ON CONTRA $\Lambda^*_I$-CONTINUOUS FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In this work, we introduce and study the classes of contra $\Lambda^*_I$-continuous, contra quasi-$\Lambda^*_I$-continuous and contra $\Lambda^*_I$-irresolute functions in a topological space endowed with an ideal. We investigate the relationships among these functions and their respective characterizations. Also, we analyze the behavior of certain topological notions under direct and inverse images of these new classes of functions.

Keywords: ideal; semi-$I$-open set; $\Lambda^*_I$-closed set; contra $\Lambda^*_I$-irresolute function.

2020 AMS Subject Classification: 54C08, 54C05, 54D10, 54D15.

1. INTRODUCTION

In 1933, Kuratowski [5] introduced the notion of local function of a set as a generalization of the closure of a set. In this sense, given a topological space $(X, \tau)$ and an ideal $I$ on $X$, the local function of a subset $A$ of $X$ with respect to $I$ and $\tau$, is the set $A^* = \{x \in X : U \cap A \notin I \forall U \in \tau\}$...
for each $U \in \tau$ such that $x \in U$. In 1990, Jankovic and Hamlett [4], used this generalization in order to define the Kuratowski operator $Cl^*$, which induces a topology $\tau^*$ finer than $\tau$. Using the operator $Cl^*$, Hatir and Noiri [3], in 2002, defined the concept of semi-$I$-open set and used it to establish some decompositions of generalized continuity. Later, in 2013, Sanabria et al. [6] used the class of semi-$I$-open sets to define and study the notions of $\Lambda^I_s$-sets and $\Lambda^I_s$-closed sets, in particular, through these notions they characterized the semi-$I$-$T_1$-spaces and the semi-$I$-$T_{1/2}$-spaces. Quite recently Sanabria et al. [7] used the class of $\Lambda^I_s$-closed sets to define and characterize new variants of continuity called $\Lambda^I_s$-continuous, quasi-$\Lambda^I_s$-continuous and $\Lambda^I_s$-irresolute functions. On the other hand, in 1996, Dontchev [1] introduced the notion of contra-continuous function in topological spaces and, established interesting results that related contra-continuity with compact spaces, $S$-closed spaces and strongly $S$-closed spaces. In this work, we introduce and characterize new variants of contra-continuous functions defined in terms of $\Lambda^I_s$-closed sets, in contrast with the variants of continuity due to Sanabria et al. [7]. Specifically, are studied the concepts of contra $\Lambda^I_s$-continuous, contra $\Lambda^I_s$-irresolute and contra quasi-$\Lambda^I_s$-continuous functions. The preservation of certain modifications of connectedness, compactness and separations are investigated, through direct and inverse images of these functions.

2. Preliminaries

Throughout this paper, $Int(A)$ and $Cl(A)$ denote the interior and the closure of a subset $A$ of a topological space $(X, \tau)$, respectively. Also $P(X)$ denote the power set of $X$. An ideal $I$ on $X$ is a nonempty subfamily of $P(X)$ which satisfies the following conditions:

(1) If $A \subset B$ and $B \in I$, then $A \in I$.

(2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Some well-known types of ideals on a topological space $(X, \tau)$ are the family of all nowhere dense subsets of $X$, the family of all closed and discrete subsets of $X$ and the family of all meager subsets of $X$.

In the sequel, $(X, \tau, I)$ (or simply $X$) denote a topological space $(X, \tau)$ with an ideal $I$ on $X$, which is simply called an ideal space. Given an ideal space $(X, \tau, I)$ and a subset $A$ of $X$, the local function of $A$ with respect to $I$ and $\tau$ is defined as $A^*(I, \tau) = \{x \in X : U \cap A \not\in I\}$.
I for each \( U \in \tau(x) \}, \) where \( \tau(x) = \{ U \in \tau : x \in U \}. \) When there is no chance for confusion, we will simply write \( A^* \) instead of \( A^*(I, \tau) \). It is a well known fact that \( Cl^*(A) = A \cup A^* \) is a Kuratowski closure operator and hence, \( \tau^*(I, \tau) = \{ U \subset X : Cl^*(X - U) = X - U \} \) is a topology on \( X \), which is finer than \( \tau \). If there is no chance to confusion, we will write \( \tau^* \) instead of \( \tau^*(I, \tau) \). The members of \( \tau^* \) are called \( \tau^*-open \) sets and their complements are called \( \tau^*-closed \) sets. It is easy to see that a subset \( A \) of \( X \) is \( \tau^*-closed \) if and only if \( A^* \subset A \).

According to [3], \( A \subset X \) is \( semi-I-open \) if \( A \subset Cl^*(Int(A)) \). The family of all semi-I-open sets of \( X \) is denoted by \( \text{SIO}(X, \tau) \) and the complement of a semi-I-open set is called \( semi-I-closed \) set. As in [6], we say that \( A \subset X \) is a \( \Lambda^*_I \)-set if \( A = \Lambda^*_I(A) \), where \( \Lambda^*_I(A) \) is the intersection of all semi-I-open subsets of \( X \) containing \( A \). According also to [6], we say that \( A \subset X \) is \( \Lambda^*_I \)-closed if \( A = \Lambda \cap F \), where \( \Lambda \) is a \( \Lambda^*_I \)-set and \( F \) is a \( \tau^*-closed \) set. In [6], the following implications were established:

\[
\text{open} \quad \implies \quad \text{semi-I-open} \quad \implies \quad \Lambda^*_I \text{-set} \\
\text{closed} \quad \implies \quad \tau^*-closed \quad \implies \quad \Lambda^*_I \text{-closed}
\]

The complement of a \( \Lambda^*_I \)-closed set is called \( \Lambda^*_I \)-open set. According to the previous implications, we have the following results.

**Lemma 2.1.** [7, Lemma 2.2] Every \( \tau^*-open \) set is \( \Lambda^*_I \)-open.

**Lemma 2.2.** [7, Lemma 2.3] Let \( \{ A_\alpha : \alpha \in \Delta \} \) be a collection of subsets of a space \((X, \tau, I)\). If \( A_\alpha \) is a \( \Lambda^*_I \)-open set for each \( \alpha \in \Delta \), then \( \bigcup \{ A_\alpha : \alpha \in \Delta \} \) is a \( \Lambda^*_I \)-open set.

### 3. Contra Continuity in Terms of \( \Lambda^*_I \)-Closed Sets

In this section, we introduce and study certain types of contra continuity in terms of \( \Lambda^*_I \)-closed sets. We establish some relationships and characterizations for these new classes of functions. Next we consider a function \( f \) defined from an ideal space \((X, \tau, I)\) to an ideal space \((Y, \sigma, J)\). First, we present the definitions and characterizations of some variants of continuity due to Sanabria et al. [7].

**Definition 3.1.** [7] A function \( f : (X, \tau, I) \longrightarrow (Y, \sigma, J) \) is called:

1. \( \Lambda^*_I \)-continuous, if \( f^{-1}(V) \) is a \( \Lambda^*_I \)-open subset of \( X \) for each open subset \( V \) of \( Y \).
(2) **Quasi-Λ^i-continuous**, if \( f^{-1}(V) \) is a \( \Lambda^i_\sigma \)-open subset of \( X \) for each \( \sigma^* \)-open set \( V \) of \( Y \).

(3) **\( \Lambda^i_\sigma \)-irresolute**, if \( f^{-1}(V) \) is a \( \Lambda^i_\sigma \)-open set of \( X \) for each \( \Lambda^i_\sigma \)-open subset \( V \) of \( Y \).

**Theorem 3.1.** [7] For a function \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \), the following statements are equivalent:

1. \( f \) is \( \Lambda^i_\sigma \)-continuous (resp. quasi-\( \Lambda^i_\sigma \)-continuous, \( \Lambda^i_\sigma \)-irresolute).
2. \( f^{-1}(B) \) is a \( \Lambda^i_\sigma \)-closed subset of \( X \) for each closed (resp. \( \sigma^* \)-closed, \( \Lambda^i_\sigma \)-closed) subset \( B \) of \( Y \).
3. For each \( x \in X \) and each open (resp. \( \sigma^* \)-open, \( \Lambda^i_\sigma \)-open) subset \( V \) of \( Y \) containing \( f(x) \) there exists a \( \Lambda^i_\sigma \)-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subset V \).

Now, we introduce a new variants of contra-continuity and study the relationships among these classes of functions and also obtain properties and characterizations of them.

**Definition 3.2.** A function \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) is called:

1. **Contra \( \Lambda^i_\sigma \)-continuous**, if \( f^{-1}(V) \) is a \( \Lambda^i_\sigma \)-closed subset of \( X \) for each open set \( V \) of \( Y \).
2. **Contra quasi-\( \Lambda^i_\sigma \)-continuous**, if \( f^{-1}(V) \) is a \( \Lambda^i_\sigma \)-closed subset of \( X \) for \( \sigma^* \)-open subset \( V \) of \( Y \).
3. **Contra \( \Lambda^i_\sigma \)-irresolute**, if \( f^{-1}(V) \) is a \( \Lambda^i_\sigma \)-closed subset of \( X \) for each \( \Lambda^i_\sigma \)-open subset \( V \) of \( Y \).

**Theorem 3.2.** Let \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) be a function. The following statements hold:

1. If \( f \) is contra \( \Lambda^i_\sigma \)-irresolute, then it is contra quasi-\( \Lambda^i_\sigma \)-continuous.
2. If \( f \) is contra quasi-\( \Lambda^i_\sigma \)-continuous, then it is contra \( \Lambda^i_\sigma \)-continuous.
3. If \( f \) is contra \( \Lambda^i_\sigma \)-irresolute, then it is contra \( \Lambda^i_\sigma \)-continuous.

**Proof.** (1) Let \( V \) be any \( \sigma^* \)-open subset of \( Y \). By Lemma 2.1, we have \( V \) is a \( \Lambda^i_\sigma \)-open subset of \( Y \) and since \( f \) is contra \( \Lambda^i_\sigma \)-irresolute, it follows that \( f^{-1}(V) \) is a \( \Lambda^i_\sigma \)-closed subset of \( X \). Therefore, \( f \) is contra quasi-\( \Lambda^i_\sigma \)-continuous.

The proof of (2) is similar to case (1) and, (3) is an immediate consequence of (1) and (2).  

In the following two examples, we show that the converses of (1) and (2) in Theorem 3.2, in general are not true.
Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, \{c\}, X\}$, $I = \{\emptyset, \{b\}, \{a, b\}\}$ and $J = \{\emptyset, \{a\}\}$. Then, the family of all $\sigma^*$-open subsets of $X$ is $\{\emptyset, X, \{a\}, \{b, c\}\}$, the family of all $\Lambda^*_j$-open subsets of $X$ is $\{\emptyset, X, \{c\}, \{b, c\}, \{a, c\}\}$, the family of all $\Lambda^*_j$-closed subsets of $X$ is $\{\emptyset, X, \{a\}, \{a, b\}, \{c\}, \{a, c\}\}$ and, we have the identity function $f : (X, \tau, I) \rightarrow (X, \sigma, J)$ is contra quasi-$\Lambda^*_j$-continuous, but is not contra $\Lambda^*_j$-irresolute, because $f^{-1}(\{b, c\})$ is not a $\Lambda^*_j$-closed subset of $X$.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{a, c\}, \{b\}\}$, $I = \{\emptyset, \{b\}\}$ and $J = \{\emptyset, \{a\}\}$. Then, the family of all $\sigma^*$-open subsets of $X$ is $\{\emptyset, X, \{b\}, \{a, c\}, \{b, c\}\}$, the family of all $\Lambda^*_j$-closed subsets of $X$ is $\{\emptyset, X, \{a\}, \{a, c\}, \{b\}\}$ and, we have the identity function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is contra $\Lambda^*_j$-continuous, but is not contra quasi-$\Lambda^*_j$-continuous, because $f^{-1}(\{c\})$ and $f^{-1}(\{b, c\})$ are not $\Lambda^*_j$-closed subsets of $X$.

From Theorem 3.2, Example 3.1 and Example 3.2, we have the following diagram, where the implications are not reversible:

\[
\text{Contra } \Lambda^*_j\text{-irresolute} \implies \text{Contra quasi-} \Lambda^*_j\text{-continuous} \implies \text{Contra } \Lambda^*_j\text{-continuous}
\]

Theorem 3.3. For a function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$, the following statements are equivalent:

1. $f$ is contra $\Lambda^*_j$-continuous.
2. $f^{-1}(B)$ is a $\Lambda^*_j$-open subset of $X$ for each closed subset $B$ of $Y$.
3. For each $x \in X$ and each closed subset $B$ of $Y$ containing $f(x)$, there exists a $\Lambda^*_j$-open subset $U$ of $X$ such that $x \in U$ and $f(U) \subset B$.

Proof. (1) $\Rightarrow$ (2) Let $B$ be any closed subset of $Y$. Then $V = Y - B$ is an open subset of $Y$ and, as $f$ is contra $\Lambda^*_j$-continuous, $f^{-1}(V)$ is a $\Lambda^*_j$-closed subset of $X$. Since $f^{-1}(V) = f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B) = X - f^{-1}(B)$, we conclude $f^{-1}(B)$ is a $\Lambda^*_j$-open subset of $X$.

(2) $\Rightarrow$ (1) Let $V$ any open subset of $Y$. Then $B = Y - V$ is a closed subset of $Y$ and, by hypothesis, $f^{-1}(B)$ is a $\Lambda^*_j$-open subset of $X$. Since $f^{-1}(B) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$, we obtain that $f^{-1}(V)$ is a $\Lambda^*_j$-closed subset of $X$ and hence, $f$ is contra $\Lambda^*_j$-continuous.

(1) $\Rightarrow$ (3) Let $x \in X$ and $B$ be any closed subset of $Y$ such that $f(x) \in B$. Then $x \in f^{-1}(B)$ and, because $f$ is contra $\Lambda^*_j$-continuous, $f^{-1}(B)$ is a $\Lambda^*_j$-open subset of $X$. If $U = f^{-1}(B)$, then $U$ is
a \( \Lambda_f^- \)-open subset of \( X \) such that \( x \in U \) and \( f(U) = f(f^{-1}(B)) \subset B \).

(3) \( \Rightarrow \) (1) Let \( B \) any closed subset of \( Y \) and let \( x \in f^{-1}(B) \). Then \( f(x) \in B \) and, by hypothesis, there exists a \( \Lambda_f^- \)-open subset \( U_x \) of \( X \) such that \( x \in U_x \) and \( f(U_x) \subset B \). Thus, \( x \in U_x \subset f^{-1}(f(U)) \subset f^{-1}(B) \) and hence, \( f^{-1}(B) = \bigcup \{ U_x : x \in f^{-1}(B) \} \). By Lemma 2.2, we obtain that \( f^{-1}(B) \) is a \( \Lambda_f^- \)-open subset of \( X \).

\[ \square \]

**Theorem 3.4.** For a function \( f : (X, \tau, I) \to (Y, \sigma, J) \), the following statements are equivalent:

(1) \( f \) is contra quasi-\( \Lambda_f^- \)-continuous.

(2) \( f^{-1}(B) \) is a \( \Lambda_f^- \)-open subset of \( X \) for each \( \sigma^* \)-closed subset \( B \) of \( Y \).

(3) For each \( x \in X \) and each \( \sigma^* \)-closed subset \( B \) of \( Y \) containing \( f(x) \), there exists a \( \Lambda_f^- \)-open subset \( U \) of \( X \) such that \( x \in U \) and \( f(U) \subset B \).

**Proof.** The proof is similar to that of Theorem 3.3. \[ \square \]

**Theorem 3.5.** For a function \( f : (X, \tau, I) \to (Y, \sigma, J) \), the following statements are equivalent:

(1) \( f \) is contra \( \Lambda_f^- \)-irresolute.

(2) \( f^{-1}(B) \) is a \( \Lambda_f^- \)-open subset of \( X \) for each \( \Lambda_f^- \)-closed subset \( B \) of \( Y \).

(3) For each \( x \in X \) and each \( \Lambda_f^- \)-closed subset \( B \) of \( Y \) containing \( f(x) \), there exists a \( \Lambda_f^- \)-open subset \( U \) of \( X \) such that \( x \in U \) and \( f(U) \subset B \).

**Proof.** The proof is similar to that of Theorem 3.3. \[ \square \]

**Theorem 3.6.** Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) and \( g : (Y, \sigma, J) \to (Z, \theta, K) \) be two functions, where \( I, J \) and \( K \) are ideals on \( X, Y \) and \( Z \), respectively. Then:

(1) \( g \circ f \) is contra \( \Lambda_f^- \)-irresolute, if \( f \) is contra \( \Lambda_f^- \)-irresolute and \( g \) is contra \( \Lambda_f^- \)-irresolute.

(2) \( g \circ f \) is contra \( \Lambda_f^- \)-irresolute, if \( f \) is contra \( \Lambda_f^- \)-irresolute and \( g \) is \( \Lambda_f^- \)-irresolute.

(3) \( g \circ f \) is \( \Lambda_f^- \)-irresolute, if \( f \) is contra \( \Lambda_f^- \)-irresolute and \( g \) is contra \( \Lambda_f^- \)-irresolute.

(4) \( g \circ f \) is contra \( \Lambda_f^- \)-continuous, if \( f \) is contra \( \Lambda_f^- \)-continuous and \( g \) is continuous.

(5) \( g \circ f \) is contra \( \Lambda_f^- \)-continuous, if \( f \) is contra \( \Lambda_f^- \)-irresolute and \( g \) is \( \Lambda_f^- \)-continuous.

(6) \( g \circ f \) is contra \( \Lambda_f^- \)-continuous, if \( f \) is \( \Lambda_f^- \)-irresolute and \( g \) is contra \( \Lambda_f^- \)-continuous.

(7) \( g \circ f \) is contra \( \Lambda_f^- \)-continuous, if \( f \) is \( \Lambda_f^- \)-continuous and \( g \) is contra continuous.

(8) \( g \circ f \) is \( \Lambda_f^- \)-continuous, if \( f \) is contra \( \Lambda_f^- \)-continuous and \( g \) is contra continuous.
(9) \( g \circ f \) is \( \Lambda_f^s \)-continuous, if \( f \) is contra \( \Lambda_f^s \)-irresolute and \( g \) is contra \( \Lambda_f^s \)-continuous.

(10) \( g \circ f \) is contra quasi-\( \Lambda_f^s \)-continuous, if \( f \) is \( \Lambda_f^s \)-irresolute and \( g \) is contra quasi-\( \Lambda_f^s \)-continuous.

(11) \( g \circ f \) is contra quasi-\( \Lambda_f^s \)-continuous, if \( f \) is contra \( \Lambda_f^s \)-irresolute and \( g \) is quasi \( \Lambda_f^s \)-continuous.

(12) \( g \circ f \) is quasi \( \Lambda_f^s \)-continuous, if \( f \) is contra \( \Lambda_f^s \)-irresolute and \( g \) is contra quasi-\( \Lambda_f^s \)-continuous.

**Proof.** (1) Let \( V \) be a \( \Lambda_f^s \)-open subset of \( Z \). Since \( g \) is contra \( \Lambda_f^s \)-irresolute, \( g^{-1}(V) \) is a \( \Lambda_f^s \)-closed subset of \( Y \) and, as \( f \) is \( \Lambda_f^s \)-irresolute, then by Theorem 3.1, we have \( f^{-1}(g^{-1}(V)) \) is a \( \Lambda_f^s \)-closed subset of \( X \). Because \( (g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = (f^{-1}(g^{-1}(V))) \), it follows that \( (g \circ f)^{-1}(V) \) is a \( \Lambda_f^s \)-closed subset of \( X \). This shows that \( g \circ f \) is contra \( \Lambda_f^s \)-irresolute.

The proofs of (2)-(12) are similar to that of (1). \( \square \)

**Theorem 3.7.** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be a function and let \( \mu \) be the product topology induced by \( \tau \) and \( \sigma \) on \( X \times Y \). Let \( g : (X, \tau, I) \to (X \times Y, \mu) \) be the graph function of \( f \) defined by \( g(x) = (x, f(x)) \) for each \( x \in X \). If \( g \) is contra \( \Lambda_f^s \)-continuous, then \( f \) is contra \( \Lambda_f^s \)-continuous.

**Proof.** Let \( V \) be any open subset of \( Y \). Then \( X \times V \) is a open subset of \( X \times Y \) and, as \( g \) is contra \( \Lambda_f^s \)-continuous, we have \( f^{-1}(V) = g^{-1}(X \times V) \) is a \( \Lambda_f^s \)-closed subset of \( X \). Therefore, \( f \) is contra \( \Lambda_f^s \)-continuous. \( \square \)

### 4. Study of Direct and Inverse Images by Contra \( \Lambda_f^s \)-Continuous Functions

In this section, we study the behavior of some modifications of classical topological notions under direct and inverse images of the new variants of contra continuity introduced in the previous section. Recall that an ideal space \( (X, \tau, I) \) is said to be \( \Lambda_f^s \)-connected (resp. \( \tau^* \)-connected), if \( X \) cannot be written as a disjoint union of two nonempty \( \Lambda_f^s \)-open (resp. \( \tau^* \)-open) sets, see [7].

**Theorem 4.1.** If \( f : (X, \tau, I) \to (Y, \sigma) \) is a surjective contra \( \Lambda_f^s \)-continuous function and \( (X, \tau, I) \) is a \( \Lambda_f^s \)-connected space having more than one element, then \( (Y, \sigma) \) is a not discrete space.

**Proof.** Suppose that \( (Y, \sigma) \) is a discrete space and let \( A \) be any nonempty proper subset of \( Y \). Then, \( A \) is a clopen subset of \( Y \) and as \( f \) is contra \( \Lambda_f^s \)-continuous, \( f^{-1}(A) \) is a \( \Lambda_f^s \)-open and \( \Lambda_f^s \)-closed subset of \( X \). Since \( (X, \tau, I) \) is a \( \Lambda_f^s \)-connected space, by [7, Theorem 4.5], the only subsets
of $X$ which are both $\Lambda^J_f$-open and $\Lambda^J_f$-closed are $\emptyset$ and $X$. Thus, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = X$. If $f^{-1}(A) = \emptyset$, then this contradicts the hypothesis that $A \neq \emptyset$ and $f$ is surjective. If $f^{-1}(A) = X$, since $f$ is surjective and $A$ is a proper subset of $Y$, there exist $y_0 \in Y - A$ and $x_0 \in X$ such that $y_0 = f(x_0)$, but then $y_0 \in A$ and $y_0 \notin A$. Again we obtain a contradiction. Therefore, $(Y, \sigma)$ is not a discrete space. 

**Theorem 4.2.** Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a surjective function. The following statements hold:

1. If $f$ is contra $\Lambda^J_f$-irresolute and $(X, \tau, I)$ is a $\Lambda^J_f$-connected space, then $(Y, \sigma, J)$ is a $\Lambda^J_f$-connected space.

2. If $f$ is contra quasi-$\Lambda^J_f$-continuous and $(X, \tau, I)$ is a $\Lambda^J_f$-connected space, then $(Y, \sigma, J)$ is a $\sigma^*$-connected space.

3. If $f$ is contra $\Lambda^J_f$-continuous and $(X, \tau, I)$ is a $\Lambda^J_f$-connected, then $(Y, \sigma)$ is a connected space.

**Proof.** (1) Suppose that $(X, \tau, I)$ is a $\Lambda^J_f$-connected space and $f : (X, \tau, I) \to (Y, \sigma, J)$ is a surjective contra $\Lambda^J_f$-irresolute function. If $(Y, \sigma, J)$ is not $\Lambda^J_f$-connected, there exist two nonempty $\Lambda^J_f$-open subsets $A$ and $B$ of $Y$ such that $A \cap B = \emptyset$ and $Y = A \cup B$, which implies that $B = Y - A$ and $A = Y - B$ are nonempty $\Lambda^J_f$-closed subsets of $Y$ and, because $f$ is contra $\Lambda^J_f$-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\Lambda^J_f$-open subsets of $X$ such that $X = f^{-1}(A) \cup f^{-1}(B)$. This contradicts the fact that $(X, \tau, I)$ is $\Lambda^J_f$-connected. Therefore, $(Y, \sigma, J)$ is $\Lambda^J_f$-connected.

The proofs of (2) and (3) are similar to that of (1). 

**Theorem 4.3.** Let $(Y, \sigma)$ be any $T_0$-space. If each contra $\Lambda^J_f$-continuous function $f : (X, \tau, I) \to (Y, \sigma)$ is constant, then $(X, \tau, I)$ is a $\Lambda^J_f$-connected space.

**Proof.** Suppose that $(X, \tau, I)$ does not is a $\Lambda^J_f$-connected space and each contra $\Lambda^J_f$-continuous function $f : (X, \tau, I) \to (Y, \sigma)$ is constant. By [7, Theorem 4.5], there exists a proper nonempty subset $A$ of $X$ which is both $\Lambda^J_f$-open and $\Lambda^J_f$-closed. Let $Y = \{a, b\}$ endowed with the topology $\sigma = \{Y, \emptyset, \{a\}, \{b\}\}$. If $f : (X, \tau, I) \to (Y, \sigma)$ is a function such that $f(A) = \{a\}$ and $f(X - A) = \{b\}$, then $f$ is a non constant contra $\Lambda^J_f$-continuous function and $(Y, \sigma)$ is a $T_0$-space, which contradicts the hypothesis. Therefore, $(X, \tau, I)$ is a $\Lambda^J_f$-connected space.
Remark 4.2. If $x \in X$ and $U$ be an open subset of $Y$ such that $f(x) \in U$. Since $(Y, \sigma)$ is regular, there exists an open subset $V$ of $X$ such that $f(x) \in V$. Now, as $f$ is contra $\Lambda^I_f$-continuous and $Cl(V)$ is a closed subset of $Y$ containing $f(x)$, then by Theorem 3.3, there exists a $\Lambda^I_f$-open subset $W$ of $X$ such that $x \in W$ and $f(W) \subset Cl(V)$. By Theorem 3.1, we obtain that $f$ is a $\Lambda^I_f$-continuous function.

\[\square\]

Definition 4.1. An ideal space $(X, \tau, I)$ is said to be $\Lambda^I_f$-normal, if for each pair of disjoint closed subsets $A$ and $B$ of $X$, there exist two disjoint $\Lambda^I_f$-open subsets $U$ and $V$ of $X$ such that $A \subset U$ and $B \subset V$.

Remark 4.1. If $(X, \tau)$ is a normal space, then $(X, \tau, I)$ is a $\Lambda^I_f$-normal space for each ideal $I$ on $X$.

Recall that a topological space $(X, \tau)$ is called ultra normal [8], if for each pair of nonempty disjoint closed subsets $A$ and $B$ of $X$, there exist two disjoint clopen subsets $U$ and $V$ of $X$ such that $A \subset U$ and $B \subset V$. The following result shows that, the inverse image of an ultra normal space under a closed injective contra $\Lambda^I_f$-continuous function is a $\Lambda^I_f$-normal space.

Theorem 4.5. If $f : (X, \tau, I) \to (Y, \sigma)$ is a closed injective contra $\Lambda^I_f$-continuous function and $(Y, \sigma)$ is an ultra normal space, then $(X, \tau, I)$ is a $\Lambda^I_f$-normal space.

Proof. Let $A$ and $B$ be two disjoint closed subsets of $X$. Since $f$ is closed injective, $f(A)$ and $f(B)$ are disjoint closed subsets of $Y$ and, as $(Y, \sigma)$ is an ultra normal space, there exist two disjoint clopen subsets $U$ and $V$ of $Y$ such that $f(A) \subset U$ and $f(B) \subset V$. Now, because $f$ is contra $\Lambda^I_f$-continuous, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $\Lambda^I_f$-closed subsets of $X$ such that $A \subset f^{-1}(U)$ and $B \subset f^{-1}(V)$. Therefore, $(X, \tau, I)$ is a $\Lambda^I_f$-normal space.

\[\square\]

Definition 4.2. An ideal space $(X, \tau, I)$ is said to be $\Lambda^I_f$-$T_2$, if for each pair of distinct points $x, y \in X$, there exist two disjoint $\Lambda^I_f$-open subsets $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$.

Remark 4.2. If $(X, \tau)$ is a $T_2$-space, then $(X, \tau, I)$ is a $\Lambda^I_f$-$T_2$-space for each ideal $I$ on $X$. 
Recall that a topological space \((X, \tau)\) is called *Urysohn* [9], if for each pair of distinct points \(x, y \in X\), there exist two open subsets \(U\) and \(V\) of \(X\) such that \(x \in U\), \(y \in V\) and \(\text{Cl}(U) \cap \text{Cl}(V) = \emptyset\). The following result shows that, the inverse image of a Urysohn space under an injective contra \(\Lambda^*_I\)-continuous function is a \(\Lambda^*_I\)-\(T_2\)-space.

**Theorem 4.6.** If \(f : (X, \tau, I) \to (Y, \sigma)\) is an injective contra \(\Lambda^*_I\)-continuous function and \((Y, \sigma)\) is an Urysohn space, then \((X, \tau, I)\) is a \(\Lambda^*_I\)-\(T_2\)-space.

**Proof.** Let \(x\) and \(y\) be two distinct points of \(X\). Since \(f\) is injective, we have \(f(x) \neq f(y)\) and, as \((Y, \sigma)\) is an Urysohn space, there exist two open subsets \(U\) and \(V\) of \(Y\) such that \(f(x) \in U\), \(f(y) \in V\) and \(\text{Cl}(U) \cap \text{Cl}(V) = \emptyset\). By Theorem 3.3, there exist two \(\Lambda^*_I\)-open subsets \(A\) and \(B\) of \(X\) such that \(x \in A\), \(y \in B\), \(f(A) \subset \text{Cl}(U)\) and \(f(B) \subset \text{Cl}(V)\). Thus, \(f(A) \cap f(B) \subset \text{Cl}(U) \cap \text{Cl}(V) = \emptyset\), which implies that \(f(A \cap B) = \emptyset\) and hence, \(A \cap B = \emptyset\). This shows that \((X, \tau, I)\) is a \(\Lambda^*_I\)-\(T_2\)-space. \(\Box\)

According to [2], we say that a topological space \((X, \tau)\) is *locally indiscrete*, if each open subset of \(X\) is closed. In the following definition some modifications of a locally indiscrete space are introduced in order to investigate related properties with the functions defined in Section 3.

**Definition 4.3.** An ideal space \((X, \tau, I)\) is said to be:

(1) *Locally \(\tau^*\)-indiscrete*, if each \(\tau^*\)-open subset of \(X\) is closed in \(X\).

(2) *Locally \(\Lambda^*_I\)-indiscrete*, if each \(\Lambda^*_I\)-open subset of \(X\) is closed in \(X\).

(3) *\(\Lambda^*_I\)-space*, if each \(\Lambda^*_I\)-open subset of \(X\) is open in \(X\).

**Proposition 4.1.** Let \((X, \tau, I)\) be an ideal space. The following statements hold:

(1) If \((X, \tau, I)\) is locally \(\Lambda^*_I\)-indiscrete, then it is locally \(\tau^*\)-indiscrete.

(2) If \((X, \tau, I)\) is locally \(\tau^*\)-indiscrete, then it is locally indiscrete.

(3) \((X, \tau, I)\) is locally \(\tau^*\)-indiscrete if and only if each \(\tau^*\)-closed subset of \(X\) is open in \(X\).

(4) \((X, \tau, I)\) is locally \(\Lambda^*_I\)-indiscrete if and only if each \(\Lambda^*_I\)-closed subset of \(X\) is open in \(X\).

(5) \((X, \tau, I)\) is \(\Lambda^*_I\)-space if and only if each \(\Lambda^*_I\)-closed subset of \(X\) is closed in \(X\).
Theorem 4.7. Let \( f : (X, \tau, I) \to (Y, \sigma) \) be a contra \( \Lambda_I^{\mathbb{J}} \)-continuous function. The following statements hold:

(1) If \((X, \tau, I)\) is locally \( \Lambda_I^{\mathbb{J}} \)-indiscrete, then \( f \) is continuous.

(2) If \((X, \tau, I)\) is a \( \Lambda_I^{\mathbb{J}} \)-space, then \( f \) is contra-continuous.

Proof. (1) Let \( B \) be a closed subset of \( Y \). Since \( f \) is contra \( \Lambda_I^{\mathbb{J}} \)-continuous, \( f^{-1}(B) \) is a \( \Lambda_I^{\mathbb{J}} \)-open subset of \( X \) and, as \((X, \tau, I)\) is locally \( \Lambda_I^{\mathbb{J}} \)-indiscrete, we obtain that \( f^{-1}(B) \) is a closed subset of \( X \). Therefore, \( f \) is a continuous function.

The proof of (2) is similar to that of (1). \( \square \)

The following result shows that, the direct image of a \( \Lambda_I^{\mathbb{J}} \)-space under a closed surjective contra \( \Lambda_I^{\mathbb{J}} \)-irresolute (resp. contra quasi-\( \Lambda_I^{\mathbb{J}} \)-continuous, contra \( \Lambda_I^{\mathbb{J}} \)-continuous) function is a locally \( \Lambda_I^{\mathbb{J}} \)-indiscrete (resp. locally \( \sigma^* \)-indiscrete, locally indiscrete) space.

Theorem 4.8. Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) be a closed surjective function. The following statements hold:

(1) If \( f \) is contra \( \Lambda_I^{\mathbb{J}} \)-irresolute and \((X, \tau, I)\) is a \( \Lambda_I^{\mathbb{J}} \)-space, then \((Y, \sigma, J)\) is locally \( \Lambda_I^{\mathbb{J}} \)-indiscrete.

(2) If \( f \) is contra quasi-\( \Lambda_I^{\mathbb{J}} \)-continuous and \((X, \tau, I)\) is a \( \Lambda_I^{\mathbb{J}} \)-space, then \((Y, \sigma, J)\) is locally \( \sigma^* \)-indiscrete.

(3) If \( f \) is contra \( \Lambda_I^{\mathbb{J}} \)-continuous and \((X, \tau, I)\) is a \( \Lambda_I^{\mathbb{J}} \)-space, then \((Y, \sigma)\) is locally indiscrete space.

Proof. Let \( V \) be a \( \Lambda_I^{\mathbb{J}} \)-open subset of \( Y \). Since \( f \) is contra \( \Lambda_I^{\mathbb{J}} \)-irresolute, \( f^{-1}(V) \) is a \( \Lambda_I^{\mathbb{J}} \)-closed subset of \( X \) and, as \((X, \tau, I)\) is a \( \Lambda_I^{\mathbb{J}} \)-space, \( f^{-1}(V) \) is a closed subset of \( X \). Now, because \( f \) is closed surjective, \( V = f(f^{-1}(V)) \) is a closed subset of \( Y \) and hence, \((Y, \sigma, J)\) is a locally \( \Lambda_I^{\mathbb{J}} \)-indiscrete space.

The proofs of (2) and (3) are similar to that of (1). \( \square \)

Recall that a topological space \((X, \tau)\) is said to be strongly \( S \)-closed [1], if each closed cover of \( X \) has a finite subcover. Now we introduce a modification of a strongly \( S \)-closed space using \( \Lambda_I^{\mathbb{J}} \)-closed sets.
**Definition 4.4.** An ideal space \((X, \tau, I)\) is said to be strongly \(S-\Lambda^j\)-closed, if each cover of \(X\) by \(\Lambda^j\)-closed sets has a finite subcover.

**Remark 4.3.** If \((X, \tau, I)\) is a strongly \(S-\Lambda^j\)-closed space, then \((X, \tau)\) is a strongly \(S\)-closed space.

According to [7], we say that an ideal space \((X, \tau, I)\) is \(\Lambda^j\)-compact (resp. \(\tau^*\)-compact), if each cover of \(X\) by \(\Lambda^j\)-open (resp. \(\tau^*\)-open) sets has a finite subcover. The following result shows that, the direct image of a strongly \(S-\Lambda^j\)-closed space under a surjective contra \(\Lambda^j\)-irresolute (resp. contra quasi-\(\Lambda^j\)-continuous, contra \(\Lambda^j\)-continuous) function, is a \(\Lambda^j\)-compact (resp. \(\sigma^*\)-compact, compact) space.

**Theorem 4.9.** Let \(f : (X, \tau, I) \rightarrow (Y, \sigma, I)\) be a surjective function. The following statements hold:

1. If \(f\) is contra \(\Lambda^j\)-irresolute and \((X, \tau, I)\) is strongly \(S-\Lambda^j\)-closed, then \((Y, \sigma, J)\) is \(\Lambda^j\)-compact.
2. If \(f\) is contra quasi-\(\Lambda^j\)-continuous and \((X, \tau, I)\) is strongly \(S-\Lambda^j\)-closed, then \((Y, \sigma, J)\) is \(\sigma^*\)-compact.
3. If \(f\) is contra \(\Lambda^j\)-continuous and \((X, \tau, I)\) is strongly \(S-\Lambda^j\)-closed, then \((Y, \sigma)\) is compact.

**Proof.** (1) Let \(\{W_\alpha : \alpha \in \Delta\}\) be a cover of \(Y\) by \(\Lambda^j\)-open sets. Since \(f\) is contra \(\Lambda^j\)-irresolute, \(\{f^{-1}(W_\alpha) : \alpha \in \Delta\}\) is a cover of \(X\) by \(\Lambda^j\)-closed sets and, as \((X, \tau, I)\) is strongly \(S-\Lambda^j\)-closed, there exists a finite subfamily \(\{f^{-1}(W_\alpha) : i = 1, \ldots, n\}\) of \(\{f^{-1}(W_\alpha) : \alpha \in \Delta\}\) such that \(X = \bigcup_{i=1}^{n} f^{-1}(W_\alpha)\). Therefore \(Y = f(X) = f \left( \bigcup_{i=1}^{n} f^{-1}(W_\alpha) \right) = \bigcup_{i=1}^{n} f(f^{-1}(W_\alpha)) = \bigcup_{i=1}^{n} W_\alpha\) and so, \((Y, \sigma, J)\) is \(\Lambda^j\)-compact.

The proofs of (2) and (3) are similar to that of (1). \(\square\)

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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