+}\xi(t) = \xi(t) + B^{-1}f(t). \tag{16}$$

It is worth to notice that the assumption $BL\mathbf{N} = L\mathbf{N}B$ implies that $B(L\mathbf{N})^k = (L\mathbf{N})^k B$, for $k = 1,2,\ldots,m-1$, and so $(L\mathbf{N})^k B^{-1} = B^{-1}(L\mathbf{N})^k$, for $k = 1,2,\ldots,m-1$. Setting $Q = B^{-1}L\mathbf{N}\mathscr{D}^{\alpha}_{0^+}$, we get for every $k = 1,2,\ldots,m-1$,

$$\begin{aligned} Q^k B^{-1} &= \left(B^{-1}L\mathbf{N}\mathscr{D}^{\alpha}_{0^+}\right)^k B^{-1} = \left(B^{-1}\right)^k (L\mathbf{N})^k B^{-1}\left(\mathscr{D}^{\alpha}_{0^+}\right)^k \\ &= \left(B^{-1}\right)^{k+1}(L\mathbf{N})^k\left(\mathscr{D}^{\alpha}_{0^+}\right)^k. \end{aligned}$$

Expressing equation (16) in term of Q we obtain

$$Q\xi(t) = \xi(t) + B^{-1}f(t). \tag{17}$$

Next, applying the operators Q^k, $k = 1,2,\ldots,m-1$, to equation (17) we get respectively

$$\begin{aligned} Q^2\xi(t) &= Q\xi(t) + QB^{-1}f(t) \\ &= \xi(t) + B^{-1}f(t) + QB^{-1}f(t), \end{aligned}$$

$$\begin{aligned} Q^3\xi(t) &= Q\xi(t) + QB^{-1}f(t) + Q^2B^{-1}f(t) \\ &= \xi(t) + B^{-1}f(t) + QB^{-1}f(t) + Q^2B^{-1}f(t), \end{aligned}$$

$$\vdots$$

$$Q^m\xi(t) = 0 = \xi(t) + \sum_{k=0}^{m-1} Q^k B^{-1} f(t).$$

So that, the unique solution to the fractional differential equation (14) is given by

$$\begin{aligned} \xi(t) &= -\sum_{k=0}^{m-1} Q^k B^{-1} f(t) \\ &= -\sum_{k=0}^{m-1}\left(B^{-1}\right)^{k+1}(L\mathbf{N})^k\left(\mathscr{D}^{\alpha}_{0^+}\right)^k f(t),\ t \geq 0, \end{aligned}$$

which completes the Lemma's proof. □

Our next step is to establish an equivalence between the fractional differential equation (1) and a couple of appropriate fractional differential equations. We have

Proposition 2. *Let* $E,\ A \in \mathscr{B}(X)$ *with* $\ker E \neq \{0\}$. *We assume the Drazin inverse* E^D *exists and* $EA = AE$. *Then, equation (1) is equivalent to the fractional differential system*

$$\begin{cases} \mathbf{C}\mathscr{D}^{\alpha}_{0_+} y(t) = Ay(t) + f_1(t), \\ \mathbf{N}\mathscr{D}^{\alpha}_{0_+} z(t) = Az(t) + f_2(t),\ t \geq 0, \end{cases} \tag{18}$$

where $\mathbf{C} = EE^DE$, $\mathbf{N} = E - \mathbf{C}$, *and*

$$\begin{aligned} y(t) &= E^DEx(t), \qquad z(t) = \left(I - E^DE\right)x(t), \\ f_1(t) &= E^DEf(t), \qquad f_2(t) = \left(I - E^DE\right)f(t). \end{aligned}$$

Moreover, the function $y(t) = E^DEx(t)$ *is a solution to the first equation of (18), if and only if, it satisfies the regular fractional differential equation*

$$\mathscr{D}^{\alpha}_{0_+} y(t) = E^DAy(t) + E^Df_1(t),\ t \geq 0. \tag{19}$$

Proof. It is worth to notice that we have $\left(E^DE\right)^2 = E^DE$, and so, applying the operator E^DE to both sides of equation (1) we obtain at once the following fractional differential equation

$$\begin{aligned} \left(E^DE\right)^2 E\mathscr{D}^{\alpha}_{0_+} x(t) &= EE^DE\mathscr{D}^{\alpha}_{0_+} \left(E^DE\right)x(t) \\ &= \mathbf{C}\mathscr{D}^{\alpha}_{0_+} y(t) \\ &= E^DEAx(t) + E^DEf(t) \\ &= Ay(t) + f_1(t), \end{aligned}$$

which is a solution to the equation $(18)_1$.

Likewise, noticing that $\left(I - E^DE\right)^2 = \left(I - E^DE\right)$, and applying the operator $\left(I - E^DE\right)$ to both sides of equation (1) we get

$$\begin{aligned} \left(I - E^DE\right)^2 E\mathscr{D}^{\alpha}_{0_+} x(t) &= E\left(I - E^DE\right)\mathscr{D}^{\alpha}_{0_+} \left(I - E^DE\right)x(t) \\ &= \mathbf{N}\mathscr{D}^{\alpha}_{0_+} z(t) \\ &= \left(I - E^DE\right)Ax(t) + \left(I - E^DE\right)f(t) \\ &= Az(t) + f_2(t). \end{aligned}$$

Therefore, $z(t)$ satisfies equation $(18)_2$.

Conversely, if $(y(t), z(t))$ satisfies the system (18), then, thanks to the linearity of the fractional derivative, the function $x(t) = y(t) + z(t)$ satisfies

$$\begin{aligned} E\mathscr{D}_{0_+}^{\alpha} x(t) &= E\mathscr{D}_{0_+}^{\alpha}(y(t) + z(t)) = E\mathscr{D}_{0_+}^{\alpha} y(t) + E\mathscr{D}_{0_+}^{\alpha} z(t) \\ &= A(y(t) + z(t)) + f_1(t) + f_2(t) \\ &= Ax(t) + f(t). \end{aligned}$$

To establish the last assertion we notice that $y(t) = E^D E x(t)$ is already a solution to the first equation of (18), and we have

$$\begin{aligned} \mathscr{D}_{0_+}^{\alpha} y(t) &= \mathscr{D}_{0_+}^{\alpha}\left(E^D E\right) x(t) = E^D E \mathscr{D}_{0_+}^{\alpha} x(t) \\ &= E^D (Ax(t) + f(t)) \\ &= E^D \left[A\left(E^D E\right) x(t) + E^D E f(t)\right] \\ &= E^D A y(t) + E^D f_1(t),\ t \geq 0. \end{aligned}$$

Conversely, multiplying (19) by $\mathbf{C}$ we obtain

$$\begin{aligned} \mathbf{C}\mathscr{D}_{0_+}^{\alpha} y(t) &= E^D E A y(t) + \mathbf{C} E^D f(t) \\ &= E^D E A\left[E^D E x(t)\right] + f_1(t) \\ &= A y(t) + f_1(t). \end{aligned}$$

□

Let us now state and prove another important result which is

Proposition 3. *Let $E,\ A \in \mathscr{B}(X)$ such that $EA = AE$ and E^D, A^D exist. Then the following assertions are equivalent*

a)

$$\ker E^D \cap \ker A^D = \{0\}. \tag{20}$$

b)

$$A^D A\left(I - E^D E\right) = I - E^D E. \tag{21}$$

Proof. **a)**$\Rightarrow$**b)**: Suppose that $\ker E^D \cap \ker A^D = \{0\}$ and set

$$B = A^D A\left(I - E^D E\right) - \left(I - E^D E\right).$$

Applying the operators A^D and E^D to the latter equation we get respectively

$$\begin{aligned} A^D B &= A^D A^D A\left(I - E^D E\right) - A^D\left(I - E^D E\right) \\ &= A^D\left(I - E^D E\right) - A^D\left(I - E^D E\right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} E^D B &= E^D A^D A\left(I - E^D E\right) - E^D\left(I - E^D E\right) \\ &= A^D A\left(E^D - E^D E^D E\right) - \left(E^D - E^D E^D E\right) \\ &= 0. \end{aligned}$$

Hence, for any $x \in X$, we have

$$A^D(Bx) = E^D(Bx) = 0,$$

that is

$$Bx \in \ker E^D \cap \ker A^D = \{0\}.$$

It follows that $Bx = 0$, for every $x \in X$, and accordingly (21) holds.

b)$\Rightarrow$**a)**: Suppose that (21) holds. Let $x \in \ker E^D \cap \ker A^D$, then

$$A^D x = E^D x = 0.$$

It follows that

$$\begin{aligned} \left(I - E^D E\right)x &= x - EE^D x = x \\ &= A\left(I - E^D E\right)A^D x = 0. \end{aligned}$$

Hence $x = 0$; therefore $\ker E^D \cap \ker A^D = \{0\}$. □

Remark 3. *i) It is not hard to check the following inclusion by using the property* $E^D = E^D E^D E$ *and* $A^D = A^D A^D A$,

$$\ker E \cap \ker A \subset \ker E^D \cap \ker A^D.$$

ii) If $E,\ A \in \mathscr{B}(X)$ *commute,* E^D, A^D *exist, with* $indE = m < \infty$ *and* $indA = k < \infty$, *and*

$$\ker E^m \cap \ker A^k = \{0\},$$

then the relation (21) holds. Indeed, applying respectively E^m *and* A^k *to the operator* B *we obtain*

$$\begin{aligned} E^m B &= E^m A^D A\left(I - E^D E\right) - E^m\left(I - E^D E\right) \\ &= A^D\left(E^m - E^D E^{m+1}\right) - A^D\left(E^m - E^D E^{m+1}\right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} A^k B &= A^k A^D A\left(I - E^D E\right) - A^k\left(I - E^D E\right) \\ &= A^k\left(I - E^D E\right) - A^k\left(I - E^D E\right) \\ &= 0. \end{aligned}$$

Reasoning as above we conclude that (21) holds.

Before tackling the general singular fractional differential equation we would like to investigate the homogeneous one, we have

Theorem 2. *Let* $E,\ A \in \mathscr{B}(X)$ *with* $\ker\, E \neq \{0\}$ *so that* $EA = AE$ *and* $indE = m$. *We assume that* E *and* A *have bounded Drazin inverses* E^D *and* A^D *obeying condition (20). Then, the general solution of*

$$E\mathscr{D}_{0_+}^{\alpha} x(t) = Ax(t), t > 0, \tag{22}$$

is given by

$$x(t) = \sum_{k=0}^{N-1} t^k \mathscr{E}_{\alpha,k+1}\left(t^{\alpha} E^D A\right) E^D E b_k, t \geq 0,$$

for some constant vectors $b_0,\ b_1,\dots,b_{N-1}\in X$, where

$$\mathscr{E}_{\alpha,\beta}\left(t\right)=\sum_{k\geq 0}\frac{1}{\Gamma\left(\alpha k+\beta\right)}t^{k}$$

is the Mittag-Leffler function of two parameters $\alpha,\beta>0$.

Proof. Define $y\left(t\right)=E^{D}Ex\left(t\right)$, then

$$\mathscr{D}_{0_{+}}^{\alpha}y\left(t\right)=E^{D}E\mathscr{D}_{0_{+}}^{\alpha}x\left(t\right)=E^{D}Ay\left(t\right),\ t\geq 0.$$

Next, applying Laplace transform to the latter equation, we obtain by virtue of the linearity of $\mathscr{L}$,

$$\begin{aligned}\mathscr{L}(D_{0+}^{\alpha}y)(p) &= p^{\alpha}(\mathscr{L}y)(p)-\sum_{k=0}^{N-1}y^{(k)}\left(0\right)p^{\alpha-k-1}\\ &= \mathscr{L}(E^{D}Ay)(p)=E^{D}A\ \mathscr{L}(y)\left(p\right).\end{aligned}$$

Setting $Y\left(p\right)=(\mathscr{L}y)(p)$, we infer

$$\left(p^{\alpha}I-E^{D}A\right)Y\left(p\right)=\sum_{k=0}^{N-1}p^{\alpha-k-1}y^{(k)}\left(0\right).$$

If $|p|>\left\Vert E^{D}A\right\Vert_{op}^{1/\alpha}$, then we get

$$\left(p^{\alpha}I-E^{D}A\right)^{-1}=\sum_{j\geq 0}p^{-\alpha(j+1)}\left(E^{D}A\right)^{j}.$$

It follows that

$$\begin{aligned}Y\left(p\right) &= \left(p^{\alpha}I-E^{D}A\right)^{-1}\sum_{k=0}^{N-1}p^{\alpha-k-1}y^{(k)}\left(0\right)\\ &= \mathscr{L}\left(\sum_{k=0}^{N-1}t^{k}\mathscr{E}_{\alpha,k+1}\left(t^{\alpha}E^{D}A\right)y^{(k)}\left(0\right)\right).\end{aligned}$$

To simplify the solution's expression we shall put throughout

$$T_{\alpha,\beta}\left(t\right):=t^{\beta-1}\mathscr{E}_{\alpha,\beta}\left(t^{\alpha}E^{D}A\right).$$

Therefore,

$$y(t)=\sum_{k=0}^{N-1}T_{\alpha,k+1}\left(t\right)y^{(k)}\left(0\right).$$

We notice that for any constant vectors $b_0, b_1, \ldots, b_{N-1} \in X$ the function

$$y(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t)\, b_k,$$

satisfies the following

$$\begin{aligned}\mathbf{C}\mathscr{D}_{0_+}^{\alpha} y(t) &= \mathbf{C}\sum_{k=0}^{N-1} \mathscr{D}_{0_+}^{\alpha} T_{\alpha,k+1}(t)\, b_k \\ &= EE^D A \sum_{k=0}^{N-1} T_{\alpha,k+1}(t)\, b_k = Ay(t).\end{aligned}$$

Hence $y(t)$ is a solution to the homogeneous equation associated with $(18)_1$.

Let us now obtain the closed form of the general solution to the equation (22).

Consider the following homogeneous equation associated with $(18)_2$

$$\mathbf{N}\mathscr{D}_{0_+}^{\alpha} z(t) = Az(t),\ t > 0 \tag{23}$$

Applying $\mathbf{N}^{m-1}$ to both sides of (23) we infer

$$\mathbf{N}^m \mathscr{D}_{0_+}^{\alpha} z(t) = 0 = A\mathbf{N}^{m-1} z(t).$$

It follows that $A^D A \mathbf{N}^{m-1} z(t) = 0$, and thanks to the assumption (20) and Proposition 3 we get

$$\begin{aligned} 0 &= A^D A \left(I - E^D E\right) \mathbf{N}^{m-1} z(t) \\ &= \left(I - E^D E\right) \mathbf{N}^{m-1} z(t) = \mathbf{N}^{m-1} z(t),\end{aligned}$$

by assuming of course that $m-1 > 0$. Hence, $\mathbf{N}^{m-1} z(t) = 0$, and continuing in this manner we arrive at the final result $\mathbf{N}z(t) = 0$, which in turn implies that

$$\mathbf{N}\mathscr{D}_{0_+}^{\alpha} z(t) = \mathscr{D}_{0_+}^{\alpha} \mathbf{N} z(t) = 0 = Az(t).$$

Finally, since $\left(I - E^D E\right) y(t) = E^D E z(t) = 0$, then $\left(I - E^D E\right) z(t) = z(t)$. It follows by virtue of Proposition 3 that

$$\begin{aligned} A^D A \left(I - E^D E\right) z(t) &= A^D \left(I - E^D E\right) A z(t) = 0 \\ &= \left(I - E^D E\right) z(t) = z(t).\end{aligned}$$

Therefore, the unique solution to the differential equation (23) is the null one. Accordingly, the general solution to the singular fractional differential equation (23) is

$$x(t) = y(t) = E^D E y(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) E^D E b_k,\ t \geq 0,$$

for some constant vectors $b_0,\ b_1,\ldots,b_{N-1} \in X$. □

We are now in the position to establish the existence and uniqueness of the solution to the singular fractional differential initial value problem (1)-(2). We have

Theorem 3. *Let $E,\ A \in \mathscr{B}(X)$ with $\ker E \neq \{0\}$ so that $AE = EA$ and $\operatorname{ind}E = m$. We assume that E and A possess bounded Drazin inverses E^D and A^D and both satisfy condition (20). Let $f \in C^N(\mathbb{R}^+;X)$ so that $T_{\alpha,\alpha} * f$ is integrable, the composite Caputo's fractional derivative $(\mathscr{D}^{\alpha}_{0^+})^i f(t),\ t > 0$, exists for every $i = 1,\ldots,m-1$, $\lim_{t\to 0^+}(\mathscr{D}^{\alpha}_{0^+})^i f^{(j)}(t)$ exists for every $i = 1,\ldots,m-1$ and $j = 0,1,\ldots,N-1$. If the initial conditions satisfy*

$$\text{(24)} \qquad \begin{aligned} v_j &= E^D E b_j - (I - E^D E) \sum_{i=0}^{m-1} (A^D E)^i A^D (\mathscr{D}^{\alpha}_{0^+})^i f^{(j)}(0^+), \\ \text{for } j &= 0,1,\ldots,N-1, \end{aligned}$$

for some constant vectors $b_j,\ j = 0,1,\ldots,N-1$, then the unique solution $x(t)$ to problem (1)-(2) has the closed form

$$\begin{aligned} x(t) &= \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) E^D E b_k + \int_0^t T_{\alpha,\alpha}(s) E^D f(t-s)\, ds \\ &\quad -(I - E^D E) \sum_{i=0}^{m-1} (A^D E)^i A^D (\mathscr{D}^{\alpha}_{0^+})^i f(t),\ t \geq 0. \end{aligned}$$

Proof. Applying the Laplace transform $\mathscr{L}$ to the equation (19), we obtain by virtue of the linearity of $\mathscr{L}$,

$$\begin{aligned} \mathscr{L}(D^{\alpha}_{0^+} y)(p) &= p^{\alpha}(\mathscr{L}y)(p) - \sum_{k=0}^{n-1} y^{(k)}(0)\, p^{\alpha-k-1} \\ &= E^D A\, \mathscr{L}(y)(p) + E^D \mathscr{L}(f)(p). \end{aligned}$$

Setting $Y(p) = (\mathscr{L}y)(p)$ and $F(p) = \mathscr{L}(f)(p)$, the latter equation becomes

$$\left(p^{\alpha} I - E^D A\right) Y(p) = \sum_{k=0}^{n-1} p^{\alpha-k-1} y^{(k)}(0) + E^D F(p).$$

Assuming that $|p| > \left\|E^D A\right\|_{op}^{1/\alpha}$ we obtain

$$\left(p^{\alpha} I - E^D A\right)^{-1} = \sum_{j\geq 0} p^{-\alpha(j+1)} \left(E^D A\right)^j.$$

It follows that

$$\begin{aligned} Y(p) &= \left(p^{\alpha} I - E^D A\right)^{-1} \sum_{k=0}^{n-1} p^{\alpha-k-1} y^{(k)}(0) + \left(p^{\alpha} I - E^D A\right)^{-1} E^D F(p) \\ &= \mathscr{L}\left(\sum_{k=0}^{n-1} T_{\alpha,k+1}(t)\, y^{(k)}(0) + T_{\alpha,\alpha}(t) * E^D f(t)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= \sum_{k=0}^{n-1} T_{\alpha,k+1}(t)\, y^{(k)}(0) + T_{\alpha,\alpha}(t) * E^D f(t) \\ &= \sum_{k=0}^{n-1} T_{\alpha,k+1}(t)\, E^D E b_k + \int_0^t T_{\alpha,\alpha}(s)\, E^D f(t-s)\, ds, \end{aligned}$$

for some constant vectors $b_j,\ j = 0, 1, \ldots, N-1$,.

Next, to solve explicitly the fractional differential equation $(18)_2$ we apply the operator A^D to both sides of the equation to get by virtue of Proposition 3,

$$\begin{aligned} A^D \mathbf{N} \mathscr{D}_{0_+}^{\alpha} z(t) &= A^D A z(t) + A^D \left(I - E^D E\right) f(t) \\ &= \left(I - E^D E\right) x(t) + A^D \left(I - E^D E\right) f(t) \\ &= z(t) + A^D \left(I - E^D E\right) f(t), \end{aligned}$$

Next, applying Lemma 2, for $B = I$ and $L = A^D$, one gets the unique solution of the latter equation which is

$$z(t) = -\left(I - E^D E\right) \sum_{i=0}^{m-1} \left(A^D \mathbf{N}\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f(t),\ t \geq 0.$$

Since we have $\mathbf{N} = E\left(I - E^D E\right)$ and $\left(I - E^D E\right)^i = \left(I - E^D E\right)$, for $i = 1, 2, \ldots, m-1$, then

$$z(t) = -\left(I - E^D E\right) \sum_{i=0}^{m-1} \left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f(t).$$

Summing up the solutions of the above subproblems $y(t)$ and $z(t)$ we obtain the unique solution to the singular fractional differential initial value problem (1)-(2), that is

$$\begin{aligned} x(t) &= y(t)+z(t) \\ &= \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) E^D E b_k + \int_0^t T_{\alpha,\alpha}(s) E^D f(t-s)\, ds \\ &\quad - \left(I - E^D E\right) \sum_{i=0}^{m-1} \left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f(t),\ t \geq 0. \end{aligned}$$

Let us now check the given initial values. Using the derivation rule regarding integrals depending upon a certain real parameter, we get, for $j = 1, \ldots, N-1$, the following

$$\begin{aligned} x^{(j)}(t) &= \sum_{k=0}^{j-1} \sum_{m\geq 1} \frac{\alpha m}{\Gamma(\alpha m + k - j + 1)} t^{\alpha m + k - j} \left(E^D A\right)^m E^D E b_k \\ &\quad + \sum_{k=j}^{N-1} \sum_{m\geq 0} \frac{\alpha m}{\Gamma(\alpha m + k - j + 1)} t^{\alpha m + k - j} \left(E^D A\right)^m E^D E b_k \\ &\quad + \sum_{k=0}^{j-1} T_{\alpha,\alpha-k}(t) E^D f^{(j-k-1)}(0) + \int_0^t T_{\alpha,\alpha}(s) E^D f^{(j)}(t-s)\, ds \\ &\quad - \left(I - E^D E\right) \sum_{i=0}^{m-1} \left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f^{(j)}(t) \end{aligned}$$

so that, letting $t \to 0^+$, we obtain

$$\begin{aligned} x^{(j)}(0) &= v_j = E^D E b_j - \left(I - E^D E\right) \sum_{i=0}^{m-1} \left(A^D E\right)^i \left(\mathscr{D}_{0^+}^{\alpha}\right)^i A^D f^{(j)}(0) \\ &\quad j = 0, 1, \ldots, N-1. \end{aligned}$$

Regarding the uniqueness of the solution (under assumption (24)), it suffices to cope with the homogeneous problem whose solution is identically zero, and accordingly the uniqueness follows. □

Remark 4. *We point out that if $f \equiv 0$, then the compatibility assumption (24) reduces merely to $v_j = E^D E v_j$, for $j = 0, 1, \ldots, N-1$. Moreover, if E is nonsingular, then $E^D E = I$, and once again, assumption (24) becomes $v_j = E^D E v_j$, for $j = 0, 1, \ldots, N-1$. Whence, we obtain as a unique solution in such a case as expected the function*

$$x(t) = \sum_{k=0}^{N-1} T_{\alpha,k+1}(t) v_k + \int_0^t T_{\alpha,\alpha}(s) E^{-1} f(t-s)\, ds, \ , \ t \geq 0.$$

4. Illustrating Examples

In order to illustrate the obtained results we consider the following examples:

Example 1. *Consider the following singular fractional differential initial value problem in* $\mathbb{R}^4$*:*

$$\begin{cases} E\mathscr{D}_{0^+}^{4/3}x(t) = Ax(t) + f(t), t > 0, \\ x_0 = \begin{pmatrix} -1, & 1, & 0, & 1 \end{pmatrix}^T, \\ x_1 = \begin{pmatrix} 1, & 1, & 1, & 1 \end{pmatrix}^T, \end{cases} \tag{25}$$

where $E,\ A \in \mathbb{R}^{4\times 4}$ *are as follows*

$$E = \frac{1}{12}\begin{pmatrix} 10 & -1 & 4 & 5 \\ 5 & -2 & -1 & 4 \\ 4 & 5 & 10 & -1 \\ -1 & 4 & 5 & -2 \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

and $f(t) = \begin{pmatrix} t^2, & t, & 0, & -t \end{pmatrix}^T$. *We notice that E and A are singular matrices whose Drazin inverses are*

$$E^D = \begin{pmatrix} 1 & -2 & 1 & -2 \\ -2 & 7 & -2 & 7 \\ 1 & -2 & 1 & -2 \\ -2 & 7 & -2 & 7 \end{pmatrix}, \qquad A^D = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Hence, the explicit representation of the solution is given by

$$\begin{aligned} x(t) &= T_{\frac{4}{3},1}(t)E^D Ex_0 + T_{\frac{4}{3},2}(t)E^D Ex_1 + \int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)E^D F(t-s)ds \\ &\quad -(I - EE^D)\sum_{i=0}^{1}(A^D E)^i A^D(\mathscr{D}_{0^+}^{4/3})^i f(t), \end{aligned}$$

with

$$-(I - EE^D)\sum_{i=0}^{1}(A^D E)^i A^D(\mathscr{D}_{0^+}^{4/3})^i f(t) = 0.$$

Therefore, the closed form of the solution to the given problem is

$$x(t)=\begin{pmatrix} T_{\frac{4}{3},1}(t)-\frac{1}{2}T_{\frac{4}{3},2}(t)+\int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2ds \\ T_{\frac{4}{3},1}(t)+T_{\frac{4}{3},2}(t)-2\int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2ds \\ T_{\frac{4}{3},1}(t)-\frac{1}{2}T_{\frac{4}{3},2}(t)+\int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2ds \\ T_{\frac{4}{3},1}(t)+T_{\frac{4}{3},2}(t)-2\int_0^t T_{\frac{4}{3},\frac{4}{3}}(s)(t-s)^2ds \end{pmatrix}, t\geq 0.$$

Our second example deals this time with a singular fractional differential initial value problem in an infinite dimensional space, namely the Banach space

$$l^2=\left\{x=(x_n)_{n\geq 1}\subset\mathbb{R}:\sum_{n\geq 1}|x_n|^2<\infty\right\},$$

endowed with the norm $\|x\|=\left(\sum_{n\geq 1}|x_n|^2\right)^{1/2}$. We have

Example 2. *Consider the following singular fractional differential initial value problem*

$$\begin{cases} E\mathscr{D}_{0_+}^{2/3}x(t)=Ax(t)+f(t),\ t>0, \\ x(0)=(1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n},\dots)\in l^2, \end{cases} \tag{26}$$

where, $f(t)=(\frac{1}{n}\sin nt)_{n\geq 1}$*,* $E,A\in\mathscr{L}(l^2)$ *are projection operators, defined respectively by*

$$Ex=(x_1,x_2,0,0,x_5,x_6,0,0,x_9,x_{10},0,0,x_{13},x_{14},0,0,\dots) \tag{27}$$

and

$$Ax=(x_1,0,x_3,x_4,x_5,0,x_7,x_8,x_9,0,x_{11},x_{12},x_{13},0,\dots). \tag{28}$$

Taking into account the projection properties, we get at once

$$E^D=E,\ A^D=A,$$

and

$$A^DA(I-E^DE)=I-E^DE.$$

Hence condition (21) in Proposition 3 is satisfied, and so, according to Theorem 3, we obtain

$$\begin{aligned} x(t) &= T_{\frac{2}{3},1}(t)Ex(0)+\int_0^t T_{\frac{2}{3},\frac{2}{3}}(s)Ef(t-s)ds \\ &\quad -(I-E)Af(t)-(I-E)EA\mathscr{D}_{0^+}^{2/3}f(t). \end{aligned}$$

We notice that

$$-(I-E)EA\mathscr{D}_{0^+}^{2/3}f(t)=0,\ t>0.$$

It follows that the closed form of the given singular fractional differential initial value problem is

$$x(t)=(x_n)_{n\geq 1},$$

where

$$x_n(t)=\begin{cases} -\frac{1}{n}\sin nt, \text{ if } n\in J=\{4k-1,4k, \text{ for } k=1,2,...\}, \\ \frac{1}{n}T_{\frac{2}{3},1}(t)+\frac{1}{n}\int_0^t T_{\frac{2}{3},\frac{2}{3}}(s)\sin n(t-s)ds, \text{ if } n\in\mathbb{N}^*\backslash J. \end{cases}$$

5. CONCLUSION

Combining the theory of Fractional Calculus and the theory of singular differential equations we have been able to establish the existence and uniqueness of the singular fractional differential initial value problem (1)-(2) by using the notion of Drazin inverses as well as the Laplace transform. The derived results are new and simpler than those published elsewhere. The novelty in our approach is the replacement of the empty intersection assumption of the kernels of the operators E and A by the empty intersection of the kernels of their Drazin inverses; two illustrative examples are presented at the end of the present article.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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