PERIODICITY OF $p$-ADIC EXPANSION OF RATIONAL NUMBER

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Abstract. In this paper we give an algorithm to calculate the coefficients of the $p$-adic expansion of a rational numbers, and we give a method to decide whether this expansion is periodic or ultimately periodic.

Keywords: $p$-adic expansion; $p$-adic number; rational number.

2010 AMS Subject Classification: 11A07, 11D88, 11Y99.

1. INTRODUCTION

It is known that in $\mathbb{R}$, an element is rational if and only if its decimal expansion is ultimately periodic. An important analogous theorem for the $p$-adic expansion of rational number, is given by the following statement (see [1]):

Theorem 1.1. The number $x \in \mathbb{Q}_p$ is rational if and only if the sequence of digits of its $p$-adic expansion is periodic or ultimately periodic.

For example, in $\mathbb{Q}_3$, the 3-adic expansion of $-\frac{1}{2}$ is $1 + 3 + 3^2 + 3^3 + ... = 111111111111$, it is clear that this expansion is purely periodic. In the second example in $\mathbb{Q}_3$, the 3-adic expansion of $\frac{11}{5}$ is given by $1 + 1.3 + 1.3^2 + 2.3^3 + 1.3^4 + 0.3^5 + ... = 111210121012101210...$. This

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expansion is ultimately periodic, with periodic block 1210. Another example in \( \mathbb{Q}_5 \), the 5-adic expansion of \( \frac{213}{7} \) is given by

\[
4 + 1.5 + 3.5^2 + 1.5^3 + 4.5^4 + 2.5^5 + 3.5^6 + 0.5^7 + 2.5^8 + \ldots = 413142302142302...
\]

This expansion is ultimately periodic, with periodic block 142302.

Evertse in [3], gave an algorithm to calculate the coefficients of \( p \)-adic expansion of an element in \( \mathbb{Z}_p \). We continue the study of the characterization of \( p \)-adic numbers (see [2]), we inspired by the works of Evertse, we propose the algorithm (1), to calculate the sequence of digits of a rational number \( \frac{c}{d} \), then we prove that this sequence defines the \( p \)-adic expansion of \( \frac{c}{d} \) (see lemma 2.2), and it satisfies the relationship (2) (see lemma 2.3). Finally, in the main theorem, we demonstrate the periodicity of the \( p \)-adic expansion of \( \frac{c}{d} \).

2. Definitions and Properties

We will recall some definitions and basic facts from \( p \)-adic numbers (see [4]). Throughout this paper \( p \) is a prime number, \( \mathbb{Q} \) is the field of rational numbers, \( \mathbb{Q}^+ \) is the field of nonnegative rational numbers and \( \mathbb{R} \) is the field of real numbers. We use \( |.\| \) to denote the ordinary absolute value, \( v_p \) the \( p \)-adic valuation and \( |.|_p \) the \( p \)-adic absolute value. The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is the completion of \( \mathbb{Q} \) with respect to the \( p \)-adic absolute value. We denote the ring of \( p \)-adic integers by \( \mathbb{Z}_p \). Every element of \( \mathbb{Q}_p \) can be expressed uniquely by the \( p \)-adic expansion

\[
\sum_{n=-j}^{+\infty} \alpha_n p^n \quad \text{with} \quad \alpha_i \in \{0, 1, \ldots, p-1\} \quad \forall i \geq -j.
\]

In \( \mathbb{Z}_p \) we have simply \( j=0 \).

Now, we give in the following definition the requested algorithm for a rational number

**Definition 2.1.** Let \( \frac{c}{d} \in \mathbb{Q}^+ \cap \mathbb{Z}_p \), with \( c \in \mathbb{N} \), \( d \in \mathbb{N}^* \), and \( (c, p) = 1 \), \( (d, p) = 1 \), \( (c, d) = 1 \).

We define the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) by

\[
\begin{cases}
\beta_0 = c \\
\alpha_i = \beta_i d^{-1} \mod p, \forall i \geq 0 \\
\beta_{i+1} = \frac{\beta_i - \alpha_i d}{p} \in \mathbb{Z}, \forall i \geq 0
\end{cases}
\]

**Lemma 2.2.** Under the hypothesis of the definition (2.1), the \( p \)-adic expansion of \( \frac{c}{d} \) is given by \( \sum_{i=0}^{+\infty} \alpha_i p^i \), with \( \alpha_i \in \{0, 1, \ldots, p-1\}, \forall i \geq 0 \). The opposite is true, i.e., if \( \frac{c}{d} = \sum_{i=0}^{+\infty} \alpha_i p^i \), then the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) satisfies the algorithm (1).
Proof. Let \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) as in the definition (2.1). We have
\[
\frac{c}{d} = \alpha_0 + \frac{\beta_1}{d}p = \alpha_0 + \alpha_1p + \frac{\beta_2}{d}p^2
\]
\[
\ldots
\]
\[
= \alpha_0 + \alpha_1p + \ldots + \alpha_np^n + \frac{\beta_{n+1}}{d}p^{n+1}
\]
So
\[
\left| \frac{c}{d} - \sum_{i=0}^{n} \alpha_i p^i \right| \leq \frac{1}{p^{n+1}}
\]
therefore
\[
\sum_{i=0}^{+\infty} \alpha_i p^i = \frac{c}{d}
\]
For the second part, we suppose \(\frac{c}{d} = \sum_{i=0}^{+\infty} \alpha_i p^i\), and we prove by recursion that the sequences \((\alpha_i)_{i \in \mathbb{N}}\) and \((\beta_i)_{i \in \mathbb{N}}\) satisfies the algorithm (1). For \(i = 0\), we have \(\frac{c}{d} = \alpha_0 \text{mod} p\), then \(\alpha_0 = cd^{-1} \text{mod} p\). Now, we suppose that \(\alpha_i = \beta_i d^{-1} \text{mod} p\) and \(\beta_{i+1} = \frac{\beta_i - \alpha_i d}{p}\), so we have
\[
\alpha_i = \beta_i d^{-1} \text{mod} p \implies \alpha_{i+1} p + \alpha_i = \beta_i d^{-1} \text{mod} p
\]
\[
\implies \alpha_{i+1} p = (\beta_i d^{-1} - \alpha_i) \text{mod} p
\]
\[
\implies \alpha_{i+1} = \left(\frac{\beta_i - \alpha_i}{p}\right) d^{-1} \text{mod} p = \beta_{i+1} d^{-1} \text{mod} p
\]
therefore \(\forall i \geq 0 : \alpha_i = \beta_i d^{-1} \text{mod} p\).

\[\square\]

Lemma 2.3. Under the hypothesis of the definition (2.1), we have
\[
c = d \left( \sum_{n=0}^{i-1} \alpha_n p^n \right) + \beta_i p^i , \quad \forall i \in \mathbb{N}^*
\]

Proof. We prove this lemma, also, by induction. For \(i = 1\), it’s obvious.
\[
d \left( \sum_{n=0}^{0} \alpha_n p^n \right) + \beta_1 p = d \alpha_0 + \left( \frac{c - \alpha_0 d}{p} \right) p = c
\]
Suppose that, the relationship is true for $i$. From (1), we have
\[ \beta_i = \alpha_i d + \beta_{i+1} p. \]
Then
\[ c = d \left( \sum_{n=0}^{i-1} \alpha_n p^n \right) + \beta_i p^i = d \left( \sum_{n=0}^{i-1} \alpha_n p^n \right) + (\beta_{i+1} p + \alpha_i d) p^i = d \left( \sum_{n=0}^{i} \alpha_n p^n \right) + \beta_{i+1} p^{i+1} \]
So, the relationship is true for all $i \in \mathbb{N}$. \qed

**Remark 2.4.** Let $r = \frac{c'}{d'} \in \mathbb{Q}^+$, but not in $\mathbb{Z}_p$, i.e. the $p$-adic expansion of $\frac{c'}{d'}$ is given by
\[ \sum_{n=-j}^{+\infty} \alpha_n p^n, \]
with $j \neq 0$ and $\alpha_i \in \{0, 1, \ldots, p - 1\}$, $\forall i \geq -j$. In this case, we can suppose
\[ c' = c \in \mathbb{N}, \ d' = p^i d \in \mathbb{N}^*, \]
with $(d, p) = 1$, and $(c, p) = 1$. So, we have $\frac{c}{d} = \sum_{n=0}^{+\infty} \alpha_n p^n$. We define a sequence $(\beta_i)_{i \in \mathbb{N}}$ by the same way

\[
\begin{cases}
\beta_0 = c = c' \\
\beta_{i+1} = \frac{\beta_i - \alpha_i d}{p} = \frac{\beta_i p^j - \alpha_i d'}{p^{i+1}} \in \mathbb{Z}
\end{cases}
\]

### 3. Main Results

To show that the algorithm (2.1) stops after a certain rank, it suffices to prove that the sequence $(|\beta_n|)_{n \in \mathbb{N}}$ is bounded or decreasing. This is the subject of the main theorem.

**Main Theorem 3.1.** The sequence $(\beta_i)_{i \in \mathbb{N}}$ given in (1) verified the following cases:

**Case 1.** If $c < d$, then
\[ 0 \leq |\beta_i| < d, \quad \forall i \in \mathbb{N} \]

**Case 2.** If $c > d$ and $p \geq 3$, we have, also, two cases:

**Case 2.1.** If $0 < \frac{c(p-1)}{2dp} < 1$, then for all $i \in \mathbb{N}^*$, we have $|\beta_i| < d$.

**Case 2.2.** If $1 < \frac{c(p-1)}{2dp}$, then for a fixed integer

\[
m = \left\lfloor \frac{\log \left( \frac{c(p-1)}{2dp} \right)}{\log p} \right\rfloor
\]
it comes that

\[
\begin{align*}
    d &< |\beta_i| < c \quad \text{for} \quad 0 \leq i < m+1 \\
    0 &\leq |\beta_i| < d \quad \text{for} \quad m+1 < i \\
    0 &\leq |\beta_i| < c \quad \text{for} \quad m+1 = i
\end{align*}
\]

Proof. We treat all cases:

Case 1. Let \( c < d \), we use the proof by induction. For \( i = 0 \) is trivial. We suppose that in the rank \( n \) we have \( |\beta_i| < d \), and we prove the inequality \( |\beta_{i+1}| < d \). Indeed, we have

\[
|\beta_{i+1}| = \left| \frac{\beta_i - \alpha_i d}{p} \right|
\]

\[
< \frac{1}{p} |\beta_i| + \frac{1}{p} |\alpha_i d|
\]

\[
< \frac{1}{p} d + \frac{p-1}{p} d = d
\]

Case 2. For \( c > d \) and \( p \geq 3 \), we prove the two following cases:

Case 2.1. We suppose \( 0 < \frac{c(p-1)}{2dp} < 1 \). Also, we prove by recurrence that \( |\beta_i| < d \). Starting with \( i = 1 \), we have

\[
0 < \frac{c(p-1)}{2dp} < 1 \iff -\frac{\alpha_0 d}{p} < \frac{c}{p} - \frac{\alpha_0 d}{p} < \frac{2d}{p-1} - \frac{\alpha_0 d}{p}
\]

So

\[
-d < -\frac{\alpha_0 d}{p} < \beta_1 < d \left( \frac{2}{p-1} - \frac{\alpha_0}{p} \right) < d
\]

Now, we assume that the property is true at rank \( i \), and we show it at rank \( i + 1 \). Indeed, we have

\[
-d < \beta_i < d \iff -d < -\frac{d(1 + \alpha_i)}{p} < \frac{\beta_i - \alpha_i d}{p} < \frac{d(1 - \alpha_i)}{p} < d
\]

then \( -d < \beta_{i+1} < d \). Which means that for every \( i \in \mathbb{N}^* \), we have \( |\beta_i| < d \).

Case 2.2. Let the integer \( m \) given in (4), we suppose that \( 1 < \frac{c(p-1)}{2dp} \).

Firstly, we will prove that for all \( 0 \leq i \leq m \) the terms \( \beta_i \) are strictly positive. Indeed, we assume that there is \( k \in \{1, \ldots, m\} \), such that \( \beta_k < 0 \). From definition (2.1), we have

\[
\frac{\beta_{k-1} - \alpha_{k-1} d}{p} < 0
\]
which means $\beta_{k-1} < dp$. Multiplying both sides by $p^{k-1}$, and applying the lemma (2.3), it comes

$$c < d \left( \sum_{n=0}^{k-2} \alpha_n p^n \right) + dp^k$$

The coefficients $\alpha_n$ are strictly less than $p$, so

$$c < dp \left( \frac{p^{k-1} - 1}{p-1} + p^{k-1} \right)$$

Then, after simplification

$$c < \frac{pd}{p-1} (p^k - 1) < \frac{2pd}{p-1} p^k$$

Thus

$$\log \left( \frac{c(p-1)}{2dp} \right) \log p < k$$

however $m + 1 \leq k$. Where does the contradiction come from. Which means that for every $0 \leq i \leq m$, we have $\beta_k > 0$.

Now, we prove the inequalities $d \leq \beta_i \leq c$ for $i \in \{0, ..., m\}$.

The inequality in law is easily proved by recurrence for all $0 \leq i \leq m$. To prove the inequality in the left, we use the absurd. We assume that, there is a positive integer $k \in \{1, ..., m\}$ such that $0 < \beta_k < d$ (the condition $d < c$ implies that $k \neq 0$). By lemma (2.3) we obtain

$$\beta_k < d \iff c < d \left( \sum_{n=0}^{k-1} \alpha_n p^n \right) + dp^k$$

So $c < dp(1 + p + ... + p^{k-1} + p^{k-1})$. Hence

$$c < \frac{dp}{p-1} (2p^k - p^{k-1} - 1) \iff c < \frac{2pd}{p-1} p^k$$

It comes that

$$\log \left( \frac{c(p-1)}{2dp} \right) \log p < k$$

However $m + 1 \leq k$, hence the contradiction. Which means that for all $0 \leq i \leq m$, we have $c \geq \beta_k \geq d$. 

For the second part of this case, we suppose there is a positive integer \( k > m + 1 \) such that \( |\beta_k| > d \), that is \( \beta_k > d \) or \( \beta_k < -d \). By lemma (2.3), we have

\[
\beta_k > d \iff c > d \left( \sum_{n=0}^{k-1} \alpha_n p^n \right) + dp^k > dp^k
\]

hence \( \frac{c(p-1)}{2dp} > \left( \frac{p-1}{2} \right) p^{k-1} > p^{k-1} \), therefore

\[
\log \left( \frac{c(p-1)}{2dp} \right) \log p > k - 1
\]

then

\[
m + 1 = \left\lfloor \frac{\log \left( \frac{c(p-1)}{2dp} \right)}{\log p} \right\rfloor + 1 > k
\]

Contradiction. For the second inequality, we have by the formula (1)

\[
\beta_k = \frac{\beta_{k-1} - \alpha_k d}{p} \leq -d
\]

then \( \beta_{k-1} \leq d(\alpha_k - p) \), however \( \alpha_k \leq p - 1 \), thus \( \beta_{k-1} \leq -d \). And so on, until \( \beta_0 = c \leq -d \), which is another contradiction. So, for all \( i \geq m + 2 \) we have \( |\beta_i| \leq d \). The last part is easily.

\[\square\]

**Example 3.2.** For \( p = 3 \), \( c = 7 \) and \( d = 11 \), the case 1 is verified (see table 1)

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
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</thead>
<tbody>
<tr>
<td>( \alpha_k )</td>
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<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
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<td>0</td>
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<td>1</td>
</tr>
<tr>
<td>( \beta_k )</td>
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<td>-9</td>
<td>-3</td>
<td>-1</td>
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<td>-3</td>
<td>-1</td>
<td>-4</td>
</tr>
</tbody>
</table>
For $p = 3$, $c = 8$ and $d = 5$, the case 2.1 is verified (see table 2)

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<td>1</td>
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<tr>
<td>$\beta_k$</td>
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<td>-1</td>
<td>-2</td>
<td>-4</td>
<td>-3</td>
<td>-1</td>
</tr>
</tbody>
</table>

For $p = 3$, $c = 17$ and $d = 5$, we have $m = 0$ and the case 2.2 is verified (see table 3)

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<tbody>
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<tr>
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<td>-1</td>
<td>-2</td>
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</tr>
</tbody>
</table>

For $p = 3$, $c = 124$ and $d = 7$, we have $m = 1$ and the case 2.2 is verified (see table 4)

<table>
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<td>-2</td>
<td>-3</td>
<td>-1</td>
<td>-5</td>
<td>-4</td>
</tr>
</tbody>
</table>

For $p = 3$, $c = 247$ and $d = 7$, we have $m = 2$ and the case 2.2 is verified (see table 5)

<table>
<thead>
<tr>
<th>$k$</th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
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<td>$\alpha_k$</td>
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<td>-1</td>
<td>-5</td>
<td>-4</td>
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<td>-2</td>
</tr>
</tbody>
</table>
In the following corollary, we give a particular case \( p = 2 \).

**Corollary 3.3.** For \( p = 2 \), the sequence \((\beta_i)_{i \in \mathbb{N}}\) given in (1) verified the same cases:

**Cas1.** If \( c < d \), then

\[
0 \leq |\beta_i| < d \quad \forall i \in \mathbb{N}
\]

**Cas2.** If \( c > d \), we have also two cases:

**Cas2.1.** If \( 0 < \frac{c}{2d} < 1 \), then for all \( i \in \mathbb{N}^* \) we have \( |\beta_i| < d \).

**Cas2.2.** If \( 1 < \frac{c}{2d} \), then for a fixed integer

\[
m = \left\lfloor \frac{\log \left( \frac{c}{2d} \right)}{\log 2} \right\rfloor
\]

it comes that

\[
\begin{align*}
\{ d \leq |\beta_i| \leq c \quad & \text{for} \quad 0 \leq i < m + 1 \\
0 \leq |\beta_i| \leq d \quad & \text{for} \quad m + 1 \leq i \\
0 \leq |\beta_i| < c \quad & \text{for} \quad m + 1 = i
\end{align*}
\]

**Proof.** The proof is similar to that of the main theorem. \( \square \)

**Example 3.4.** For \( p = 2, c = 5 \) and \( d = 9 \), the case 1 is verified (see table 6)

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_k )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \beta_k )</td>
<td>5</td>
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<td>-1</td>
<td>-5</td>
<td>-7</td>
<td>-8</td>
<td>-4</td>
<td>-2</td>
<td>-1</td>
<td>-5</td>
<td>-7</td>
<td>-8</td>
<td>-4</td>
<td>-2</td>
<td>-1</td>
<td>-5</td>
</tr>
</tbody>
</table>

For \( p = 2, c = 5 \) and \( d = 3 \), the case 2.1 is verified (see table 7)

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_k )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \beta_k )</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
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<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>
For $p = 2$, $c = 7$ and $d = 3$, we have $m = 0$ and the case 2.2 is verified (see table 8)

**Table 8: Case 2.2 for m=0**

<table>
<thead>
<tr>
<th>$k$</th>
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<th>1</th>
<th>2</th>
<th>3</th>
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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
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</tr>
<tr>
<td>$\beta_k$</td>
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<td>-2</td>
<td>-1</td>
<td>-2</td>
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<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

For $p = 2$, $c = 13$ and $d = 3$, we have $m = 1$ and the case 2.2 is verified (see table 9)

**Table 9: Case 2.2 for m=1**

<table>
<thead>
<tr>
<th>$k$</th>
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<th>1</th>
<th>2</th>
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<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
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<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k$</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_k$</td>
<td>13</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
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<td>-1</td>
<td>-2</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

For $p = 2$, $c = 25$ and $d = 3$, we have $m = 2$ and the case 2.2 is verified (see table 10)

**Table 10: Case 2.2 for m=2**

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_k$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_k$</td>
<td>25</td>
<td>11</td>
<td>4</td>
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<td>-2</td>
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<td>-1</td>
</tr>
</tbody>
</table>

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**


