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TRACIAL TOPOLOGICAL RANK ZERO AND STABLE RANK ONE FOR CERTAIN TRACIAL APPROXIMATION C*-ALGEBRAS

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Abstract. We show that let $\mathscr{P}$ be a class of unital C*-algebras which have tracial topological rank zero (stable rank one). Then A has tracial topological rank zero (stable rank one) for any simple unital C*-algebra $A \in \mathrm{WTA}\mathscr{P}$.

Keywords: C*-algebras; stable rank one; SP-property.

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1. Introduction

Inspired by Lin's tracial approximation by interval algebras in [20], Elliott and Niu in [7] considered the natural notion of tracial approximation by other classes of C*-algebras. Let $\mathscr{P}$ be a class of unital C*-algebras. Then the class of C*-algebras which can be tracially approximated by C*-algebras in $\mathscr{P}$, denoted by $\mathrm{TA}\mathscr{P}$, is defined as follows. A simple unital C^*-algebra A is said to belong to the class $\mathrm{TA}\mathscr{P}$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any element $a \geq 0$, there is a projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ and $B \in \mathscr{P}$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$, and

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

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Let $\mathscr{P}$ be a class of finite dimensional C*-algebras. Then the class of C*-algebras which can be tracially approximated by C*-algebras in $\mathscr{P}$ is called tracial topological rank zero and denoted by $\mathrm{TR(A)} = 0$.

Hirshberg and Orovitz introduce the tracially $\mathscr{Z}$-absorbing in [18], they show that tracially $\mathscr{Z}$-absorbing is equivalence $\mathscr{Z}$-stability for separable simple amenable unital C*-algebra in [18].

Inspired by Hirshberg and Orovitz's tracial $\mathscr{Z}$-absorbing, Fu introduced some type finite tracial nuclear dimension in his doctoral dissertation in [13] and introduced certain tracial approximation C*-algebras in [14], and he show that finite tracial nuclear dimension implies tracially $\mathscr{Z}$-absorbing for separable, exact simple C*-algebra with non-empty tracial state space.

Inspired by Fu's finite tracial nuclear dimension and the general tracial topological rank one in [6], Fan and Yang introduced certain weak tracial approximation by a class of unital C*-algebras in [27]. Let $\mathscr{P}$ be a class of unital C*-algebras. Then the class of unital simple C*-algebras which can be weak tracial approximation by $\mathscr{P}$ is denote by WTA$\mathscr{P}$, A simple unital C*-algebra A is said to belong to the class WTA$\mathscr{P}$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a of A, there exist a unital C*-subalgebra B of A with $B \in \mathscr{P}$ and completely positive contractive linear maps $\varphi : A \to A$ and $\psi : A \to B$ with $\varphi(A) \perp B$, i.e., $\varphi(A)B = 0$, such that

(1) $\varphi(1) \precsim a$, and

(2) $\|x - \varphi(x) - \psi(x)\| < \varepsilon$, for any $x \in F$.

In this paper, let $\mathscr{P}$ be a class of unital C*-algebras which have stable rank one (tracial topological rank zero). Then A has stable rank one (tracial topological rank zero) for any simple unital C*-algebra $A \in$ WTA$\mathscr{P}$.

2. Preliminaries

Recall that a unital C*-algebra A is said to have stable rank one, written $tsr(A) = 1$, if the set of invertible elements is dense in A.

Recall that a C^*-algebra A has SP property, if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Let a and b be positive elements of a C^*-algebra A. We write $[a] \leq [b]$ if there is a partial isometry $v \in A^{**}$ with $vv^* = P_a$ such that, for every $0 \leq c \in \mathrm{Her}(a)$, $cv \in A$ and $v^*cv \in \mathrm{Her}(b)$. ($[a] \leq [b]$ implies that a is Cuntz subequivalent to b, i.e. $a \lesssim b$. If A has stable rank one then, by [2], $[a] \leq [b]$ if $a \lesssim b$ but even in this case the preorder relation $[a] \leq [b]$ is not necessarily an order relation.) We write $[a] = [b]$ if, for some v as above, $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \leq [b]$ if in addition there are n mutually orthogonal positive elements $b_1,\ b_2,\ \cdots,\ b_n \in \mathrm{Her}(b)$ such that $[a] \leq [b_i]$, $i = 1,\ 2,\ \cdots,\ n$ (see Definition 1.1 of [23], Definition 3.2 of [22], or Definition 3.5.2 of [21].)

Let $0 < \sigma \leq 1$ be two positive numbers. Define

$$f_\sigma(t) = \begin{cases} 1 & if\ t \geq \sigma \\ \frac{2t-\sigma}{\sigma} & if\ \sigma/2 \leq t \leq \sigma \\ 0 & if\ 0 < t \leq \sigma/2. \end{cases}$$

Let A be a C^*-algebra, and let $\mathrm{M}_n(A)$ denote the C^*-algebra of $n \times n$ matrices with entries elements of A. Let $\mathrm{M}_\infty(A)$ denote the algebraic inductive limit of the sequence $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is the canonical embedding as the upper left-hand corner block. Let $\mathrm{M}_\infty(A)_+$ (resp. $\mathrm{M}_n(A)_+$) denote the positive elements of $\mathrm{M}_\infty(A)$ (resp. $\mathrm{M_n}(A)$). For positive elements a and b of $\mathrm{M}_\infty(A)$, write $a \oplus b$ to denote the element $\mathrm{diag}(a,\ b)$, which is also positive of $\mathrm{M}_\infty(A)$. Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \lesssim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \lesssim b$ and $b \lesssim a$. We write $\langle a \rangle$ for the equivalence class of a.

Hirshberg and Orovitz introduce the tracially $\mathscr{Z}$-absorbing in [18], they show that tracially $\mathscr{Z}$-absorbing is equivalence $\mathscr{Z}$-stability for separable simple amenable unital C^*-algebra in [18].

Inspired by Hirshberg and Orovitz's tracial $\mathscr{Z}$-absorbing, some finite tracial nuclear dimensions were introduced by Fu in his doctoral dissertation in [13].

Definition 2.1. *([13].) A unital* C^**-algebra A is said to have type* III *tracial nuclear dimension at most m, denote* $\mathrm{T}^3\dim_{\mathrm{nuc}}(A) \leq m$*, if for any* $\varepsilon > 0$*, any finite subset* $F \subseteq A$*, any nonzero positive element a of A, there exist a unital* C^**-subalgebra B of A with* $\dim_{\mathrm{nuc}}(B) \leq m$ *and contractive completely positive linear maps* $\varphi : A \to A$ *and* $\psi : A \to B$ *with* $\varphi(A) \perp B$*, i.e.,* $\varphi(A)B = 0$*, such that*

(1) $\varphi(1) \lesssim a$*, and*

(2) $\|x - \varphi(x) - \psi(x)\| < \varepsilon$*, for any* $x \in F$*.*

Inspired by Fu's finite tracial nuclear dimension and the general tracial topological rank one in [6], Fan and Yang introduced certain weak tracial approximation by a class of unital C^*-algebras in [27].

Let $\mathscr{P}$ be a class of unital C^*-algebras. Then the class of unital C^*-algebras which can be weak tracial approximated by C^*-algebras in $\mathscr{P}$, denoted by $\mathrm{WTA}\mathscr{P}$, is defined as follows.

Definition 2.2. *([27].) A unital* C^**-algebra A is said to belong to the class* $\mathrm{WTA}\mathscr{P}$*, if for any* $\varepsilon > 0$*, any finite subset* $F \subseteq A$*, any nonzero positive element a of A, there exist a unital* C^**-subalgebra B of A with* $B \in \mathscr{P}$ *and completely positive contractive linear maps* $\varphi : A \to A$ *and* $\psi : A \to B$ *with* $\varphi(A) \perp B$*, i.e.,* $\varphi(A)B = 0$*, such that*

(1) $\varphi(1) \lesssim a$*, and*

(2) $\|x - \varphi(x) - \psi(x)\| < \varepsilon$*, for any* $x \in F$*.*

Let $\mathscr{P}$ be a class of unital C^*-algebras such that $\dim_{\mathrm{nuc}} \leq n$ for any $B \in \mathscr{P}$, then $A \in \mathrm{WTA}\mathscr{P}$ if and only if $\mathrm{T}^3\dim_{\mathrm{nuc}}(A) \leq n$.

Theorem 2.3. *([1], [18], [25], [26].) Let A be a stably finite* C^**-algebra.*

(1) *Let* $a,\ b \in A_+$ *and* $\varepsilon > 0$ *be such that* $\|a - b\| < \varepsilon$*. Then there is a contraction d in A with* $(a - \varepsilon)_+ = dbd^*$*.*

(2) *Let* $a,\ p$ *be positive elements in* $\mathrm{M}_\infty(A)$ *with p a projection. If* $p \lesssim a$*, then there is b in* $\mathrm{M}_\infty(A)_+$ *such that* $bp = 0$ *and* $b + p \sim a$*.*

(3) *The following conditions are equivalent:* $(1)'$ $a \lesssim b$*,* $(2)'$ *for any* $\varepsilon > 0$*,* $(a - \varepsilon)_+ \lesssim b$*, and* $(3)'$ *for any* $\varepsilon > 0$*, there is* $\delta > 0$*, such that* $(a - \varepsilon)_+ \lesssim (b - \delta)_+$*.*

(4) *Let a be a purely positive element of A (i.e., a is not Cuntz equivalent to a projection). Let* $\delta > 0$*, and let* $f \in C_0(0,1]$ *be a non-negative function with* $f = 0$ *on* $(\delta, 1)$*,* $f > 0$ *on* $(0, \delta)$*, and* $\|f\| = 1$*. We have* $f(a) \neq 0$ *and* $(a-\delta)_+ + f(a) \lesssim a$*.*

(5) *Let* $a, b \in A$ *satisfy* $0 \leq a \leq b$*. Let* $\varepsilon > 0$*, then* $(a-\varepsilon)_+ \lesssim (b-\varepsilon)_+$*.*

The following Theorem is lemma 3.3 in [16].

Theorem 2.4. *([6]) Let* $1 > \varepsilon > 0$ *and* $1 > \sigma > 0$ *be given. There exists* $\delta > 0$ *satisfying the following condition: If A is a* C^**-algebra, and if* $x, y \in A_+$ *are such that* $0 \leq x, y \leq 1$ *and*

$$\|x - y\| < \delta,$$

then there exists a partial isometry $w \in A^{**}$ *with* $ww^* f_\sigma(x) = f_\sigma(x) ww^* = f_\sigma(x)$*,* $w\mathrm{Her}(f_\sigma(x))w^* \subset \mathrm{Her}(y)$ *and*

$$w^* cw \in \overline{yAy}, \|w^* cw - c\| < \varepsilon \|c\|$$

for all $c \in \overline{f_\sigma(x) A f_\sigma(x)}$*.*

3. Main Results

The technique in the proof of the following Theorem is take from [13] or from [14].

Theorem 3.1. *If the class* $\mathscr{P}$ *is closed under tensoring with matrix algebras, or closed under passing to hereditary* C^**-subalgebras, then the class* $\mathrm{WTA}\mathscr{P}$ *is closed under tensoring with matrix algebras or passing to unital hereditary* C^**-subalgebras.*

Proof. (**I**) Write $B = qAq$ for some projection $q \in A$. We will prove that $B \in \mathrm{WTA}\mathscr{P}$.

Take $\varepsilon = \frac{\varepsilon}{16}, \sigma = (\frac{\varepsilon}{32})^2$, there exits $\delta_1 > 0$ which satisfy Theorem 2.4.

Take $\varepsilon = \frac{\delta_1}{4}, \sigma = (\frac{\delta_1}{4})^2$, there exits $\delta_2 > 0$ which satisfy Theorem 2.4.

For any finite subset $F \subseteq B$ contain a nonzero positive element $a \in B_+$, since $A \in \mathrm{WTA}\mathscr{P}$, for $G = F \cup \{q\}$, any $\delta_2 > 0$, there exist a unital C^*-subalgebra B of A with $B \in \mathscr{P}$ and completely positive contractive linear maps $\varphi' : A \to A$ and $\psi' : A \to C$ with $\varphi'(A) \perp C$, i.e., $\varphi'(A)C = 0$, such that

(1) $\varphi'(1) \lesssim a$, and

(2) $\|x-\varphi'(x)-\psi'(x)\|<\delta_2$, for any $x\in F$.

Let $q'=\varphi'(q)+\psi'(q)$, then $\|q-q'\|<\delta_2$.

By Theorem 2.4, there exists a partial isometry $w\in A^{**}$ with $ww^*f_{(\frac{\delta_1}{4})^2}(q')=f_{(\frac{\delta_1}{4})^2}(q')ww^*=f_{(\frac{\delta_1}{4})^2}(q')$, $w\mathrm{Her}_A(f_{(\frac{\delta_1}{4})^2}(q'))w^*\subset \mathrm{Her}_A(q')$ and

$$w^*cw\in\overline{q'Aq'},\|w^*cw-c\|<\frac{\delta_1}{4}\|c\|$$

for all $c\in\overline{f_{(\frac{\delta_1}{4})^2}(q')Af_{(\frac{\delta_1}{4})^2}(q')}$.

Since $\|\varphi'(q)-f_{(\frac{\delta_1}{4})^2}(q')\varphi'(q)f_{(\frac{\delta_1}{4})^2}(q')\|<\frac{\delta_1}{4}$, then $\|\varphi'(q)-wf_{(\frac{\delta_1}{4})^2}(q')\ \varphi'(q)f_{(\frac{\delta_1}{4})^2}(q')w^*\|<\frac{\delta_1}{2}$ (since $\|w^*cw-c\|<\frac{\delta_1}{4}\|c\|$ for all $c\in\overline{f_{(\frac{\delta_1}{4})^2}(q')Af_{(\frac{\delta_1}{4})^2}(q')}$).

Let $\overline{q}=wf_{(\frac{\delta_1}{4})^2}(q')\varphi'(q)f_{(\frac{\delta_1}{4})^2}(q')w^*$, then we have $\|\overline{q}-\varphi'(q)\|<\frac{\delta_1}{2}$.

By Theorem 2.4, there exists a partial isometry $v\in A^{**}$ with $vv^*f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))=f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))vv^*=f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))$, $v\mathrm{Her}_A(f_{(\frac{\varepsilon}{32})^2}(\varphi'(q)))v^*\subset \mathrm{Her}_A(\overline{q})$ and

$$v^*cv\in\overline{\varphi'(q)A\varphi'(q)},\|v^*cv-c\|<\frac{\varepsilon}{16}\|c\|$$

for all $c\in\overline{f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))Af_{(\frac{\varepsilon}{32})^2}(\varphi'(q))}$.

Since $\|\psi'(q)-f_{(\frac{\delta_1}{4})^2}(q')\psi'(q)f_{(\frac{\delta_1}{4})^2}(q')\|<\frac{\delta_1}{4}$, then $\|\psi'(q)-wf_{(\frac{\delta_1}{4})^2}(q')\ \psi'(q)f_{(\frac{\delta_1}{4})^2}(q')w^*\|<\frac{\delta_1}{2}$ (since $\|w^*cw-c\|<\frac{\delta_1}{4}\|c\|$ for all $c\in\overline{f_{(\frac{\delta_1}{4})^2}(q)Af_{(\frac{\delta_1}{4})^2}(q)}$).

Let $\overline{\overline{q}}=wf_{(\frac{\delta_1}{4})^2}(q')\psi'(q)f_{(\frac{\delta_1}{4})^2}(q')w^*$, then we have $\|\overline{\overline{q}}-\psi'(q)\|<\frac{\delta_1}{2}$.

By Theorem 2.4, there exists a partial isometry $u\in A^{**}$ with $uu^*f_{(\frac{\varepsilon}{32})^2}(\psi'(q))=f_{(\frac{\varepsilon}{32})^2}(\psi'(q))uu^*=f_{(\frac{\varepsilon}{32})^2}(\psi'(q))$, $u\mathrm{Her}_A(f_{(\frac{\varepsilon}{32})^2}(\psi'(q)))u^*\subset \mathrm{Her}_A(\overline{\overline{q}})$ and

$$u^*cu\in\overline{\overline{q}A\overline{q}},\|u^*cu-c\|<\frac{\varepsilon}{16}\|c\|$$

for all $c\in\overline{f_{(\frac{\varepsilon}{32})^2}(\psi'(q))Af_{(\frac{\varepsilon}{32})^2}(\psi'(q))}$.

Define $D=u\mathrm{Her}_C(f_{(\frac{\varepsilon}{32})^2}(q)\psi(q')\ f_{(\frac{\varepsilon}{32})^2}(q'))u^*\subset u\mathrm{Her}_A(f_{(\frac{\varepsilon}{32})^2}(q')\ \psi(q)f_{(\frac{\varepsilon}{32})^2}(q'))u^*\subset B$, we have $D\cong \mathrm{Her}_C((f_{(\frac{\varepsilon}{32})^2}(q')\psi(q)f_{(\frac{\varepsilon}{32})^2}(q'))$, then $D\in\Omega$.

We define $\varphi:A\to A$ by taking x to $vf_{(\frac{\varepsilon}{32})^2}(\varphi'(q))\varphi'(x)f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))v^*$ and $\psi:A\to D$ by taking x to $uf_{(\frac{\varepsilon}{32})^2}(\psi'(q))\psi'(x)f_{(\frac{\varepsilon}{32})^2}(\psi'(q))u^*$, then we have φ and ψ are completely positive contractive linear maps with $\varphi(A)\perp D$, i.e., $\varphi(A)D=0$.

We have (1)

$$
\begin{aligned}
&\|x-\varphi(x)-\psi(x)\| \\
&\le \|x-\varphi'(x)-\psi'(x)\|+\|\varphi(x)-\varphi'(x)\|+\|\psi(x)-\psi'(x)\| \\
&\le 3\varepsilon+\|vf_{(\frac{\varepsilon}{32})^2}(\varphi'(q))\varphi'(x)f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))v^*-f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))\varphi'(x)f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))\| \\
&+\|\varphi'(x)-f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))\varphi'(x)f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))\| \\
&\|vf_{(\frac{\varepsilon}{32})^2}(\psi'(q))\psi'(x)f_{(\frac{\varepsilon}{32})^2}(\psi'(q))v^*-f_{(\frac{\varepsilon}{32})^2}(\psi'(q))\psi'(x)f_{(\frac{\varepsilon}{32})^2}(\psi'(q))\| \\
&+\|\psi'(x)-vf_{(\frac{\varepsilon}{32})^2}(\psi'(q))\psi'(x)f_{(\frac{\varepsilon}{32})^2}(\psi'(q))v^*\|\le 7\varepsilon.
\end{aligned}
$$

(2) $\varphi(q)=vf_{\sigma}(\frac{\varepsilon}{32})^2(\varphi'(q))\varphi'(q)f_{(\frac{\varepsilon}{32})^2}(\varphi'(q))v^*\precsim\varphi'(q)\precsim b$ in A, since B is a hereditary C*-subalgebra of A, then we have $\varphi(q)\precsim b$ in B.

(**II**) For any finite subset $F\subseteq \mathrm{M}_n(A)$ contains a nonzero positive element $b\in \mathrm{M}_n(A)_+$, any $\varepsilon>0$, as the same argument as Theorem 3.7.3 in [21], there are mutually orthogonal and mutually equivalent projections $e_1,e_2,\cdots,e_n$ in $\mathrm{Her}(b)$ such that each of them is equivalent to a projection $e_0\in A$.

Take $G=\{a_{ij}:(a_{ij})_{n\times n}\in F\}$. For $\delta>0$, since $A\in \mathrm{WTA}\mathscr{P}$, there exist a unital C*-subalgebra B of A with $B\in\mathscr{P}$ and completely positive contractive linear maps $\varphi:A\to A$ and $\psi:A\to B$ with $\varphi(A)\perp B$, i.e., $\varphi(A)B=0$, such that

$(1)'$ $\varphi(1)\precsim e_0$, and

$(2)'$ $\|x-\varphi(x)-\psi(x)\|<\delta$, for any $x\in F$.

Define $\Phi:=\varphi\otimes id:A\otimes M_n\to A\otimes \mathrm{M}_n$ and $\Psi:\psi\otimes id:A\otimes \mathrm{M}_n\to B\otimes \mathrm{M}_n$, if we take δ sufficiently small, then, we have

(1) $\varphi(1_{A\otimes \mathrm{M}_n})=\sum 1\otimes e_{i,i}\precsim\sum e_0\otimes e_{i,i}\precsim b$, and

(2) $\|x-\varphi(x)-\psi(x)\|<\varepsilon$, for any $x\in F$.

□

Theorem 3.2. *Let $\mathscr{P}$ be a class of unital* C**-algebras which have tracial topological rank zero. Then A has tracial topological rank zero for any simple infinite dimensional unital* C**-algebra $A\in \mathrm{WTA}\mathscr{P}$.*

Proof. We need to show that for any $\varepsilon > 0$, any finite subset F of A, any nonzero positive element b of A, there exist a projection $p \in A$ and unital C*- subalgebra D of A and D is finite dimensional algebra with $1_D = p$ such that

(1) $\|px - xp\| < \varepsilon$ for any $x \in F$,

(2) $\|pxp \in_\varepsilon D$ for any $x \in F$, and

(3) $[1-p] \leq [b]$.

Since A is an infinite dimensional simple unital C*-algebra there exist non-zero positive elements $b_1,\ b_2 \in A_+$, such that $b_1 b_2 = 0$ and $b_1 + b_2 \lesssim b$.

Since $A \in$ WTA$\mathscr{P}$, for $\varepsilon > 0$, finite subset $F \cup \{1_A\}$ of A, non-zero positive element b_1 of A, there exist a unital C*- subalgebra B of A with $B \in$ WTA$\mathscr{P}$ and $1_D = q$, and completely positive contractive linear maps $\varphi' : A \to A$ and $\psi' : A \to B$ with $\varphi'(A) \perp B$, such that

$(1)'$ $\varphi'(1_A) \lesssim b_1$,

$(2)'$ $\|x - \varphi'(x) - \psi'(x)\| < \varepsilon$, for any $x \in F$ and

$(3)'$ $\|1_A - \varphi'(1_A) - \psi'(1_A)\| < \varepsilon$.

By $(2)'$, we have

$$\begin{aligned}\varepsilon &> \|x - \varphi'(x) - \psi'(x)\|\\ &\geq \|qxq - q\varphi'(x)q - q\psi'(x)q\|\\ &= \|qxq - \psi'(x)\|\end{aligned}$$

and

$$\begin{aligned}\varepsilon &> \|x - \varphi'(x) - \psi'(x)\|\\ &\geq \|(1-q)x(1-q) - (1-q)\varphi'(x)(1-q) - (1-q)\psi'(x)(1-q)\|\\ &= \|(1-q)x(1-q) - \varphi'(x)\|.\end{aligned}$$

Therefore, we have $\|x - qxq - (1-q)x(1-q)\| < \varepsilon$.

By Theorem 2.3 (1), and by $(3)'$, we have $((1_A - \psi'(1_A)) - \varepsilon)_+ \lesssim \varphi'(1_A)$, i.e., $1_A - q \lesssim \varphi'(1_A)$.

Since $1_A - q \leq 1 - \psi'(1_A)$, by Theorem 2.3 (5), we have $((1_A - q) - \varepsilon)_+ \lesssim ((1_A - \psi'(1_A)) - \varepsilon)_+ \lesssim \varphi'(1_A)$. So, $1_A - q \lesssim \varphi'(1_A)$ (since $1_A - q$ is a projection).

Since B has topological rank zero, for $G = \{\psi'(x), x \in F\}$, any $\varepsilon > 0$, there exist a projection $p \in A$ and unital C*- subalgebra D of A and D is finite dimensional algebra with $1_D = p$ such that

$(1)''$ $\|p\psi'(x)-\psi(x)'p\|<\varepsilon$ for any $x\in F$,

$(2)''$ $\|p\psi'(x)p\in_{\varepsilon} D$ for any $x\in F$, and

$(3)''$ $[q-p]\leq[b_2]$.

Therefore, we have

(1) $\|px-p(qxq-(1-q)x(1-q))\|<\varepsilon$, i.e., $\|px-pqxq\|<\varepsilon$, then we have $\|px-p\psi'(x)\|<2\varepsilon$, as the same argument we have $\|xp-\psi'(x)p\|<2\varepsilon$, by $(1)''$, we have $\|px-xp\|<4\varepsilon$.

(2) $\|pxp-p\psi'(x)p\|\leq\|pxp-p(qxq-(1-q)x(1-q))p\|+\|pqxqp-p\psi'(x)p\|<2\varepsilon$, then we have $\|pxp\in_{2\varepsilon} D$ for any $x\in F$, and

(3) $1-p=1-q+p-q\lesssim\varphi'(1_A)+p-q\lesssim b_1+b_2\lesssim b$, since $tsr(A)=1$, we have $[1-p]\leq[b]$. □

Theorem 3.3. *Let $\mathscr{P}$ be a class of unital* C*-*algebras which have stable rank one. Then A has stable rank one for any simple stably finite infinite dimensional unital* C*-*algebra $A\in\mathrm{WTA}\mathscr{P}$ and A has* SP *property.*

Proof. Let $x\in A$. For any $\varepsilon>0$, we will show that there exists an invertible element $y\in A$ such that $\|x-y\|<\varepsilon$.

Since A is stably finite, we may assume that x is not one-sided invertible. Since A is a simple unital and to show x is a norm limit of invertible in A, it suffice to show that ux is a norm limit of invertible elements (for some unitary $u\in A$), by Lemma 3.6.9 in [21], we may assume that there exist a nonzero positive element $cx=xc=0$.

Since A is simple infinite dimensional and has SP property, there exist nonzero projections $p_1,p_2\in\mathrm{Her(c)}$.

By Theorem 3.1, we have $(1-p_1)A(1-p_1)\in\mathrm{WTA}\mathscr{P}$.

For $F=\{(1-p_1)x(1-p_1),1-p_1\}$, and $\varepsilon>0$, since $(1-p_1)A(1-p_1)\in\mathrm{WTA}\mathscr{P}$, there there exist a unital C*- subalgebra D of $(1-p_1)A(1-p_1)$ with $D\in\mathscr{P}$, $1_D=q$ and completely positive contractive linear maps $\varphi:(1-p_1)A(1-p_1)\to(1-p_1)A(1-p_1)$ and $\psi:(1-p_1)A(1-p_1)\to D$ with $\varphi((1-p_1)A(1-p_1))\perp D$, i. e., $\varphi((1-p_1)A(1-p_1))D=0$, such that

(1) $\varphi(1-p_1)\lesssim p_1$,

(2) $\|(1-p_1)x(1-p_1)-\varphi((1-p_1)x(1-p_1))-\psi((1-p_1)x(1-p_1))\|<\varepsilon$, and

(3) $\|1-p_1-\varphi(1-p_1)-\psi(1-p_1)\|<\varepsilon$.

By Theorem 2.3 (1), and by (3), we have $((1-p_1-\psi(1-p_1))-\varepsilon)_+\lesssim\varphi(1-p_1)$, since $\psi(1-p_1))\leq q$, also by Theorem 2.3 (5), we have $((1-p_1-q)-\varepsilon)_+\lesssim((1-p_1-\psi(1-p_1))-\varepsilon)_+\lesssim\varphi(1-p_1)\lesssim p_1$. So we have $1-p_1-q\lesssim p_1$ (since $1-p_1-q$ is a projection).

By (2) and $\varphi((1-p_1)A(1-p_1))D=0$, we have

$$\begin{aligned}
&\varepsilon>\|(1-p_1)x(1-p_1)-\varphi((1-p_1)x(1-p_1))-\psi((1-p_1)x(1-p_1))\|\\
&\geq\|q(1-p_1)x(1-p_1)q-q\varphi((1-p_1)x(1-p_1))q-q\psi((1-p_1)x(1-p_1))q\|\\
&=\|q(1-p_1)x(1-p_1)q-q\psi((1-p_1)x(1-p_1))q\|\\
&=\|q(1-p_1)x(1-p_1)q-\psi((1-p_1)x(1-p_1))\|
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon>\|(1-p_1)x(1-p_1)-\varphi((1-p_1)x(1-p_1))-\psi((1-p_1)x(1-p_1))\|\\
&\geq\|(1-p_1-q)(1-p_1)x(1-p_1)(1-p_1-q)-\\
&(1-p_1-q)\varphi((1-p_1)x(1-p_1))(1-p_1-q)\\
&-(1-p_1-q)\psi((1-p_1)x(1-p_1))(1-p_1-q)\|\\
&=\|(1-p_1-q)(1-p_1)x(1-p_1)(1-p_1-q)\\
&-(1-p_1-q)\psi((1-p_1)x(1-p_1))(1-p_1-q)\|\\
&=\|(1-p_1-q)(1-p_1)x(1-p_1)(1-p_1-q)-\varphi((1-p_1)x(1-p_1))\|.
\end{aligned}$$

Therefore, we have $\|x-\psi((1-p_1)x(1-p_1))-\varphi((1-p_1)x(1-p_1))\|<2\varepsilon$, since $\|x-qxq-(1-p_1-q)x(1-p_1-q)\|<\varepsilon$.

Since $\psi((1-p_1)x(1-p_1))\in D$ and $tsr(D)=1$ there exist an invertible element $y_1\in D$ such that $\|\psi((1-p_1)x(1-p_1))-y_1\|<\varepsilon$.

Since $1-p_1-q\lesssim p_1$. Let $v\in A$ such that $v^*v=1-p_1-q$ and $vv^*\leq p_1$. Set $y_2=\varphi((1-p_1)x(1-p_1))+(\varepsilon/32)v+(\varepsilon/32)v^*+(\varepsilon/8)(p_1-vv^*)$, Then we have $\varphi((1-p_1)x(1-p_1))+(\varepsilon/32)v+(\varepsilon/32)v^*$ is invertible in $((1-p_1-q)+vv^*)A((1-p_1-q)+vv^*)$. So y_2 is invertible in $(1-p_1-q)A(1-p_1-q)$. Hence y_1+y_2 is invertible in A. Therefore, we have $\|x-y_1+y_2\|<4\varepsilon$. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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