TRACIAL TOPOLOGICAL RANK ZERO AND STABLE RANK ONE FOR CERTAIN TRACIAL APPROXIMATION C*-ALGEBRAS

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Abstract. We show that let \( \mathcal{P} \) be a class of unital C*-algebras which have tracial topological rank zero (stable rank one). Then \( A \) has tracial topological rank zero (stable rank one) for any simple unital C*-algebra \( A \in \text{WTA} \mathcal{P} \).

Keywords: C*-algebras; stable rank one; SP-property.

2010 AMS Subject Classification: 46L35, 46L05, 46L80.

1. Introduction

Inspired by Lin’s tracial approximation by interval algebras in [20], Elliott and Niu in [7] considered the natural notion of tracial approximation by other classes of C*-algebras. Let \( \mathcal{P} \) be a class of unital C*-algebras. Then the class of C*-algebras which can be tracially approximated by C*-algebras in \( \mathcal{P} \), denoted by \( \text{TA} \mathcal{P} \), is defined as follows. A simple unital C*-algebra \( A \) is said to belong to the class \( \text{TA} \mathcal{P} \) if, for any \( \epsilon > 0 \), any finite subset \( F \subseteq A \), and any element \( a \geq 0 \), there is a projection \( p \in A \) and a C*-subalgebra \( B \) of \( A \) with \( 1_B = p \) and \( B \in \mathcal{P} \) such that

\[
\begin{align*}
(1) & \quad \|xp - px\| < \epsilon \text{ for all } x \in F, \\
(2) & \quad pxp \in \epsilon B \text{ for all } x \in F, \text{ and } \\
(3) & \quad 1 - p \text{ is Murray-von Neumann equivalent to a projection in } \overline{aAa}.
\end{align*}
\]

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Received February 3, 2021

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Let $\mathcal{P}$ be a class of finite dimensional $C^*$-algebras. Then the class of $C^*$-algebras which can be tracially approximated by $C^*$-algebras in $\mathcal{P}$ is called tracial topological rank zero and denoted by $\text{TR}(A) = 0$.

Hirshberg and Orovitz introduce the tracially $\mathcal{Z}$-absorbing in [18], they show that tracially $\mathcal{Z}$-absorbing is equivalence $\mathcal{Z}$-stability for separable simple amenable unital $C^*$-algebra in [18].

Inspired by Hirshberg and Orovitz’s tracial $\mathcal{Z}$-absorbing, Fu introduced some type finite tracial nuclear dimension in his doctoral dissertation in [13] and introduced certain tracial approximation $C^*$-algebras in [14], and he show that finite tracial nuclear dimension implies tracially $\mathcal{Z}$-absorbing for separable, exact simple $C^*$-algebra with non-empty tracial state space.

Inspired by Fu’s finite tracial nuclear dimension and the general tracial topological rank one in [6], Fan and Yang introduced certain weak tracial approximation by a class of unital $C^*$-algebras in [27]. Let $\mathcal{P}$ be a class of unital $C^*$-algebras. Then the class of unital simple $C^*$-algebras which can be weak tracial approximation by $\mathcal{P}$ is denote by $\text{WTA}_{\mathcal{P}}$, A simple unital $C^*$-algebra $A$ is said to belong to the class $\text{WTA}_{\mathcal{P}}$ if, for any $\epsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element $a$ of $A$, there exist a unital $C^*$-subalgebra $B$ of $A$ with $B \in \mathcal{P}$ and completely positive contractive linear maps $\phi : A \to A$ and $\psi : A \to B$ with $\phi(A) \perp B$, i.e., $\phi(A)B = 0$, such that

1. $\phi(1) \preceq a$, and
2. $\|x - \phi(x) - \psi(x)\| < \epsilon$, for any $x \in F$.

In this paper, let $\mathcal{P}$ be a class of unital $C^*$-algebras which have stable rank one (tracial topological rank zero). Then $A$ has stable rank one (tracial topological rank zero) for any simple unital $C^*$-algebra $A \in \text{WTA}_{\mathcal{P}}$.

2. Preliminaries

Recall that a unital $C^*$-algebra $A$ is said to have stable rank one, written $\text{tsr}(A) = 1$, if the set of invertible elements is dense in $A$.

Recall that a $C^*$-algebra $A$ has SP property, if every nonzero hereditary $C^*$-subalgebra of $A$ contains a nonzero projection.
Let $a$ and $b$ be positive elements of a $C^*$-algebra $A$. We write $[a] \leq [b]$ if there is a partial isometry $v \in A^{**}$ with $vv^* = P_a$ such that, for every $0 \leq c \in \text{Her}(a)$, $cv \in A$ and $v^*cv \in \text{Her}(b)$. $([a] \leq [b]$ implies that $a$ is Cuntz subequivalent to $b$, i.e. $a \precsim b$. If $A$ has stable rank one then, by [2], $[a] \leq [b]$ if $a \precsim b$ but even in this case the preorder relation $[a] \leq [b]$ is not necessarily an order relation.) We write $[a] = [b]$ if, for some $v$ as above, $v^*\text{Her}(a)v = \text{Her}(b)$.

Let $n$ be a positive integer. We write $n[a] \leq [b]$ if in addition there are $n$ mutually orthogonal positive elements $b_1, b_2, \cdots, b_n \in \text{Her}(b)$ such that $[a] \leq [b_i], i = 1, 2, \cdots, n$ (see Definition 1.1 of [23], Definition 3.2 of [22], or Definition 3.5.2 of [21].)

Let $0 < \sigma \leq 1$ be two positive numbers. Define

$$f_\sigma(t) = \begin{cases} 
1 & \text{if } t \geq \sigma \\
\frac{2t-\sigma}{\sigma} & \text{if } \sigma/2 \leq t \leq \sigma \\
0 & \text{if } 0 < t \leq \sigma/2.
\end{cases}$$

Let $A$ be a $C^*$-algebra, and let $M_n(A)$ denote the $C^*$-algebra of $n \times n$ matrices with entries elements of $A$. Let $M_\infty(A)$ denote the algebraic inductive limit of the sequence $(M_n(A), \phi_n)$, where $\phi_n : M_n(A) \to M_{n+1}(A)$ is the canonical embedding as the upper left-hand corner block. Let $M_\infty(A)_+$ (resp. $M_n(A)_+$) denote the positive elements of $M_\infty(A)$ (resp. $M_n(A)$). For positive elements $a$ and $b$ of $M_\infty(A)$, write $a \oplus b$ to denote the element $\text{diag}(a, b)$, which is also positive of $M_\infty(A)$. Given $a, b \in M_\infty(A)_+$, we say that $a$ is Cuntz subequivalent to $b$ (written $a \precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $M_\infty(A)$ such that

$$\lim_{n \to \infty} \|v_nbv_n^* - a\| = 0.$$ 

We say that $a$ and $b$ are Cuntz equivalent (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. We write $\langle a \rangle$ for the equivalence class of $a$.

Hirshberg and Orovitz introduce the tracially $\mathcal{Z}$-absorbing in [18], they show that tracially $\mathcal{Z}$-absorbing is equivalence $\mathcal{Z}$-stability for separable simple amenable unital $C^*$-algebra in [18].

Inspired by Hirshberg and Orovitz’s tracial $\mathcal{Z}$-absorbing, some finite tracial nuclear dimensions were introduced by Fu in his doctoral dissertation in [13].
Definition 2.1. ([13].) A unital C*-algebra A is said to have type III tracial nuclear dimension at most m, denote $T^3 \dim_{\text{nuc}}(A) \leq m$, if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element $a$ of $A$, there exist a unital C*-subalgebra $B$ of $A$ with $\dim_{\text{nuc}}(B) \leq m$ and contractive completely positive linear maps $\varphi : A \to A$ and $\psi : A \to B$ with $\varphi(A) \perp B$, i.e., $\varphi(A)B = 0$, such that

1. $\varphi(1) \preccurlyeq a$, and
2. $\|x - \varphi(x) - \psi(x)\| < \varepsilon$, for any $x \in F$.

Inspired by Fu’s finite tracial nuclear dimension and the general tracial topological rank one in [6], Fan and Yang introduced certain weak tracial approximation by a class of unital C*-algebras in [27].

Let $\mathcal{P}$ be a class of unital C*-algebras. Then the class of unital C*-algebras which can be weak tracial approximated by C*-algebras in $\mathcal{P}$, denoted by $\text{WTA} \mathcal{P}$, is defined as follows.

Definition 2.2. ([27].) A unital C*-algebra A is said to belong to the class $\text{WTA} \mathcal{P}$, if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element $a$ of $A$, there exist a unital C*-subalgebra $B$ of $A$ with $B \in \mathcal{P}$ and completely positive contractive linear maps $\varphi : A \to A$ and $\psi : A \to B$ with $\varphi(A) \perp B$, i.e., $\varphi(A)B = 0$, such that

1. $\varphi(1) \preccurlyeq a$, and
2. $\|x - \varphi(x) - \psi(x)\| < \varepsilon$, for any $x \in F$.

Let $\mathcal{P}$ be a class of unital C*-algebras such that $\dim_{\text{nuc}} \leq n$ for any $B \in \mathcal{P}$, then $A \in \text{WTA} \mathcal{P}$ if and only if $T^3 \dim_{\text{nuc}}(A) \leq n$.

Theorem 2.3. ([1], [18], [25], [26].) Let $A$ be a stably finite C*-algebra.

1. Let $a, b \in A_+$ and $\varepsilon > 0$ be such that $\|a - b\| < \varepsilon$. Then there is a contraction $d$ in $A$ with $(a - \varepsilon)_+ = dbd^*$.

2. Let $a, p$ be positive elements in $M_\infty(A)$ with $p$ a projection. If $p \preccurlyeq a$, then there is $b$ in $M_\infty(A)_+$ such that $bp = 0$ and $b + p \sim a$.

3. The following conditions are equivalent: (1)' $a \preccurlyeq b$, (2)' for any $\varepsilon > 0$, $(a - \varepsilon)_+ \preccurlyeq b$, and (3)' for any $\varepsilon > 0$, there is $\delta > 0$, such that $(a - \varepsilon)_+ \preccurlyeq (b - \delta)_+$. 
(4) Let \(a\) be a purely positive element of \(A\) (i.e., \(a\) is not Cuntz equivalent to a projection). Let \(\delta > 0\), and let \(f \in C_0(0, 1)\) be a non-negative function with \(f = 0\) on \((\delta, 1)\), \(f > 0\) on \((0, \delta)\), and \(\|f\| = 1\). We have \(f(a) \neq 0\) and \((a - \delta)_+ + f(a) \lesssim a\).

(5) Let \(a, b \in A\) satisfy \(0 \leq a \leq b\). Let \(\varepsilon > 0\), then \((a - \varepsilon)_+ \lesssim (b - \varepsilon)_+\).

The following Theorem is lemma 3.3 in [16].

**Theorem 2.4.** ([6]) Let \(1 > \varepsilon > 0\) and \(1 > \sigma > 0\) be given. There exists \(\delta > 0\) satisfying the following condition: If \(A\) is a \(C^*\)-algebra, and if \(x, y \in A_+\) are such that \(0 \leq x, y \leq 1\) and

\[
\|x - y\| < \delta,
\]

then there exists a partial isometry \(w \in A^{**}\) with \(ww^*f_\sigma(x) = f_\sigma(x)ww^* = f_\sigma(x)\), \(w\text{Her}(f_\sigma(x))w^* \subset \text{Her}(y)\) and

\[
w^*cw \in \overline{yAy}, \|w^*cw - c\| < \varepsilon\|c\|
\]

for all \(c \in f_\sigma(x)Af_\sigma(x)\).

3. **Main Results**

The technique in the proof of the following Theorem is take from [13] or from [14].

**Theorem 3.1.** If the class \(\mathcal{P}\) is closed under tensoring with matrix algebras, or closed under passing to hereditary \(C^*\)-subalgebras, then the class \(\text{WTA}\mathcal{P}\) is closed under tensoring with matrix algebras or passing to unital hereditary \(C^*\)-subalgebras.

**Proof.** (I) Write \(B = qaq\) for some projection \(q \in A\). We will prove that \(B \in \text{WTA}\mathcal{P}\).

Take \(\varepsilon = \frac{\varepsilon}{16}, \sigma = (\frac{\varepsilon}{32})^2\), there exits \(\delta_1 > 0\) which satisfy Theorem 2.4.

Take \(\varepsilon = \frac{\delta_1}{4}, \sigma = (\frac{\delta_1}{4})^2\), there exits \(\delta_2 > 0\) which satisfy Theorem 2.4.

For any finite subset \(F \subseteq B\) contain a nonzero positive element \(a \in B_+\), since \(A \in \text{WTA}\mathcal{P}\), for \(G = F \cup \{q\}\), any \(\delta_2 > 0\), there exist a unital \(C^*\)-subalgebra \(B\) of \(A\) with \(B \in \mathcal{P}\) and completely positive contractive linear maps \(\varphi' : A \to A\) and \(\psi' : A \to C\) with \(\varphi'(A) \perp C\), i.e., \(\varphi'(A)C = 0\), such that

1. \(\varphi'(1) \lesssim a\), and
We define $\phi(q) = \psi(q) + \psi'(q)$, then $\|q - q'\| < \delta_2$.

By Theorem 2.4, there exists a partial isometry $w \in A^{**}$ with $ww^* f_{(\frac{\delta}{4})^2}(q') = f_{(\frac{\delta}{4})^2}(q')w w^* = f_{(\frac{\delta}{4})^2}(q')$, $w \text{Her}_A(f_{(\frac{\delta}{4})^2}(q')) w^* \subset \text{Her}_A(q')$ and

$$w^* cw \in \overline{q' A q'}, \|w^* cw - c\| < \frac{\delta_1}{4} \|c\|$$

for all $c \in f_{(\frac{\delta}{4})^2}(q')Af_{(\frac{\delta}{4})^2}(q')$.

Since $\|\phi(q) - f_{(\frac{\delta}{4})^2}(q')\psi(q)\| < \frac{\delta_1}{4}$, then $\|\phi(q) - w f_{(\frac{\delta}{4})^2}(q')\psi(q)\| < \frac{\delta_1}{4}$ (since $\|w^* cw - c\| < \frac{\delta_1}{4} \|c\|$ for all $c \in f_{(\frac{\delta}{4})^2}(q')Af_{(\frac{\delta}{4})^2}(q')$).

Let $\overline{q} = w f_{(\frac{\delta}{4})^2}(q')\psi(q)w^*$, then we have $\|\overline{q} - \phi(q)\| < \frac{\delta_1}{2}$.

By Theorem 2.4, there exists a partial isometry $v \in A^{**}$ with $vv^* f_{(\frac{\delta}{4})^2}(\phi(q)) = f_{(\frac{\delta}{4})^2}(\phi(q))vv^* = f_{(\frac{\delta}{4})^2}(\phi(q))$, $v \text{Her}_A(f_{(\frac{\delta}{4})^2}(\phi(q)))v^* \subset \text{Her}_A(q)$ and

$$v^* cv \in \overline{\phi(q)A\phi(q)}, \|v^* cv - c\| < \frac{\delta_1}{16} \|c\|$$

for all $c \in f_{(\frac{\delta}{4})^2}(\phi(q))Af_{(\frac{\delta}{4})^2}(\phi(q))$.

Since $\|\psi'(q) - f_{(\frac{\delta}{4})^2}(q')\psi(q)\| < \frac{\delta_1}{4}$, then $\|\psi'(q) - w f_{(\frac{\delta}{4})^2}(q')\psi'(q)w^*\| < \frac{\delta_1}{2}$ (since $\|w^* cw - c\| < \frac{\delta_1}{4} \|c\|$ for all $c \in f_{(\frac{\delta}{4})^2}(q')Af_{(\frac{\delta}{4})^2}(q')$).

Let $\overline{q} = w f_{(\frac{\delta}{4})^2}(q')\psi'(q)w^*$, then we have $\|\overline{q} - \psi'(q)\| < \frac{\delta_1}{2}$.

By Theorem 2.4, there exists a partial isometry $u \in A^{**}$ with $uu^* f_{(\frac{\delta}{4})^2}(\psi'(q)) = f_{(\frac{\delta}{4})^2}(\psi'(q))uu^* = f_{(\frac{\delta}{4})^2}(\psi'(q))$, $u \text{Her}_A(f_{(\frac{\delta}{4})^2}(\psi'(q)))u^* \subset \text{Her}_A(q)$ and

$$u^* cu \in \overline{qAq}, \|u^* cu - c\| < \frac{\delta_1}{16} \|c\|$$

for all $c \in f_{(\frac{\delta}{4})^2}(\psi'(q))Af_{(\frac{\delta}{4})^2}(\psi'(q))$.

Define $D = u \text{Her}_C(f_{(\frac{\delta}{4})^2}(\psi(q)) f_{(\frac{\delta}{4})^2}(\phi(q)))u^* \subset u \text{Her}_A(f_{(\frac{\delta}{4})^2}(\psi(q)) f_{(\frac{\delta}{4})^2}(\phi(q)))u^* \subset B$, we have $D \cong \text{Her}_C((f_{(\frac{\delta}{4})^2}(\psi(q))) f_{(\frac{\delta}{4})^2}(\phi(q)))$, then $D \in \Omega$.

We define $\phi : A \to A$ by taking $x$ to $vf_{(\frac{\delta}{4})^2}(\phi(q))\phi(q) f_{(\frac{\delta}{4})^2}(\phi(q))v^*$ and $\psi : A \to D$ by taking $x$ to $uf_{(\frac{\delta}{4})^2}(\psi(q))\psi(q) f_{(\frac{\delta}{4})^2}(\psi(q))u^*$, then we have $\phi$ and $\psi$ are completely positive contractive linear maps with $\phi(A) \perp D$, i.e., $\phi(A)D = 0$. 

(2) $\|x - \phi'(x) - \psi'(x)\| < \delta_2$, for any $x \in F$. 

Let $q' = \phi'(q) + \psi'(q)$, then $\|q - q'\| < \delta_2$. 

By Theorem 2.4, there exists a partial isometry $w \in A^{**}$ with $ww^* f_{(\frac{\delta}{4})^2}(q') = f_{(\frac{\delta}{4})^2}(q')w w^* = f_{(\frac{\delta}{4})^2}(q')$, $w \text{Her}_A(f_{(\frac{\delta}{4})^2}(q')) w^* \subset \text{Her}_A(q')$ and

$$w^* cw \in \overline{q' A q'}, \|w^* cw - c\| < \frac{\delta_1}{4} \|c\|$$

for all $c \in f_{(\frac{\delta}{4})^2}(q')Af_{(\frac{\delta}{4})^2}(q')$. 

Since $\|\phi(q) - f_{(\frac{\delta}{4})^2}(q')\phi(q)\| < \frac{\delta_1}{4}$, then $\|\phi(q) - w f_{(\frac{\delta}{4})^2}(q')\phi(q)\| < \frac{\delta_1}{4}$ (since $\|w^* cw - c\| < \frac{\delta_1}{4} \|c\|$ for all $c \in f_{(\frac{\delta}{4})^2}(q')Af_{(\frac{\delta}{4})^2}(q')$).

Let $\overline{q} = w f_{(\frac{\delta}{4})^2}(q')\phi(q)w^*$, then we have $\|\overline{q} - \phi(q)\| < \frac{\delta_1}{2}$.

By Theorem 2.4, there exists a partial isometry $v \in A^{**}$ with $vv^* f_{(\frac{\delta}{4})^2}(\phi(q)) = f_{(\frac{\delta}{4})^2}(\phi(q))vv^* = f_{(\frac{\delta}{4})^2}(\phi(q))$, $v \text{Her}_A(f_{(\frac{\delta}{4})^2}(\phi(q)))v^* \subset \text{Her}_A(q)$ and

$$v^* cv \in \overline{\phi(q)A\phi(q)}, \|v^* cv - c\| < \frac{\delta_1}{16} \|c\|$$

for all $c \in f_{(\frac{\delta}{4})^2}(\phi(q))Af_{(\frac{\delta}{4})^2}(\phi(q))$.

Define $D = u \text{Her}_C(f_{(\frac{\delta}{4})^2}(\psi(q)) f_{(\frac{\delta}{4})^2}(\phi(q)))u^* \subset u \text{Her}_A(f_{(\frac{\delta}{4})^2}(\psi(q)) f_{(\frac{\delta}{4})^2}(\phi(q)))u^* \subset B$, we have $D \cong \text{Her}_C((f_{(\frac{\delta}{4})^2}(\psi(q))) f_{(\frac{\delta}{4})^2}(\phi(q)))$, then $D \in \Omega$.

We define $\phi : A \to A$ by taking $x$ to $vf_{(\frac{\delta}{4})^2}(\phi(q))\phi(q) f_{(\frac{\delta}{4})^2}(\phi(q))v^*$ and $\psi : A \to D$ by taking $x$ to $uf_{(\frac{\delta}{4})^2}(\psi(q))\psi(q) f_{(\frac{\delta}{4})^2}(\psi(q))u^*$, then we have $\phi$ and $\psi$ are completely positive contractive linear maps with $\phi(A) \perp D$, i.e., $\phi(A)D = 0$. 


We have (1)

\[ \|x - \varphi(x) - \psi(x)\| \]

\[ \leq \|x - \varphi'(x) - \psi'(x)\| + \|\varphi(x) - \varphi'(x)\| + \|\psi(x) - \psi'(x)\| \]

\[ \leq 3\varepsilon + \|vf_1(\frac{\varepsilon}{3^2})\varphi'(q)\varphi'(x)f_1(\frac{\varepsilon}{3^2})\varphi'(q)v^* - f_1(\frac{\varepsilon}{3^2})\varphi'(q)\varphi'(x)f_1(\frac{\varepsilon}{3^2})\varphi'(q)\| \]

\[ + \|\varphi'(x) - f_1(\frac{\varepsilon}{3^2})\varphi'(q)\varphi'(x)f_1(\frac{\varepsilon}{3^2})\varphi'(q)\| \]

\[ \leq 7\varepsilon. \]

(2) \( \varphi(q) = vf_\sigma(\frac{\varepsilon}{3^2})\varphi'(q)\varphi'(x)f_1(\frac{\varepsilon}{3^2})\varphi'(q)v^* \lesssim \varphi'(q) \lesssim b \) in \( A \), since \( B \) is a hereditary \( C^* \)-subalgebra of \( A \), then we have \( \varphi(q) \lesssim b \) in \( B \).

\textbf{(II)} For any finite subset \( F \subseteq M_n(A) \) contains a nonzero positive element \( b \in M_n(A)_+ \), any \( \varepsilon > 0 \), as the same argument as Theorem 3.7.3 in [21], there are mutually orthogonal and mutually equivalent projections \( e_1, e_2, \ldots, e_n \) in \( \text{Her}(b) \) such that each of them is equivalent to a projection \( e_0 \in A \).

Take \( G = \{ a_{ij} : (a_{ij})_{n \times n} \in F \} \). For \( \delta > 0 \), since \( A \in \text{WTA}\mathcal{P} \), there exist a unital \( C^* \)-subalgebra \( B \) of \( A \) with \( B \in \mathcal{P} \) and completely positive contractive linear maps \( \varphi : A \to A \) and \( \psi : A \to B \) with \( \varphi(A) \perp B \), i.e., \( \varphi(A)B = 0 \), such that

(1)' \( \varphi(1) \lesssim e_0 \), and

(2)' \( \|x - \varphi(x) - \psi(x)\| < \delta \), for any \( x \in F \).

Define \( \Phi := \varphi \otimes id : A \otimes M_n \to A \otimes M_n \) and \( \Psi : \psi \otimes id : A \otimes M_n \to B \otimes M_n \), if we take \( \delta \) sufficiently small, then, we have

(1) \( \varphi(1_{A \otimes M_n}) = \sum 1 \otimes e_{i,i} \lesssim \sum e_0 \otimes e_{i,i} \lesssim b \), and

(2) \( \|x - \varphi(x) - \psi(x)\| < \varepsilon \), for any \( x \in F \).

\[ \square \]

\textbf{Theorem 3.2.} Let \( \mathcal{P} \) be a class of unital \( C^* \)-algebras which have tracial topological rank zero. Then \( A \) has tracial topological rank zero for any simple infinite dimensional unital \( C^* \)-algebra \( A \in \text{WTA}\mathcal{P} \).
Proof. We need to show that for any $\varepsilon > 0$, any finite subset $F$ of $A$, any nonzero positive element $b$ of $A$, there exist a projection $p \in A$ and unital $C^*$-subalgebra $D$ of $A$ and $D$ is finite dimensional algebra with $1_D = p$ such that

1. $\|px - xp\| < \varepsilon$ for any $x \in F$,
2. $\|pxp\| \leq \varepsilon D$ for any $x \in F$, and
3. $[1 - p] \leq [b]$.

Since $A$ is an infinite dimensional simple unital $C^*$-algebra there exist non-zero positive elements $b_1, b_2 \in A_+$, such that $b_1b_2 = 0$ and $b_1 + b_2 \leq b$.

Since $A \in WTA\mathcal{P}$, for $\varepsilon > 0$, finite subset $F \cup \{1_A\}$ of $A$, non-zero positive element $b_1$ of $A$, there exist a unital $C^*$-subalgebra $B$ of $A$ with $B \in WTA\mathcal{P}$ and $1_B = q$, and completely positive contractive linear maps $\varphi' : A \to A$ and $\psi' : A \to B$ with $\varphi'(A) \perp B$, such that

1. $\varphi'(1_A) \leq b_1$,
2. $\|x - \varphi'(x) - \psi'(x)\| < \varepsilon$, for any $x \in F$ and
3. $\|1_A - \varphi'(1_A) - \psi'(1_A)\| < \varepsilon$.

By (2)', we have

$$\varepsilon > \|x - \varphi'(x) - \psi'(x)\|$$
$$\geq \|qxq - q\varphi'(x)q - q\psi'(x)q\|$$
$$= \|qxq - \psi'(x)\|$$

and

$$\varepsilon > \|x - \varphi'(x) - \psi'(x)\|$$
$$\geq \|(1 - q)x(1 - q) - (1 - q)\varphi'(x)(1 - q) - (1 - q)\psi'(x)(1 - q)\|$$
$$= \|(1 - q)x(1 - q) - \varphi'(x)\|.$$

Therefore, we have $\|x - qxq - (1 - q)x(1 - q)\| < \varepsilon$.

By Theorem 2.3 (1), and by (3)', we have $((1_A - \psi'(1_A)) - \varepsilon)_+ \lesssim \varphi'(1_A)$, i.e., $1_A - q \lesssim \varphi'(1_A)$.

Since $1_A - q \leq 1 - \varphi'(1_A)$, by Theorem 2.3 (5), we have $((1_A - q) - \varepsilon)_+ \lesssim ((1_A - \varphi'(1_A)) - \varepsilon)_+ \lesssim \varphi'(1_A)$. So, $1_A - q \lesssim \varphi'(1_A)$ (since $1_A - q$ is a projection).

Since $B$ has topological rank zero, for $G = \{\psi'(x), x \in F\}$, any $\varepsilon > 0$, there exist a projection $p \in A$ and unital $C^*$-subalgebra $D$ of $A$ and $D$ is finite dimensional algebra with $1_D = p$ such that
\[(1)'' \| p\psi'(x) - \psi(x)'p \| < \varepsilon \text{ for any } x \in F, \]
\[(2)'' \| p\psi'(x)p \| \in \varepsilon D \text{ for any } x \in F, \text{ and} \]
\[(3)'' \| q - p \| \leq [b_2]. \]

Therefore, we have
\[(1) \| px - p(\text{for any simple stably finite infinite dimensional unital } C^*\text{-algebras which have stable rank one. Then } A \text{ has stable rank one for any simple stably finite infinite dimensional unital } C^*\text{-algebra } A \in \text{WTA} \mathcal{P} \text{ and } A \text{ has SP property}. \]

\[\text{Theorem 3.3. Let } \mathcal{P} \text{ be a class of unital } C^*\text{-algebras which have stable rank one. Then } A \text{ has stable rank one for any simple stably finite infinite dimensional unital } C^*\text{-algebra } A \in \text{WTA} \mathcal{P} \text{ and } A \text{ has SP property.} \]

\[\text{Proof. Let } x \in A. \text{ For any } \varepsilon > 0, \text{ we will show that there exists an invertible element } y \in A \text{ such that } \|x - y\| < \varepsilon. \]

\[\text{Since } A \text{ is stably finite, we may assume that } x \text{ is not one-sided invertible. Since } A \text{ is a simple unital and to show } x \text{ is a norm limit of invertible in } A, \text{ it suffice to show that } ux \text{ is a norm limit of invertible elements (for some unitary } u \in A), \text{ by Lemma 3.6.9 in [21], we may assume that there exist a nonzero positive element } cx = xc = 0. \]

\[\text{Since } A \text{ is simple infinite dimensional and has SP property, there exist nonzero projections } p_1, p_2 \in \text{Her}(c). \]

By Theorem 3.1, we have \((1 - p_1)A(1 - p_1) \in \text{WTA} \mathcal{P}. \]

For \(F = \{(1 - p_1)x(1 - p_1), 1 - p_1\}, \text{ and } \varepsilon > 0, \text{ since } (1 - p_1)A(1 - p_1) \in \text{WTA} \mathcal{P}, \text{ there exist a unital } C^*\text{-subalgebra } D \text{ of } (1 - p_1)A(1 - p_1) \text{ with } D \in \mathcal{P}, 1_D = q \text{ and completely positive contractive linear maps } \psi : (1 - p_1)A(1 - p_1) \to (1 - p_1)A(1 - p_1) \text{ and } \psi : (1 - p_1)A(1 - p_1) \to D \text{ with } \| \psi((1 - p_1)A(1 - p_1))D = 0, \text{ such that} \]
\[(1) \psi(1 - p_1) \lesssim p_1, \]
\[(2) \| (1 - p_1)x(1 - p_1) - \varphi((1 - p_1)x(1 - p_1)) - \psi((1 - p_1)x(1 - p_1)) \| < \varepsilon, \text{ and} \]
(3) \( \|1 - p_1 - \varphi(1 - p_1) - \psi(1 - p_1)\| < \varepsilon. \)

By Theorem 2.3 (1), and by (3), we have \( ((1 - p_1 - \psi(1 - p_1)) - \varepsilon)_+ \lesssim \varphi(1 - p_1), \) since \( \psi(1 - p_1) \leq q, \) also by Theorem 2.3 (5), we have \( ((1 - p_1 - q) - \varepsilon)_+ \lesssim ((1 - p_1 - \psi(1 - p_1)) - \varepsilon)_+ \lesssim \varphi(1 - p_1) \lesssim p_1. \) So we have \( 1 - p_1 - q \lesssim p_1 \) (since \( 1 - p_1 - q \) is a projection).

By (2) and \( \varphi((1 - p_1)A(1 - p_1))D = 0, \) we have

\[
\varepsilon > \|(1 - p_1)x(1 - p_1) - \varphi((1 - p_1)x(1 - p_1)) - \psi((1 - p_1)x(1 - p_1))\| \\
\geq \|(1 - p_1 - q)(1 - p_1)x(1 - p_1)(1 - p_1 - q) - (1 - p_1 - q)\varphi((1 - p_1)x(1 - p_1))(1 - p_1 - q) - (1 - p_1 - q)\psi((1 - p_1)x(1 - p_1))(1 - p_1 - q)\| \\
= \|(1 - p_1 - q)(1 - p_1)x(1 - p_1)(1 - p_1 - q) - (1 - p_1 - q)\psi((1 - p_1)x(1 - p_1))(1 - p_1 - q)\| \\
= \|(1 - p_1 - q)(1 - p_1)x(1 - p_1)(1 - p_1 - q) - \varphi((1 - p_1)x(1 - p_1))\|. 
\]

Therefore, we have \( \|x - \psi((1 - p_1)x(1 - p_1)) - \varphi((1 - p_1)x(1 - p_1))\| < 2\varepsilon, \) since \( \|x - qxq - (1 - p_1 - q)x(1 - p_1 - q)\| < \varepsilon. \)

Since \( \psi((1 - p_1)x(1 - p_1)) \in D \) and \( tsr(D) = 1 \) there exist an invertible element \( y_1 \in D \) such that \( \|\psi((1 - p_1)x(1 - p_1)) - y_1\| < \varepsilon. \)

Since \( 1 - p_1 - q \lesssim p_1. \) Let \( v \in A \) such that \( v^*v = 1 - p_1 - q \) and \( vv^* \leq p_1. \) Set \( y_2 = \varphi((1 - p_1)x(1 - p_1)) + (\varepsilon/32)v + (\varepsilon/32)v^* + (\varepsilon/8)(p_1 - vv^*). \) Then we have \( \varphi((1 - p_1)x(1 - p_1)) + (\varepsilon/32)v + (\varepsilon/32)v^* \) is invertible in \( ((1 - p_1 - q) + vv^*)A((1 - p_1 - q) + vv^*). \) So \( y_2 \) is invertible in \( (1 - p_1 - q)A(1 - p_1 - q). \) Hence \( y_1 + y_2 \) is invertible in \( A. \) Therefore, we have \( \|x - y_1 + y_2\| < 4\varepsilon. \) \( \square \)
FUNDING

This research is supported by National Natural Science Foundation of China (No.11501357).

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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