ON SOME PROPERTIES OF $I^K$-CONVERGENCE

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Abstract. In this paper, we study $I^K$-convergent sequences and observe that various properties of usual convergence are exhibited by $I^K$-convergence in the set of real numbers $\mathbb{R}$. Subsequently, we prove the Sandwich Theorem for $I^K$-convergent sequences in $\mathbb{R}$. We also introduce $I^K$-convergence field and study its various properties.

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1. INTRODUCTION

An ideal on a set $\mathcal{I}$ is a collection of subsets of $\mathcal{I}$ closed under finite unions and subset inclusion. Two basic ideals are $\text{Fin}$ and $\mathcal{I}_0$ on $\mathbb{N}$ defined as $\text{Fin} :=$ collection of all finite subsets of $\mathbb{N}$ and $\mathcal{I}_0 :=$ subsets of $\mathbb{N}$ with density 0. For a subset $A$ of $\mathbb{N}$, $A \in \mathcal{I}_0$ if and only if
\[
\limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = 0.
\]

An ideal $\mathcal{I}$ is a $P$-ideal if it is $\sigma$-directed modulo finite sets, i.e., for every sequence $(A_n)$ of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all $n \in \mathbb{N}$. For an ideal $\mathcal{I}$ in $P(\mathbb{N})$, we observe two additional subsets of $P(\mathbb{N})$ namely $\mathcal{I}^*$, $\mathcal{I}^+$ where $\mathcal{I}^* := \{A \subset \mathbb{N} : A^c \in \mathcal{I}\}$, the filter dual of $\mathcal{I}$ and $\mathcal{I}^+ :=$ collection of all subsets of $\mathbb{N}$ which does not belong to $\mathcal{I}$.

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The ideal convergence of a sequence of real numbers was introduced by Kostyrko et al. [8], as a generalisation of the existing notions of convergence. For an ideal $\mathcal{I}$, two modes of ideal convergence are denoted by $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence.

**Definition 1.1.** Let $X$ be a topological space. Then a sequence $x = \{x_n\}_{n \in \mathbb{N}}$ is said to be $\mathcal{I}$-convergent to $\xi$, denoted by $x_n \to_\mathcal{I} \xi$, if $\{n : x_n \notin U\} \in \mathcal{I}^+$, $\forall$ neighborhoods $U$ of $\xi$.

**Definition 1.2.** Let $X$ be a topological space. A sequence $x = \{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ is said to be $\mathcal{I}^*$-convergent to $\xi$ if and only if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{I}^*$ (i.e. $\mathbb{N} \setminus M \in \mathcal{I}$), such that $\lim_{k \to \infty} (x_{m_k}) = \xi$.

It may be observed that these two definitions arose from two equivalent definitions of usual convergence. Kostyrko et al. [8] showed that $\mathcal{I}^*$-convergence coincide $\mathcal{I}$-convergence for an admissible $P$-ideal $\mathcal{I}$, where admissible ideals contain elements in $\text{Fin}$.

In 2011, Macaj and Sleziak [5] defined the $\mathcal{I}^\mathcal{K}$-convergence of function in a topological space. Comparisons of $\mathcal{I}^\mathcal{K}$-convergence with ideal convergence [8] are studied by many authors [7, 6] in last decade. Some of the definitions and results of [2, 5] are listed below for further reference. Here $X$ is a topological space and $S$ is a set.

i) [5] A function $f : S \to X$ is called $\mathcal{I}^\mathcal{K}$-convergent to a point $x \in X$ if there exist $M \in \mathcal{I}^*$ such that the function $g : S \to X$ such that

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{K}$-convergent to $x$.

ii) [5] A function $f : S \to X$ is called $\mathcal{I}^\mathcal{K}^*$-convergent to a point $x \in X$ if there exist $M \in \mathcal{I}^*$ such that the function $g : S \to X$ such that

$$g(s) = \begin{cases} f(s), & s \in M \\ x, & s \notin M \end{cases}$$

is $\mathcal{K}^*$-convergent to $x$.

**Lemma 1.3.** [5, Lemma 2.1] If $\mathcal{I}$ and $\mathcal{K}$ are two ideals on $\mathbb{N}$ and $f : S \to X$ is a function such that $\mathcal{K} - \lim f = x$, then $\mathcal{I}^\mathcal{K} - \lim f = x$.

**Proposition 1.4.** [5, Lemma 2.1] Let $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be ideals on a set $S$ such that $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and $X$ be a topological space. Then for any function $f : S \to X$ we have
A sequence \( \{x_n\} \in X \) is said to be \( \mathcal{I} \)-bounded for an ideal \( \mathcal{I} \), if there exists \( M > 0 \) such that \( \{k \in \mathbb{N} : x_k > M\} \in \mathcal{I} \).

**Result 1.5.** [5, Result 3.3] If a sequence is \( \mathcal{I} \)-convergent, then it is \( \mathcal{I} \)-bounded.

**Theorem 1.6.** [1, Theorem 4.1] If a series \( \sum x_n \) is \( \mathcal{I} \)-convergent, then there exists a subset \( P = \{n_1, n_2, \ldots\} \) such that \( P \in \mathcal{I} \) and \( \sum x_n \) is convergent.

Throughout this paper we deal with the ideals \( \mathcal{I} \) containing \( Fin \) and \( S \notin \mathcal{I} \).

### 2. \( \mathcal{I} \mathcal{K} \)-CONVERGENT SEQUENCES

There are certain properties of \( \mathcal{I} \mathcal{K} \)-convergent sequences that can be shown straightway from usual convergence setup. Following results are obvious, we prefer to skip some of the proofs.

**Theorem 2.1.** If a sequence is \( \mathcal{I} \mathcal{K} \)-convergent then it is \( \mathcal{I} \cup \mathcal{K} \)-bounded, provided \( \mathcal{I} \cup \mathcal{K} \) is an ideal.

**Proof.** Let a sequence \( x = \{x_n\} \) is \( \mathcal{I} \mathcal{K} \)-convergent. Subsequently, we can observe that \( x \) is \( \mathcal{I} \cup \mathcal{K} \)-convergent. That means by Theorem 1.5, \( x \) is \( \mathcal{I} \cup \mathcal{K} \)-bounded. \( \square \)

**Result 2.2.** Let \( \mathcal{I} \) and \( \mathcal{K} \) be two ideals on \( \mathbb{N} \). \( \{x_n\}, \{y_n\} \) be two sequences such that \( x_n \leq y_n \) for all \( n \in \mathcal{K} \). Then

1. \( \mathcal{I} \mathcal{K} - \lim x_n = \infty \implies \mathcal{I} \mathcal{K} - \lim y_n = \infty. \)
2. \( \mathcal{I} \mathcal{K} - \lim y_n = -\infty \implies \mathcal{I} \mathcal{K} - \lim x_n = -\infty. \)

**Result 2.3.** Let \( \mathcal{I}_1, \mathcal{I}_2, \mathcal{K}_1, \mathcal{K}_2 \) be ideals on \( \mathbb{N} \) such that \( \mathcal{I}_1 \subseteq \mathcal{I}_2 \) and \( \mathcal{K}_1 \subseteq \mathcal{K}_2 \). Also \( \{x_n\}, \{y_n\} \) be two sequences such that \( x_n \leq y_n \) for all \( n \in \mathcal{K} \). Then

1. \( \mathcal{I} \mathcal{K}_1 - \lim x_n = \infty \implies \mathcal{I} \mathcal{K}_2 - \lim y_n = \infty. \)
2. \( \mathcal{I} \mathcal{K}_1 - \lim y_n = -\infty \implies \mathcal{I} \mathcal{K}_2 - \lim x_n = -\infty. \)
3. \( \mathcal{I}_1 \mathcal{K} - \lim x_n = \infty \implies \mathcal{I}_2 \mathcal{K} - \lim y_n = \infty. \)
(4) $I_K^\mathcal{K} - \lim y_n = -\infty \implies I_K^\mathcal{K} - \lim x_n = -\infty$.

**Proof.** Using Proposition 1.4, above results are immediate. □

**Theorem 2.4.** Let $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$ be real sequences such that $x_n \leq y_n \leq z_n$ for all $n \in K$, where $K \in \mathcal{K}^\ast$. If $I_K^\mathcal{K} - \lim x = L = I_K^\mathcal{K} - \lim z$ then $I_K^\mathcal{K} - \lim y = L$.

**Proof.** For a given $\varepsilon > 0$, Then, for $x = \{x_n\}$, $z = \{z_n\}$ there exist $M_1, M_2 \in I^\ast$ such that the sets

$$B_x = \{n \in M_1 : |x_n - L| \geq \varepsilon\},$$

$$B_z = \{n \in M_2 : |y_n - L| \geq \varepsilon\}$$

belong to $\mathcal{K}$. Then, for the set $M = M_1 \cap M_2 \in I^\ast$, we have the sets

$$B_x' = \{n \in M : |x_n - L| \geq \varepsilon\},$$

$$B_z' = \{n \in M : |y_n - L| \geq \varepsilon\}$$

belong to $\mathcal{K}$. Therefore, for $M \in I^\ast$, we have $B_y' \subseteq (B_x' \cup B_z') \cap K$ and hence the set

$$B_y' = \{n \in M : |z_n - L| \geq \varepsilon\}$$

is in $\mathcal{K}$. It follows that $\{y_n\}$ is $I_K^\mathcal{K}$-convergent to $L$. □

Following results are immediate, so we prefer to omit the proofs.

**Result 2.5.** Let $x_n \geq \alpha$ for all $n \in K(\subseteq \mathbb{N})$ with $K \in \mathcal{K}$. If $I_K^\mathcal{K} - \lim x_n = L$, then $L \geq \alpha$.

**Result 2.6.** Let $x_n \leq y_n$, for all $n \in I(\in \mathcal{I})$.

1. If $I_K^\mathcal{K} - \lim x_n$ and $I_K^\mathcal{K} - \lim y_n$ exist then $I_K^\mathcal{K} - \lim x_n \leq I_K^\mathcal{K} - \lim y_n$.
2. If $I_K^\mathcal{K} - \lim y_n \leq B$, then $I_K^\mathcal{K} - \lim x_n \leq B$.

**Result 2.7.** Let $x_n > 0$ for all $n \in K(K \in \mathcal{K})$ and $x_n \neq 0$ for all $n \in \mathbb{N}$, then $I_K^\mathcal{K} - \lim x_n = \infty$ if and only if $I_K^\mathcal{K} - \lim x_n^{-1} = 0$.

**Result 2.8.** If $I_K^\mathcal{K} - \lim x_n = L$, then $I_K^\mathcal{K} - \lim |x_n| = |L|$ but the converse is not true.
3. $\mathcal{I}^\mathcal{K}$-CONVERGENT SERIES

In this section, we introduce the notion of $\mathcal{I}^\mathcal{K}$-convergence for series of real or complex numbers which unifies and generalizes different notions of convergence of series.

**Definition 3.1.** A series $\sum_{k=1}^{\infty} x_k$ is said to be $\mathcal{I}^\mathcal{K}$-convergent if the sequence of its partial sums $(s_n)$, where $s_n = x_1 + x_2 + \ldots + x_n$ is $\mathcal{I}^\mathcal{K}$-convergent.

**Theorem 3.2.** If a series $\sum x_n$ is $\mathcal{I}^\mathcal{K}$-convergent, then there exists a subset $P = \{n_1, n_2, \ldots\}$ such that $\sum x_n$ is $\mathcal{I}^\mathcal{K}$-convergent provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.

**Proof.** We observe that if a series $\sum x_n$ is $\mathcal{I}^\mathcal{K}$-convergent, then it follows that $\sum x_n$ is $\mathcal{I} \cup \mathcal{K}$-convergent to the same limit. Then by Theorem 1.6, we have a set $P = \{n_1, n_2, \ldots\} \in \mathcal{I} \cup \mathcal{K}$ such that $\sum x_n$ is convergent. $\square$

**Result 3.3.** The series $\sum z_n$ with complex terms is $\mathcal{I}^\mathcal{K}$-convergent if and only if the real part and the imaginary part is $\mathcal{I}^\mathcal{K}$-convergent.

**Result 3.4.** If $\sum x_n$ and $\sum y_n$ be two $\mathcal{I}^\mathcal{K}$-convergent series then for any complex numbers $\alpha$ and $\beta$, we have the series $\sum (\alpha x_n + \beta y_n)$ is $\mathcal{I}^\mathcal{K}$-convergent to $\alpha \sum x_n + \beta \sum y_n$.

4. $\mathcal{I}^\mathcal{K}$-CONVERGENCE FIELD

**Definition 4.1.** A convergence field of $\mathcal{I}^\mathcal{K}$-convergence is a set defined as

$$F(\mathcal{I}^\mathcal{K}) = \{x = (x_n) \in l_\infty : \text{there exist } \mathcal{I}^\mathcal{K} - \lim x \in \mathbb{R}\}.$$ 

$l_\infty$ denote the space of all bounded complex valued sequences with $||.||_\infty$ norm.

Now define a function $g : F(\mathcal{I}^\mathcal{K}) \to \mathbb{R}$ such that

$$g(x) = \mathcal{I}^\mathcal{K} - \lim x, \text{ for all } x \in F(\mathcal{I}^\mathcal{K}).$$

**Theorem 4.2.** The function $g : F(\mathcal{I}^\mathcal{K}) \to \mathbb{R}$ is Lipschitz function and hence uniformly continuous.

**Proof.** Let $x, y \in F(\mathcal{I}^\mathcal{K})$ and $x \neq y$. That means $||x - y|| > 0$. So, there exist $M_1 \in \mathcal{I}^*$ such that
ON SOME PROPERTIES OF $\mathcal{F}^{X}$-CONVERGENCE

$$A_x = \{ n \in M_1 : |x_n - g(x)| \geq ||x - y|| \} \in \mathcal{K}$$

and also there exist $M_2 \in \mathcal{I}^*$ such that

$$A_y = \{ n \in M_2 : |y_n - g(y)| \geq ||x - y|| \} \in \mathcal{K}.$$  

Then for $M_1 \cap M_2 = M \in \mathcal{I}^*$, the sets

$$A_x = \{ n \in M : |x_n - g(x)| \geq ||x - y|| \},$$

$$A_y = \{ n \in M : |y_n - g(y)| \geq ||x - y|| \}$$

belong to $\mathcal{K}$. Thus

$$A_x' = \{ n \in M : |x_n - g(x)| < ||x - y|| \},$$

$$A_y' = \{ n \in M : |y_n - g(y)| < ||x - y|| \}$$

belong to $\mathcal{K}^*$. So, $A = A_x' \cap A_y' \in \mathcal{K}^*$. Now taking $n$ in $A$, we have

$$|g(x) - g(y)| \leq |g(x) - x_n| + |x_n - y_n| + |y_n - g(y)| \leq 3||x - y||.$$  

This implies that $g$ is a Lipchitz function.

Theorem 4.3. If $x, y \in F(\mathcal{F}^{X})$ then $xy \in F(\mathcal{F}^{X})$ and $g(xy) = g(x)g(y)$.  

Proof. Let $\varepsilon > 0$. Then there exist $M_1, M_2 \in \mathcal{I}^*$ such that the sets

$$B_x = \{ n \in M_1 : |x_n - g(x)| < \varepsilon \},$$

$$B_y = \{ n \in M_2 : |y_n - g(y)| < \varepsilon \}$$

belong to $\mathcal{K}^*$. Then, for $M = M_1 \cap M_2 \in \mathcal{I}^*$, the following sets

$$B_x = \{ n \in M : |x_n - g(x)| < \varepsilon \},$$

$$B_y = \{ n \in M : |y_n - g(y)| < \varepsilon \}$$

belong to $\mathcal{K}^*$. Now,

$$|x_n y_n - g(x)g(y)| = |x_n y_n - x_n g(y) + x_n g(y) - g(x)g(y)|$$

$$\leq |x_n||y_n - g(y)| + |g(y)||x_n - g(x)|.$$  

As $F(\mathcal{F}^{X}) \subseteq l_\infty$, there exist $N \in \mathbb{R}$ such that $|x_n| < N$ and $|g(y)| < N$. Thus, we get
\[ |x_n y_n - g(x)g(y)| \leq N\varepsilon + N\varepsilon = 2N\varepsilon, \]

for all \( n \in B_x \cap B_y \in \mathcal{K}^* \). Hence \( xy \in F(\mathcal{I}\mathcal{K}) \) and \( g(xy) = g(x)g(y) \).

\( \square \)

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**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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