ON COMPLETELY HOMOGENEOUS $L$-TOPOLOGICAL SPACES

PINKY$^1$,*, T.P. JOHNSON$^2$

$^1$Department of Mathematics, Cochin University of Science and Technology, Cochin-682022, India
$^2$Applied Science and Humanities Division, School of Engineering, Cochin University of Science and Technology, Cochin-682022, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we study completely homogeneous $L$-topological spaces on a non-empty set $X$ when membership lattice $L$ is a complete chain.

Keywords: $L$-topology; completely homogeneous $L$-topological spaces; join; meet.

2010 AMS Subject Classification: 03G10, 54A99.

1. INTRODUCTION

For a topological property $P$ and a set $X$, let $P(X)$ denote the collection of all topologies on $X$ with property $P$. Then $P(X)$ is a partially ordered set under the natural order of set inclusion. A topological space $(X, \mathcal{I})$ with property $P$ is minimum $P$ (maximum $P$) if $P(X)$ is non-empty and $\mathcal{I}$ is a minimum (maximum) element the set $P(X)$. In 1970, Roland E. Larson characterizes all minimum and maximum $P$ spaces [4]. He proved that a topological space $(X, \mathcal{I})$ is minimum $P$ (maximum $P'$) for some topological property $P$ ($P'$) if and only if it is completely homogeneous, where completely homogeneous means that every one-to-one function of $X$ onto itself is a homeomorphism. He then proved that the only completely homogeneous topologies on a set $X$

*Corresponding author
E-mail address: malikpinky1989@gmail.com

Received January 31, 2021
are the indiscrete topology, the discrete topology and those topologies in which the closed sets are the space \( X \) and all subsets of \( X \) of cardinality less than \( m \), where \( m \) is an infinite cardinal not greater than the cardinality of \( X \).

T.P. Johnson has defined the concept of a complete homogeneous fuzzy topological space in an analogous way and studied some of its properties [2]. P. Sini et al. have characterized completely homogeneous \( L \)-topological spaces when \( X \) is a finite set and \( L = \{0, a, 1\} \), where \( a \neq 0, 1 \) [5].

However, we consider an equivalence relation \( R \) on the set of all completely homogeneous \( L \)-topologies on a non-empty set \( X \) when membership lattice \( L \) is a complete chain and investigate all disjoint equivalence classes with respect to the relation \( R \).

2. Preliminaries

Throughout this paper, \( X \) stands for a non-empty set, \( L \) for a complete chain with the least element 0 and the greatest element 1, \( S(X) \) stands for the set of all permutations of the set \( X \). The constant function in \( L^X \), taking value \( \alpha \) is denoted by \( \alpha \) and \( x_\gamma \), where \( \gamma \neq 0 \) denotes the \( L \)-fuzzy point defined by

\[ x_\gamma(y) = \begin{cases} 
\gamma & \text{if } y = x \\
0 & \text{otherwise}
\end{cases} \]

Any \( f \in L^X \) is called as an \( L \)-subset of \( X \). The following are some important definition reported in [3, 6]:

**Definition 2.1.** Let \( \delta \) be a non-empty subset of \( L^X \). We call \( \delta \) an \( L \)-topology on \( X \), if \( \delta \) satisfies the following conditions:

1. \( 0, 1 \in \delta \).
2. If \( f, g \in \delta \), then \( f \land g \in \delta \).
3. If \( \delta_1 \subseteq \delta \), then \( \bigvee_{f \in \delta_1} f \in \delta \).

The pair \((L^X, \delta)\) is called an \( L \)-topological space. The elements of \( \delta \) are said to be open \( L \)-subsets of \( X \).

**Definition 2.2.** Let \( X \) and \( Y \) be two sets and \( \theta : X \to Y \) be a function. Then for any \( L \)-subset \( g \) in \( X \), \( \theta(g) \) is an \( L \)-subset in \( Y \) defined by

\[ \theta(g)(y) = \begin{cases} 
\sup \{ g(z) : z \in \theta^{-1}(y) \} & \text{if } \theta^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{cases} \]

where \( \theta^{-1}(y) = \{ x \in X : \theta(x) = y \} \).
For an $L$-subset $f$ in $Y$, we define
\[ \theta^{-1}(f)(x) = f|\theta(x)], \forall x \in X. \]
Obviously $\theta^{-1}(f)$ is an $L$-subset in $X$.

**Definition 2.3.** Let $(X, \delta)$ and $(Y, \delta')$ be two $L$-topological spaces. Then a function $\theta : X \to Y$ is said to be $L$-continuous if $\theta^{-1}(g) \in \delta$ for every $g \in \delta'$ and $\theta$ is said to be open if $\theta(f) \in \delta'$ for every $f \in \delta$.

**Definition 2.4.** Let $(X, \delta)$ and $(Y, \delta')$ be two $L$-topological spaces. Then a bijection $\theta : X \to Y$ is said to be $L$-homeomorphism if both $\theta$ and $\theta^{-1}$ are $L$-continuous.

By an $L$-homeomorphism of $(X, \delta)$, we mean an $L$-homeomorphism from $(X, \delta)$ to itself. The set of all $L$-homeomorphism of an $L$-topological space $(X, \delta)$ onto itself is a group under composition, which is a subgroup of the group of all permutations on the set $X$. It is called the group of $L$-homeomorphisms of $(X, \delta)$.

**Definition 2.5.** An $L$-topological space $(X, \delta)$ is called a completely homogeneous space if every bijection of $X$ onto itself is an $L$-homeomorphism.

**Notations:**
- $|A|$ stands for the cardinality of a given set $A$.
- If $(X, \delta)$ is an $L$-topological space, then define
  1. $\overline{\delta} = \delta \setminus \{0, 1\}$.
  2. $\mathcal{R}_f = \{f(x) : x \in X\}$.
  3. $\mathcal{L}_f^X = \{g : X \to \mathcal{R}_f\}$.
  4. $\mathcal{R}_{\overline{\delta}} = \{f(x) : x \in X \text{ and } f \in \overline{\delta}\}$.
- For any $A \subseteq L$, define $\mathcal{L}_A^X = \{f : X \to A\}$.
- For any $H \subseteq L$, define $H^* = \{\alpha \in L : \alpha = \bigvee_{\gamma \in M} \gamma, \text{ where } M \subseteq H\}$. Then $H$ is said to be closed with respect to arbitrary join if $H^* \subseteq H$ i.e. if $H$ contains all the possible join of its elements.

3. **Completely Homogeneous $L$-Topological Spaces**

**Definition 3.1.** Let $\text{CHLT}(X)$ be the collection of all completely homogeneous $L$-topologies on $X$. 

Let $\delta_1, \delta_2 \in \text{CHLT}(X)$ and define the relation $R$ on the set $\text{CHLT}(X)$ as:

$\delta_1 R \delta_2$ if and only if $|\Re_{\delta_1}| = |\Re_{\delta_2}|$.

Clearly, $R$ is an equivalence relation.

For $0 \leq m \leq |L|$, define

$$[m] \text{(class of } m) = \{ \delta : \delta \text{ is a completely homogeneous } L\text{-topology on } X \text{ and } |\Re_{\delta}| = m \}.$$  

**Definition 3.2.** Let $A \subseteq L$ be any subset and $|A| > 1$. Then a subset $M \subseteq A$ is called a $c$-subset of $A$ if

(i) $M^* \subseteq M$.

(ii) $|M| > 1$.

(iii) if $\alpha, \beta \in M$ and $\alpha < \gamma < \beta$ for some $\gamma \in A$, then $\gamma \in M$.

**Definition 3.3.** Two $c$-subsets $\triangle_i$ and $\triangle_j$ of a subset $A \subseteq L$ are said to be distinct if $\exists$ at least one $\alpha_i \in \triangle_i$ and $\alpha_j \in \triangle_j$ such that $\alpha_i \notin \triangle_j$ and $\alpha_j \notin \triangle_i$.

**4. Completely Homogeneous $L$-Topological Spaces When $X$ Is a Finite Set**

Throughout this section, $X$ stands for a finite set.

**Theorem 4.1.** Let $X$ be a finite set and $L$ be a complete chain. Then $(X, \delta)$ is a completely homogeneous $L$-topological space if and only if $L^X \delta \subseteq \delta, \forall f \in \delta$.

**Proof.** First suppose that $(X, \delta)$ is a completely homogeneous $L$-topological space.

Let $k_1 = \bigwedge_{k \in \Re_f} k$ and $x_{k_1}(y) = \begin{cases} k_i & \text{if } y = x \\ k_1 & \text{otherwise} \end{cases}$.

We claim that $x_{k_1} \in \delta, \forall k \in \Re_f$.

Clearly, $k_1 \in \Re_f$. Since $k_1 \in \Re_f, \exists$ an element $x_0 \in X$ such that $f(x_0) = k_1$. For each $x \in X \setminus \{x_0\}$, define $f_x = f o h_x$, where $h_x : X \to X$ is defined as:

$h_x(y) = \begin{cases} x_0 & \text{if } y = x \\ x & \text{if } y = x_0 \\ y & \text{otherwise} \end{cases}$

Then $f_x(y) = \begin{cases} k_1 & \text{if } y = x \\ f(x) & \text{if } y = x_0 \\ f(y) & \text{otherwise} \end{cases}$.
Clearly, \( f_x \in \delta, \forall x \in X \setminus \{x_0\} \).

Let \( k \in \Re_f \). So \( \exists \) an element \( z \in X \) such that \( f(z) = k \).

Now \( \bigwedge_{x \in X \setminus \{z\}} f_x(y) = \begin{cases} 
  k & \text{if } y = z \\
  k_1 & \text{otherwise}
\end{cases} \).

\( \Rightarrow \bigwedge_{x \in X \setminus \{z\}} f_x(y) = x'_k \in \delta. \)

Hence \( L^X_{\Re_f} \subset \delta, \forall f \in \delta \).

Conversely, suppose that \( L^X_{\Re_f} \subset \delta, \forall f \in \delta \). Then \( f\circ h \in \delta, \forall f \in \delta \) and \( \forall h \in S(X) \). Hence \( \delta \) is a completely homogeneous \( L \)-topology on \( X \).

**Remark 4.2.** It can be checked that following are the disjoint equivalence classes with respect to the relation \( R \) when \( X \) is a finite set :

- \([0]\) contains only one completely homogeneous \( L \)-topology \( \{0, 1\} \).
- \([1]\) contains only one type of completely homogeneous \( L \)-topologies \( \{0, 1, \alpha\} \), where \( \alpha \in L \setminus \{0, 1\} \) i.e.
  \( [1] = \{ \{0, 1, \alpha\} : \alpha \in L \setminus \{0, 1\} \} \).
- \([2]\) contains following two types of completely homogeneous \( L \)-topologies :
  (i) \( \{ \{0, 1, \alpha_1, \alpha_2\} : \alpha_1, \alpha_2 \in L \setminus \{0, 1\} \} \).
  (ii) \( \{ \{0, 1\}, g : g \in L^X_H, \text{ where } H \subseteq L \text{ and } |H| = 2 \} \).
- For \( m \geq 3 \), \([m]\) contains following three types of completely homogeneous \( L \)-topologies :
  (i) \( \{ \{0, 1, \alpha\} : \alpha \in H_1, \text{ where } H_1 \subseteq L \setminus \{0, 1\} \text{ such that } H^*_1 \subseteq H_1 \text{ and } |H_1| = m \} \).
  (ii) \( \{ \{0, 1\}, g : g \in L^X_{H_2}, \text{ where } H_2 \subseteq L \text{ such that } H^*_2 \subseteq H_2 \text{ and } |H_2| = m \} \).
  (iii) Let \( H \subseteq L \) be any subset such that \( H^* \subseteq H \) and \( |H| = m \). Consider a family \( \triangle_i, i \in \Omega \) of distinct \( c \)-subsets of \( H \).
  Let \( E = \{ \beta \in H : \beta \not\in \bigcup_{i \in \Omega} \triangle_i \} \).
  The \( L \)-topologies of the form \( \{ \{0, 1, \beta, f : \beta \in E \text{ and } f \in \bigcup_{i \in \Omega} L^X_{\triangle_i} \} \). \)

**Theorem 4.3.** Let \( X \) be a finite set and \( L \) be a complete chain. If \( F \) is a completely homogeneous \( L \)-topology on \( X \), then \( F \) is equal to one of the \( L \)-topologies defined in the remark 4.2.
Proof. Let \(|\mathcal{R}_F| = m\). If \(m = 0\), then clearly \(F = \{0, 1\}\). So, assume that \(m > 0\).

Case 1: If \(F\) contains only constant \(L\)-subsets, then \(F = \{0, 1, \alpha : \alpha \in \mathcal{R}_F\}\).

Case 2: If \(L_X^X \subseteq F\), then \(F = \{0, 1, g : g \in L_X^X\}\).

Case 3: Suppose Case 1 and Case 2 do not hold.

Let \(f \in F\) be any non-constant \(L\)-subset.

We claim that \(f \in \mathcal{L}_X^X\) for some \(c\)-subset \(\Delta \subseteq \mathcal{R}_F\).

Let \(\Delta \subseteq \mathcal{R}_F\) be a subset such that

(i) \(\mathcal{R}_f \subseteq \Delta\).

(ii) \(L_X^X \subseteq F\).

(iii) \(\Delta^* \subseteq \Delta\).

(iv) \(\Delta\) is not properly contained in any proper subset of \(\mathcal{R}_F\) satisfying above three properties.

Let \(\alpha, \beta \in \Delta\) and \(\gamma \in \mathcal{R}_F\) such that \(\alpha < \gamma < \beta\).

\(\gamma \in \mathcal{R}_F\) and \(L_X^X \subseteq F, \forall g \in F \implies \gamma \in F\).

Let \(\gamma \notin \Delta\). Since \(L\) is a chain and \(L_X^X \subseteq F\), it is easy to see that \(T = \Delta \cup \{\gamma\}\) satisfies properties (i)-(iii) and \(\Delta \subseteq T\), a contradiction \(\implies \gamma \in \Delta \implies \Delta\) is a \(c\)-subset.

Therefore, corresponding to every \(L\)-subset \(g\) of \(F\), \(\exists a c\)-subset \(\nabla \subseteq \mathcal{R}_F\) such that \(g \in L_X^X \subseteq F\).

Let \(\Delta_i, i \in \Omega\) be the collection of those distinct \(c\)-subsets of \(\mathcal{R}_F\) such that \(L_X^X \subseteq F, \forall i \in \Omega\) and for every non-constant \(L\)-subset \(h \in F, h \in L_X^X\) for at-least one \(i \in \Omega\).

Let \(E = \{\beta \in \mathcal{R}_F : \beta \notin \bigcup_{i \in \Omega} \Delta_i\}\).

Thus \(F = \{0, 1, \beta, f : \beta \in E\) and \(f \in \bigcup_{i \in \Omega} L_X^X\}\).

\(\implies\) If \(F\) is a completely homogeneous \(L\)-topology on a finite set \(X\), then \(F\) is equal to one of the \(L\)-topologies defined in remark 4.2.

5. Completely Homogeneous \(L\)-Topological Spaces When \(X\) Is a Countable Set

Throughout this section, \(X\) stands for a countable set.

Remark 5.1. It can be checked that following are the disjoint equivalence classes with respect to the relation \(R\) when \(X\) is a countable set:

- \([0]\) contains only one completely homogeneous \(L\)-topology \(\{0, 1\}\).
• [1] contains only one type of completely homogeneous $L$-topologies $\{0, 1, \alpha\}$, where $\alpha \in L \setminus \{0, 1\}$ i.e.
$[1] = \{\{0, 1, \alpha\} : \alpha \in L \setminus \{0, 1\}\}$. 

• [2] contains four types of completely homogeneous $L$-topologies:
  (i) $\{\{0, 1, \alpha, \beta\} : \alpha, \beta \in L \setminus \{0, 1\}\}$.
  (ii) $\{0, 1, g : g \in L^X_H, \text{where } H \subseteq L \text{ and } |H| = 2\}$.

Let $\alpha_1, \alpha_2 \in L$ be two arbitrary elements such that $\alpha_1 < \alpha_2$ and $g \in L^X$ be defined by
$$g(x) = \begin{cases} 
\alpha_1 & \text{for at-most finitely many } x \in X, \\
\alpha_2 & \text{otherwise}
\end{cases}.$$

(iii) $L$-topologies generated by the sets of the form $\{0, 1, goh : h \in S(X)\}$.
(iv) $L$-topologies generated by the sets of the form $\{0, 1, \alpha_1, goh : h \in S(X)\}$.

• For $m \geq 3, [m]$ contains following types of completely homogeneous $L$-topologies:
  (i) $\{0, 1, \alpha : \alpha \in H_1, \text{where } H_1 \subseteq L \setminus \{0, 1\} \text{ such that } H_1^* \subseteq H_1 \text{ and } |H_1| = m\}$.
  (ii) $\{0, 1, g : g \in L^X_H, \text{where } H \subseteq L \text{ such that } H^* \subseteq H \text{ and } |H| = m\}$.
  (iii) Let $H \subseteq L$ be any subset such that $H^* \subseteq H$ and $|H| = m$.

Consider a $c$-subset $\triangle \subseteq H$, choose an arbitrary element $\gamma \in \triangle$ and define:
$$P_1 = \{\alpha \in \triangle : \alpha < (\leq) \gamma\},$$
$$P_2 = \{\beta \in \triangle : \gamma < (>) \beta\},$$
$$L_\triangle = \left\{ f \in L^X : f(x) \in P_1 \text{ for finitely many } x \in X \right\},$$
$$L_{\triangle, C} = L_\triangle \cup \{\alpha : \alpha \in C \subseteq P_1\}.$$

Consider a family $\triangle_i, i \in \Omega$ of distinct $c$-subsets of $H$ and corresponding to each $c$-subset $\triangle_i, i \in \Omega$, choose exactly one set from the set $\{L_{\triangle_i}, L_{\triangle_i, C}\}$ and denote that set by $\Sigma_{\triangle_i}$.

Let $\mathcal{E} = \{\beta \in H : \beta \notin \bigcup_{i \in \Omega} \triangle_i\}$.

The $L$-topologies of the form $\{0, 1, \beta, f : \beta \in \mathcal{E} \text{ and } f \in \Sigma_{\triangle_i}, i \in \Omega\}$.
**Theorem 5.2.** Let $X$ be a countable set and $L$ be a complete chain. If $F$ is a completely homogeneous $L$-topology on $X$, then $F$ is equal to one of the $L$-topologies defined in the remark 5.1.

**Proof.** Let $|\mathcal{R}_F| = m$. If $m = 0$, then clearly $F = \{0, 1\}$. So, assume that $m > 0$.

- **Case 1:** If $F$ contains only constant $L$-subsets, then $F = \{0, 1, \alpha : \alpha \in \mathcal{R}_F\}$.
- **Case 2:** If $\mathcal{L}_X \subseteq F$, then $F = \{0, 1, g : g \in \mathcal{L}_X\}$.
- **Case 3:** Suppose Case 1 and Case 2 do not hold.

Let $f \in F$ be any non-constant $L$-subset.

Let $\triangle \subseteq \mathcal{R}_F$ be a subset such that

1. $\mathcal{R}_f \subseteq \triangle$.
2. $\triangle^* \subseteq \triangle$.
3. For any two elements $\alpha, \beta \in \triangle$, there exists an $L$-subset $g \in F$ such that $g(x) = \alpha, g(y) = \beta$ for some $x, y \in X$.
4. $\triangle$ is not properly contained in any proper subset of $\mathcal{R}_F$ satisfying above three properties.

Let $\alpha, \beta \in \triangle$ and $\gamma \in \mathcal{R}_F$ such that $\alpha < \gamma < \beta$.

Let $\gamma \notin \triangle$. $\mathcal{R}_F$ and $F$ is a completely homogeneous $L$-topological space so there exists an $L$-subset $h_1 \in F$ such that $h_1(x) = h_1(y) = \gamma$ for some $x, y \in X$.

Since $\alpha, \beta \in \triangle \Rightarrow$ there exists an $L$-subset $h_2 \in F$ such that $h_2(x) = \alpha, h_2(y) = \beta$.

Then $(h_1 \wedge h_2)(x) = \alpha$ and $(h_1 \wedge h_2)(y) = \gamma$.

$(h_1 \vee h_2)(x) = \gamma$ and $(h_1 \vee h_2)(y) = \beta$.

In the same way, it can be shown that for any two elements $\eta_1, \eta_2 \in T = \triangle \cup \{\gamma\}$, there exists an $L$-subset $g \in F$ such that $g(x) = \eta_1, g(y) = \eta_2$ for some $x, y \in X$ and $\mathcal{R}_f \subseteq T$, a contradiction $\Rightarrow \gamma \in \triangle \Rightarrow \triangle$ is a $c$-subset.

- **Case (i):** If $\mathcal{L}^X_\triangle \subseteq F$, then $f \in \mathcal{L}^X_\triangle$.
- **Case (ii):** Let $\mathcal{L}^X_\triangle \notin F$.

Let $\mathcal{D} = \{h \in F : \mathcal{R}_h \subseteq \triangle\}$.

$\mathcal{L}^X_\triangle \notin F \Rightarrow \exists$ some element(s) $\lambda \in \triangle$ such that if $h \in \mathcal{D}$ and $h(x) = \lambda$ for some $x \in X$, then $h(y) = \lambda$ for at-most finitely many $y \in X$.

Let $\mathcal{P} = \{\lambda \in \triangle :$ if $\lambda \in \mathcal{R}_h$ for some $h \in \mathcal{D}$, then $h(x) = \lambda$ for at-most finitely many $x \in X\}$. 
It can be checked that

(i) if $\alpha, \beta \in \mathcal{P}$ and $\eta \in \Delta$ such that $\alpha < \eta < \beta$, then $\eta \in \mathcal{P}$.

(ii) if $\alpha \in \mathcal{P}$ and $\eta \in \Delta$ such that $\eta < \alpha$, then $\eta \in \mathcal{P}$.

$$L_\Delta = \begin{cases} 
  g \in L^X : & g(x) \in \mathcal{P} \text{ for at-most finitely many } x \in X \\
  g(x) \in \Delta \setminus \mathcal{P} \text{ otherwise}.
\end{cases}$$

Now two cases arise:

**Case (a)**: when $\alpha \not\in F, \forall \alpha \in \mathcal{P}$.

Then $L_\Delta \subseteq F$.

**Case (b)**: when $\alpha \in F$ for all / some $\alpha \in \mathcal{P}$.

Let $\mathcal{C} = \{\alpha \in \mathcal{P} : \alpha \in F\}$ and $L_{\Delta, \mathcal{C}} = L_\Delta \cup \mathcal{C} \subseteq F$.

Thus either $f \in L_\Delta$ or $f \in L_{\Delta, \mathcal{C}}$.

Let $\Delta_i, i \in \Omega$ be the collection of those distinct $c$-subsets of $\mathcal{R}_\mathcal{T}$ such that corresponding to each $c$-subset $\Delta_i, i \in \Omega$, exactly one set from the set $\{L_{\Delta_i}^X, L_{\Delta_i}, L_{\Delta_i, \mathcal{C}}\}$ denoted by $\Sigma_{\Delta_i} \subset F, \forall i \in \Omega$ and for every non-constant $L$-subset $h \in F, h \in \Sigma_{\Delta_i}$ for at-least one $i \in \Omega$.

Let $\mathcal{E} = \{\beta \in \mathcal{R}_\mathcal{T} : \beta \not\in \bigcup_{i \in \Omega} \Delta_i\}$.

Thus $F = \left\{0, 1, \beta, f : \beta \in \mathcal{E} \text{ and } f \in \Sigma_{\Delta_i}, i \in \Omega\right\}$.

$\Rightarrow$ If $F$ is a completely homogeneous $L$-topology on a countable set $X$, then $F$ is equal to one of the $L$-topologies defined in the remark 5.1.

**6. Completely Homogeneous $L$-Topological Spaces When $X$ Is an Uncountable Set**

Throughout this section, $X$ stands for an uncountable set.

**Remark 6.1.** It can be checked that following are the disjoint equivalence classes with respect to the relation $R$ when $X$ is an uncountable set:

- $[0]$ contains only one completely homogeneous $L$-topology $\{0, 1\}$.

- $[1]$ contains only one type of completely homogeneous $L$-topologies $\{0, 1, \alpha\}$, where $\alpha \in L \setminus \{0, 1\}$ i.e.

  $$[1] = \{\{0, 1, \alpha\} : \alpha \in L \setminus \{0, 1\}\}.$$

- $[2]$ contains following types of completely homogeneous $L$-topologies:


(i) \( \{0, 1, \alpha, \beta\} : \alpha, \beta \in L \setminus \{0, 1\} \).

(ii) \( \{0, 1, g : g \in \mathcal{L}^X_H, \text{where } H \subseteq L \text{ and } |H| = 2\} \).

Let \( \alpha_1, \alpha_2 \in L \) be two arbitrary elements such that \( \alpha_1 < \alpha_2 \) and \( g_1, g_2 \in L^X \) be defined by
\[
g_1(x) = \begin{cases} 
\alpha_1 & \text{for at-most finitely many } x \in X \\
\alpha_2 & \text{otherwise}
\end{cases}
\]
and \( g_2(x) = \begin{cases} 
\alpha_1 & \text{for at-most countably many } x \in X \\
\alpha_2 & \text{otherwise}
\end{cases} \).

(iii) \( L \)-topologies generated by the sets of the form
\( \{0, 1, g_1h : h \in S(X)\} \).

(iv) \( L \)-topologies generated by the sets of the form
\( \{0, 1, \alpha_1, g_1h : h \in S(X)\} \).

(v) \( L \)-topologies generated by the sets of the form
\( \{0, 1, g_2h : h \in S(X)\} \).

(vi) \( L \)-topologies generated by the sets of the form
\( \{0, 1, \alpha_1, g_2h : h \in S(X)\} \).

- For \( m \geq 3, [m] \) contains following types of completely homogeneous \( L \)-topologies:

  (i) \( \{0, 1, \alpha : \alpha \in H_1, \text{where } H_1 \subseteq L \setminus \{0, 1\} \text{ such that } H_1^* \subseteq H_1 \text{ and } |H_1| = m\} \).

  (ii) \( \{0, 1, g : g \in \mathcal{L}^X_{H^*}, \text{where } H \subseteq L \text{ such that } H^* \subseteq H \text{ and } |H| = m\} \).

  (iii) Let \( H \subseteq L \) be any subset such that \( H^* \subseteq H \text{ and } |H| = m \).

Consider a \( c \)-subset \( \triangle \subseteq H \), choose an arbitrary element \( \gamma \in \triangle \) and define:
\[
\mathbb{P}_1 = \{\alpha \in \triangle : \alpha < (\leq) \gamma\}, \\
\mathbb{P}_2 = \{\beta \in \triangle : \gamma \leq (>) \beta\}, \\
\mathbb{P}_1^* = \mathbb{P}_1 \setminus \{\gamma\}, \\
\mathbb{P}_2^* = \mathbb{P}_2 \setminus \{\gamma\}, \\
\mathbb{L}^1_{\triangle} = \begin{cases} 
f \in L^X : f(x) \in \mathbb{P}_1 \text{ for at-most finitely many } x \in X \\
f \in \mathbb{P}_2 \text{ otherwise}
\end{cases}.
\]
\[ L^2_\Delta = \begin{cases} f \in L^X : & f(x) \in P_1 \text{ for at-most countably many } x \in X, \\ f(x) \in P_2 \text{ otherwise} \end{cases} \]

\[ L^3_\Delta = \begin{cases} f \in L^X : & f(x) \in P^*_1 \text{ for at-most finitely many } x \in X \\ f(x) = \gamma \text{ for at-most countably many } x \in X, \\ f(x) \in P^*_2 \text{ otherwise} \end{cases} \]

\[ L^1_{\Delta,C} = L^1_\Delta \cup \{ \alpha : \alpha \in C \subseteq P_1 \}, \]

\[ L^2_{\Delta,C} = L^2_\Delta \cup \{ \alpha : \alpha \in C \subseteq P_1 \}, \]

\[ L^3_{\Delta,C} = L^3_\Delta \cup \{ \alpha : \alpha \in C \subseteq P^*_1 \cup \{ \gamma \} \}. \]

Consider a family \( \Delta_i, i \in \Omega \) of distinct \( c \)-subsets of \( H \) and corresponding to each \( c \)-subset \( \Delta_i, i \in \Omega \), choose exactly one set from the set \( \{ L^X_{\Delta_i}, L^k_{\Delta_i}, L^k_{\Delta_i,C} : k = 1, 2, 3 \} \) and denote that set by \( \Sigma_{\Delta_i} \).

Let \( E = \{ \beta \in H : \beta \notin \bigcup_{i \in \Omega} \Delta_i \} \).

The \( L \)-topologies of the form \( \{ 0, 1, \beta, f : \beta \in E \text{ and } f \in \Sigma_{\Delta_i}, i \in \Omega \} \).

**Theorem 6.2.** Let \( X \) be an uncountable set and \( L \) be a complete chain. If \( F \) is a completely homogeneous \( L \)-topology on \( X \), then \( F \) is equal to one of the \( L \)-topologies defined in remark 6.1.

**Proof.** Let \(| \mathcal{R}_F | = m \). If \( m = 0 \), then clearly \( F = \{ 0, 1 \} \). So, assume that \( m > 0 \).

**Case 1:** If \( F \) contains only constant \( L \)-subsets, then \( F = \{ 0, 1, \alpha : \alpha \in \mathcal{R}_F \} \).

**Case 2:** If \( L^X_F \subseteq F \), then \( F = \{ 0, 1, g : g \in L^X_F \} \).

**Case 3:** Suppose Case 1 and Case 2 do not hold.

Let \( f \in F \) be any non-constant \( L \)-subset.

Let \( \Delta \subseteq \mathcal{R}_F \) be a subset such that

(i) \( \mathcal{R}_f \subseteq \Delta \).

(ii) \( \Delta^* \subseteq \Delta \).

(iii) for any two elements \( \alpha, \beta \in \Delta, \exists \text{ an } L \text{-subset } g \in F \text{ such that } g(x) = \alpha, g(y) = \beta \) for some \( x, y \in X \).

(iv) \( \Delta \) is not properly contained in any proper subset of \( \mathcal{R}_F \) satisfying above three properties.

In the same way, as in theorem 5.2, it can be shown that \( \Delta \) is a \( c \)-subset.
Case (i) : If $\mathcal{L}^X_\triangle \subset F$, then $f \in \mathcal{L}^X_\triangle$.

Case (ii) : Let $\mathcal{L}^X_\triangle \nsubseteq F$.

Let $\mathbb{D} = \{ h \in F : \mathcal{R}_h \subseteq \triangle \}$.

$\mathcal{L}^X_\triangle \nsubseteq F \Rightarrow \exists$ some element(s) $\lambda \in \triangle$ such that if $h \in \mathbb{D}$ and $h(x) = \lambda$ for some $x \in X$, then $h(y) = \lambda$ for at-most finitely/countably many $x \in X$.

Let $\mathbb{P}_1 = \{ \lambda \in \triangle : \lambda \in \mathcal{R}_h$ for some $h \in \mathbb{D}$, then $h(x) = \lambda$ for at-most finitely many $x \in X \}$. and $\mathbb{P}_2 = \{ \eta \in \triangle : \eta \in \mathcal{R}_g$ for some $g \in \mathbb{D}$, then $g(x) = \eta$ for at-most countably many $x \in X \}$. Clearly, $\mathbb{P}_1 \subseteq \mathbb{P}_2$.

It can be checked that

(i) if $\alpha, \beta \in \mathbb{P}_1(\mathbb{P}_2)$ and $\eta \in \triangle$ such that $\alpha < \eta < \beta$, then $\eta \in \mathbb{P}_1(\mathbb{P}_2)$.

(ii) if $\alpha \in \mathbb{P}_1$ and $\eta \in \triangle$ such that $\eta < \alpha$, then $\eta \in \mathbb{P}_1$.

Let $\mathbb{L}_\triangle = \begin{cases} g \in \mathcal{L}^X : & g(x) \in \mathbb{P}_1 \text{ for at-most finitely many } x \in X \\ g(x) \in \mathbb{P}_2 \setminus \mathbb{P}_1 \text{ for at-most countably many } x \in X \\ g(x) \in \triangle \setminus \{ \mathbb{P}_2 \} \text{ otherwise} \end{cases}$

Now two cases arise:

Case (a) : when $\alpha \notin F, \forall \alpha \in \mathbb{P}_2$.

Then $\mathbb{L}_\triangle \subseteq F$.

Case (b) : when $\alpha \in F$ for all / some $\alpha \in \mathbb{P}_2$.

Let $\mathbb{C} = \{ \alpha \in \mathbb{P}_2 : \alpha \in F \}$ and $\mathbb{L}_{\triangle,\mathbb{C}} = \mathbb{L}_\triangle \cup \mathbb{C}$.

Thus either $f \in \mathbb{L}_\triangle$ or $f \in \mathbb{L}_{\triangle,\mathbb{C}}$.

Let $\triangle_i, i \in \Omega$ be the collection of those distinct $c$-subsets of $\mathcal{R}_\tau$ such that corresponding to each $c$-subset $\triangle_i, i \in \Omega$, exactly one set from the set $\{ \mathcal{L}^X_{\triangle_i}, \mathbb{L}_{\triangle_i}, \mathbb{L}_{\triangle_i,\mathbb{C}} \}$ denoted by $\Sigma_{\triangle_i} \subset F, \forall i \in \Omega$ and for every non-constant $L$-subset $h \in F, h \in \Sigma_{\triangle_i}$ for-at-least one $i \in \Omega$.

Let $\mathbb{E} = \{ \beta \in \mathcal{R}_\tau : \beta \notin \bigcup_{i \in \Omega} \triangle_i \}$.

Thus $F = \{ 0, 1, \beta, f : \beta \in \mathbb{E} \text{ and } f \in \Sigma_{\triangle_i}, i \in \Omega \}$.

$\Rightarrow$ If $F$ is a completely homogeneous $L$-topology on an uncountable set $X$, then $F$ is equal to one of the $L$-topologies defined in the remark 6.1.
ACKNOWLEDGEMENT

The first author wishes to thank the University Grants Commission, India for giving financial support.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES