PAIR DIFFERENCE CORDIAL LABELING OF GRAPHS

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Abstract. Let \( G = (V, E) \) be a \((p, q)\) graph.

Define
\[
\rho = \begin{cases} 
  \frac{p}{2}, & \text{if } p \text{ is even} \\
  \frac{p-1}{2}, & \text{if } p \text{ is odd}
\end{cases}
\]
and \( L = \{\pm 1, \pm 2, \pm 3, \ldots, \pm \rho\} \) called the set of labels.

Consider a mapping \( f : V \rightarrow L \) by assigning different labels in \( L \) to the different elements of \( V \) when \( p \) is even and different labels in \( L \) to \( p-1 \) elements of \( V \) and repeating a label for the remaining one vertex when \( p \) is odd.

The labeling as defined above is said to be a pair difference cordial labeling if for each edge \( uv \) of \( G \) there exists a labeling \( |f(u) - f(v)| \) such that \( |\Delta_{f_1} - \Delta_{f_1'}| \leq 1 \), where \( \Delta_{f_1} \) and \( \Delta_{f_1'} \) respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph \( G \) for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of path, cycle, complete graph, star, bistar, comb.

Keywords: path; cycle; complete graph; star; bistar; comb.

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1. **Introduction**

In this paper we consider only finite, undirected and simple graphs. The notion of difference cordial labeling of a graph was introduced and studied some properties of difference cordial labeling in [4]. The difference cordial labeling behavior of several graphs like path, cycle, star etc have been investigated in [4]. In this paper we introduce the pair difference cordial labeling and investigate pair difference cordial labeling behavior of path, cycle, star, comb and bistar graph.

2. **Preliminaries**

**Definition 2.1.** The ladder $L_n$ is the product graph $P_n \times K_2$ with $2n$ vertices and $3n - 2$ edges.

**Definition 2.2.** The graph obtained by joining two disjoint cycles $u_1u_2, \cdots u_mu_1$ and $v_1v_2, \cdots v_nv_1$ with an edge $u_1v_1$ is called dumbbell graph and it is denoted by $Db(m, n)$.

3. **Pair Difference Cordial Labeling**

**Definition 3.1.** Let $G = (V, E)$ be a $(p, q)$ graph.

Define

$$\rho = \begin{cases} 
\frac{p}{2}, & \text{if } p \text{ is even} \\
\frac{p-1}{2}, & \text{if } p \text{ is odd} 
\end{cases}$$

and $L = \{ \pm 1, \pm 2, \pm 3, \cdots, \pm \rho \}$ called the set of labels.

Consider a mapping $f : V \rightarrow L$ by assigning different labels in $L$ to the different elements of $V$ when $p$ is even and different labels in $L$ to $p-1$ elements of $V$ and repeating a label for the remaining one vertex when $p$ is odd. The labeling as defined above is said to be a pair difference cordial labeling if for each edge $uv$ of $G$ there exists a labeling $|f(u) - f(v)|$ such that $|\Delta f_1 - \Delta f'_1| \leq 1$, where $\Delta f_1$ and $\Delta f'_1$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph $G$ for which there exists a pair difference cordial labeling is called a pair difference cordial graph.
Theorem 3.1. If $G$ is a $(p,q)$ pair difference cordial graph then

$$
q \leq \begin{cases} 
2p-3 & \text{if } p \text{ is even} \\
2p-1 & \text{if } p \text{ is odd}
\end{cases}
$$

Proof. Case 1. $p$ is even.

The maximum number of edges with the label 1 among the vertex labels $1, 2, 3, \cdots, \frac{p}{2}$ respectively is $\frac{p}{2} - 1$. Also the maximum number of edges with the label 1 among the vertex labels $-1, -2, -3, \cdots, -\frac{p}{2}$ respectively is $\frac{p}{2} - 1$. Therefore $\Delta f_1 \leq \left(\frac{p}{2} - 1\right) + \left(\frac{p}{2} - 1\right) = p - 2$. That is $\Delta f_1 \leq p - 2$. This implies $\Delta f_1^c \geq q - p + 2 \rightarrow (1)$.

Type 1. $\Delta f_1^c = \Delta f_1 + 1$.

By (1), $q - p + 2 \leq \Delta f_1^c$,

$$
\leq \Delta f_1 + 1 \\
\leq p - 1. \text{This implies } q \leq 2p - 3. \rightarrow (2)
$$

Type 2. $\Delta f_1^c = \Delta f_1 - 1$.

By (1), $q - (p - 2) \leq \Delta f_1^c$,

$$
\leq \Delta f_1 - 1, \\
\leq p - 3. \text{This implies } q \leq 2p - 5. \rightarrow (3)
$$

Type 3. $\Delta f_1^c = \Delta f_1$.

By (1), $q - (p - 2) \leq \Delta f_1^c$,

$$
\leq \Delta f_1, \\
\leq p - 2.
$$

This implies $q \leq 2p - 4 \rightarrow (4)$. By (2),(3),(4), $q \leq 2p - 3$. 


**Theorem 3.2.** The path $P_n$ is pair difference cordial for all values of $n$ except $n \neq 3$.

**Proof.** Let $P_n$ be the path $u_1u_2\cdots u_n$.

**Case. 1** $n$ is odd.

There are two cases arises.

**Subcase. 1** $n = 4t + 1, t \in N \cup \{0\}$.

Assign the labels 1, 2 to the vertices $u_1, u_2$ respectively and assign the labels $-1, -2$ respectively to the vertices $u_3, u_4$. Next assign the labels 3, 4 respectively to the vertices $u_5, u_6$ and assign the labels $-3, -4$ to the vertices $u_7, u_8$ respectively. Proceeding like this until we reach the vertex $u_{n-1}$. Finally assign the label $-2$ to the vertex $u_n$. Note that the vertices $u_{n-4}, u_{n-3}$ get the labels $\frac{n-3}{2}, \frac{n-1}{2}$ respectively and the vertices $u_{n-2}, u_{n-1}$ receive the labels $-\frac{n-3}{2}, -\frac{n-1}{2}$ respectively. This vertex labeling gives the pair difference cordial labeling of path $P_n$, since $\Delta f_1 = \Delta f'_1 = \frac{n-1}{2}$.

**Subcase. 2** $n = 4t + 3, t \in N$.

Assign the labels 1, 2 respectively to the vertices $u_1, u_2$ and assign the label $-1, -2$ to the vertices $u_3, u_4$ respectively. Next assign the labels 3, 4 respectively to the vertices $u_5, u_6$ and assign the labels $-3, -4$ to the vertices $u_7, u_8$ respectively. Proceeding like this until we reached $u_{n-3}$. Assign the label $-\frac{n-3}{2}$ to the vertex $u_n$. Finally assign the labels $\frac{n-1}{2}, -\frac{n-1}{2}$ respectively to the vertices $u_{n-2}, u_{n-1}$. Note that the vertices $u_{n-6}, u_{n-5}$ received the labels $\frac{n-5}{2}, \frac{n-3}{2}$ respectively and the vertices $u_{n-4}, u_{n-3}$ get the labels $-\frac{n-5}{2}, -\frac{n-3}{2}$ respectively. This vertex labeling gives the pair difference cordial labeling of path $P_n$, since $\Delta f_1 = \Delta f'_1 = \frac{n-1}{2}$.

**Subcase. 3** $n = 3$.

Suppose $f$ is a pair difference cordial of $P_3$, then $\Delta f_1 = 0$ and $\Delta f'_1 = 2$. This contradicts $P_3$ is not pair difference cordial.
Case. 2 \( n \) is even.

There are two cases arises.

Subcase. 1 \( n = 4t, t \in \mathbb{N} \).

Assign the labels 1,2 to the vertices \( u_1, u_2 \) respectively and assign the labels \(-1, -2\) to the vertices \( u_3, u_4 \) respectively.Next assign the labels 3,4 to the vertices \( u_5, u_6 \) respectively and assign the labels \(-3, -4\) respectively to the vertices \( u_7, u_8 \).Proceeding like this until we reach the vertex \( u_n \).Note that the vertices \( u_{n-3}, u_{n-2} \) respectively receive the labels \( \frac{n-2}{2}, \frac{n}{2} \) and the vertices \( u_{n-1}, u_n \) get the labels \(-\frac{n-2}{2}, -\frac{n}{2}\) respectively.

This vertex labeling gives a pair difference cordial labeling of the path \( P_n \), since \( \Delta f_1 = \frac{n}{2}, \Delta f_1^c = \frac{n-2}{2} \).

Subcase. 2 \( n = 4t + 2, t \in \mathbb{N} \cup \{0\} \).

Assign the labels 1,2 respectively to the vertices \( u_1, u_2 \).Now assign the labels \(-1, -2\) to the vertices \( u_3, u_4 \) respectively.Next assign the label 3,4 respectively to the vertices \( u_5, u_6 \) and assign the label \(-3, -4\) to the vertices \( u_7, u_8 \).Proceeding like this until we reach the vertex \( u_{n-2} \).Finally assign the labels \( \frac{n}{2}, -\frac{n}{2} \) to the vertices \( u_{n-1}, u_n \) respectively.Note that the vertices \( u_{n-5}, u_{n-4} \) get the label \( \frac{n-4}{2}, \frac{n-2}{2} \) respectively and the vertices \( u_{n-3}, u_{n-2} \) receive the labels \(-\frac{n-4}{2}, -\frac{n-2}{2}\) respectively.

This vertex labeling gives the pair difference cordial labeling of path \( P_n \), since \( \Delta f_1 = \frac{n-2}{2}, \Delta f_1^c = \frac{n}{2} \).

Remark. \( P_3 \) is difference cordial but not pair difference cordial [4].

Corollary 3.2.1. The cycle \( C_n \) is pair difference cordial if and only if \( n > 3 \).

Proof. Let \( C_n \) be the cycle \( u_1 u_2 \cdots u_n u_1 \). The function \( f \) in the theorem 3.3 is also a pair difference cordial labeling of the cycle \( C_n \).

□
**Theorem 3.3.** The star $K_{1,n}$ is pair difference cordial if and only if $3 \leq n \leq 6$.

**Proof.** Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}, E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. The graph $K_{1,n}$ has $n+1$ vertices and $n$ edges.

**Case 1.** $3 \leq n \leq 6$.

Table 1 shows that the star $K_{1,n}, 3 \leq n \leq 6$ is pair difference cordial.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$u$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
<th>$u_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td>3</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td>3</td>
<td>-3</td>
</tr>
</tbody>
</table>

**Table 1**

**Case 2.** $n \geq 6$.

Suppose $f$ is a pair difference cordial labeling of $K_{1,n}$. Assume $f(u) = l$, To get the edge label 1, the only possibly is that the pendant vertices receive the label $l-1$ or $l+1$.

**Subcase 1.** $n$ is odd.

In this case, $\Delta_{f_1} \leq 2$. This implies $\Delta_{f_1} - \Delta_{f_1^c} \geq n - 4 > 1$, a contradiction.

**Subcase 2.** $n$ is even.

In this case, we may use one vertex label as twice. This implies $\Delta_{f_1} \leq 3$. Therefore $\Delta_{f_1} - \Delta_{f_1^c} \geq n - 6 > 1$, a contradiction.

□

**Remark.** The star $K_{1,6}$ is pair difference cordial but not difference cordial[4].

**Corollary 3.3.1.** The complete graph $K_p$ is pair difference cordial if and only if $p \leq 2$.

**Proof.** **Case 1.** $p \leq 2$.

By theorem 3.3, $K_1, K_2$ is pair difference cordial.
Case 2. $3 \leq p \leq 5$.

The Table 2 shows that $K_3, K_4, K_5$ is not pair difference cordial.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$\Delta f_1^c$</th>
<th>$\Delta f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

**Table 2**

Case 2. $p \geq 6$.

Suppose $K_p$ is pair difference cordial. By theorem 3.2, $\binom{p}{2} \leq 2p + 1$. This implies $\binom{p(p-1)}{2} \leq 2p + 1$, a contradiction to $p \geq 6$.

**Theorem 3.4.** The comb $P_n \oplus K_1$ is a pair difference cordial for all values of $n$.

**Proof.** Let $V(P_n \oplus K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \oplus K_1) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n-1\}$.

Define a map $f : V(P_n \oplus K_1) \to \{\pm 1, \pm 2, \cdots, \pm n\}$ by

$f(u_i) = i, 1 \leq i \leq n$, and $f(v_i) = -i, 1 \leq i \leq n$. Then $\Delta f_1 = n - 1, \Delta f_1^c = n$.

**Theorem 3.5.** $K_2 + mK_1$ is pair difference cordial if and only if $m = 2$.

**Proof.** Let $V(K_2 + mK_1) = \{u, v, u_i : 1 \leq i \leq m\}$ and $E(K_2 + mK_1) = \{uu_i, vu_i : 1 \leq i \leq m\} \cup \{uv\}$.

**Case 1.** $m = 2$.

Define $f(u) = -1, f(v) = 1$ and $f(u_1) = 2, f(u_2) = -2$. Then $\Delta f_1^c = 3, \Delta f_1 = 2$.

**Case 2.** $m \geq 3$.

Suppose $f$ is a pair difference cordial. Assume $f(u) = l_1$ and $f(v) = l_2$. To get the edge label 1, the only possibly is that the vertices with degree two receive the label $l_1 - 1$ or $l_1 + 1$ and $l_2 - 1$ or $l_2 + 1$.

**Subcase 1.** $m$ is even.

In this case $\Delta f_1 \leq 2, \Delta f_1^c \geq 2m - 1$. This implies $\Delta f_1^c - \Delta f_1 \geq 2m - 3 > 1$, a contradiction.
Subcase 2. \( m \) is odd.

In this case we may use one vertex label as twice. This implies \( \Delta f_i \leq 3, \Delta f_i^c \geq 2m - 2 \). Therefore \( \Delta f_i^c - \Delta f_i \geq 2m - 5 > 1 \), a contradiction.

\[ \square \]

**Theorem 3.6.** The bistar \( B_{1,n} \) is pair difference cordial if and only if \( 2 \leq n \leq 6 \).

**Proof.** Let \( V(B_{1,n}) = \{ u, v, u_1, v_i : 1 \leq i \leq n \} \) and \( E(B_{1,n}) = \{ uu_1, vv_i, uv : 1 \leq i \leq n \} \).

**Case 1.** \( 2 \leq n \leq 6 \). Define \( f(u) = 2, f(u_1) = 1, f(v) = -2 \) and Table 3 shows that the bistar \( B_{1,n} \), \( 2 \leq n \leq 6 \) is pair difference cordial.

\[
\begin{array}{cccccc}
 n & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
2 & -1 & 2 & & & & \\
3 & -1 & 3 & -3 & & & \\
4 & -1 & -3 & 1 & 3 & & \\
5 & -1 & -3 & -4 & 3 & 4 & \\
6 & -1 & -3 & -4 & 3 & 4 & 1
\end{array}
\]

Table 3

**Case 2.** \( n \geq 7 \).

Suppose \( f(u) = l_1, f(v) = l_2 \), then the maximum value of \( \Delta f_i \) is attained when \( f(u_1) = l_1 - 1, f(v_i) = l_2 - 1, f(v_j) = l_2 + 1 \) for some \( i \) and \( j \). Therefore \( \Delta f_i \leq 1 + 2 = 3 \). That is \( \Delta f_i \leq 3 \). This implies \( \Delta f_i^c \geq n + 2 - 3 \). Therefore \( \Delta f_i^c \geq n - 1 \). Hence \( \Delta f_i^c - \Delta f_i \geq n - 1 - 3 > 1 \), a contradiction.

\[ \square \]

**Theorem 3.7.** The bistar \( B_{m,n}, (m \geq 2, n \geq 2) \) is pair difference cordial if and only if \( m + n \leq 9 \).

**Proof.** Let \( V(B_{m,n}) = \{ u, v, u_i, v_j : 1 \leq i \leq m \leq j \leq n \} \) and \( E(B_{3,n}) = \{ uu_i, vv_j, uv : 1 \leq i \leq n, 1 \leq j \leq n \} \).

There are two cases arises.
**Case 1.** $m + n \leq 9$.

There are two subcases.

**Subcase 1.** $n = m = 2$.

Define $f(u) = 1, f(v) = -1, f(u_1) = 2, f(v_1) = -3, f(v_2) = -2, f(v_2) = 3$. Here $\Delta f_i = 2$ and $\Delta f_i = 3$.

**Subcase 2.** $n > 2, m > 2$.

Define $f : \{\pm 1, \pm 2, \pm 3, \ldots, \pm \frac{m+n}{2}\}$ by $f(u) = 2, f(v) = -2, f(u_1) = 1, f(u_2) = 3, f(v_1) = -1, f(v_2) = -3$. Next assign the remaining labels to the remaining vertices in any order.

**Case 2.** $m + n \geq 10$.

There are two subcases.

**Subcase 1.** $m + n$ is even.

Suppose $f(u) = l_1, f(v) = l_2$, then the maximum value of $\Delta f_i$ is attained when $f(u_i) = l_1 - 1, f(u_j) = l_1 + 1$ for some $i$ and $j$. Therefore $\Delta f_i \leq 2 + 2 = 4$. This implies that $\Delta f_i \geq m + n - 4$. Therefore $\Delta f_i \geq m + n - 3$. Hence $\Delta f_i - \Delta f_i \geq m + n - 7$, a contradiction.

**Subcase 2.** $m + n$ is odd.

When $m + n$ is odd, either $m$ or $n$ is odd. Hence one vertex label is repeated. Therefore $\Delta f_i \leq 3 + 2$. That is $\Delta f_i \leq 5$. This implies $\Delta f_i \geq m + n - 4$. Hence $\Delta f_i - \Delta f_i \geq m + n - 9 > 1$, a contradiction.

Therefore $B_{m,n}, m + n \geq 10$ is not pair difference cordial.

\[\square\]

**Theorem 3.8.** The ladder graph $P_2 \times P_n$ is pair difference cordial for all values of $n$.

**Proof.** Let $V(P_2 \times P_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_2 \times P_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_i v_i : 1 \leq i \leq n\}$.

**Case 1.** $n = 2$.

Let $P_2 \times P_2 \cong C_4$, is pair difference cordial by theorem 3.3.

**Case 2.** $n \geq 3$.

First we assign the labels $-1, -2, -3, \ldots, -n$ to the vertices $u_1, u_2, u_3, \ldots, u_n$ respectively. Now consider the vertices $v_i, (1 \leq i \leq n)$. There are four cases arises.
Subcase 1. \( n \equiv 0 \pmod{4} \).
Assign the labels 1, 2 to the vertices \( v_1, v_2 \) respectively. Next assign the labels 3, 5 respectively to the vertices \( v_3, v_4 \) and assign the labels 4, 6 to the vertices \( v_5, v_6 \) respectively. Now assign the labels 7, 9 to the vertices \( v_7, v_8 \) respectively and assign the labels 8, 10 to the vertices \( v_9, v_{10} \) respectively. Proceeding like this until we reach \( v_n \). Note that in this process the vertex \( v_n \) get the label \( n - 1 \).

Subcase 2. \( n \equiv 1 \pmod{4} \).
As in Subcase 1, assign the labels to the vertices \( v_i \), \( 1 \leq i \leq n \). Here the vertex \( v_n \) receive the label \( n - 1 \).

Subcase 3. \( n \equiv 2 \pmod{4} \).
Assign the labels to the vertices \( v_i \), \( 1 \leq i \leq n \) as in Subcase 1. In this case the vertex \( v_n \) get the label \( n \).

Subcase 4. \( n \equiv 3 \pmod{4} \).
Similar to Subcase 1 assign the labels to the vertices \( v_i \), \( 1 \leq i \leq n \). Note that the vertex \( v_n \) receive the label \( n \).

The Table 4 given below establish that this vertex labeling \( f \) is a pair difference cordial of \( P_n \times P_2 \).

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( \Delta_{f_c} )</th>
<th>( \Delta_{f_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n ) is odd</td>
<td>( \frac{3n-3}{2} )</td>
<td>( \frac{3n-1}{2} )</td>
</tr>
<tr>
<td>( n ) is even</td>
<td>( \frac{3n-2}{2} )</td>
<td>( \frac{3n-2}{2} )</td>
</tr>
</tbody>
</table>

\( \square \)

**Theorem 3.9.** The dumbbell graph \( Db(n, n) \) is pair difference cordial for all values \( n \).

**Proof.** The vertex set and the edge set of \( Db(n, n) \) is given in definition 2.2.
There are four cases arises.
Case 1. $n \equiv 0 \pmod{4}$.
Assign the labels 1, 2 respectively to the vertices $u_1, u_2$ then assign the labels 4, 3 to the vertices $u_3, u_4$. Secondly assign the labels 5, 6 to the vertices $u_5, u_6$ then assign the labels 8, 7 to the vertices $u_7, u_8$. Proceeding like this until we reach the vertex $u_n$. Note that in this the vertex $u_{n-1}$ get the label $n-1$. Next assign the label to the vertices $v_i, 1 \leq i \leq n$. Assign the labels $-1, -2$ to the vertices $v_1, v_2$ then assign the labels $-4, -3$ to the vertices $v_3, v_4$. Secondly assign the labels $-5, -6$ to the vertices $v_5, v_6$ then assign the labels $-8, -7$ to the vertices $v_7, v_8$. Proceeding like this until we reach the vertex $v_n$. Note that in this the vertex $v_n$ receive the label $-n+1$.

Case 2. $n \equiv 1 \pmod{4}$.
Assign the labels 1, 2, 3 to the vertices $u_1, u_2, u_3$ then assign the labels 5, 4 to the vertices $u_4, u_5$. Secondly assign the labels 6, 7 to the vertices $u_6, u_7$ then assign the labels 9, 8 to the vertices $u_8, u_9$. Proceeding like this until we reach the vertex $u_n$. Note that in this the vertex $u_n$ receive the label $n-1$. As in case 1 assign the label to the vertices $v_i, 1 \leq i \leq n$. Note that in this the vertex $v_{n-1}, v_n$ get the label $-n+2, -n$.

Case 3. $n \equiv 2 \pmod{4}$.
As in case 1 assign the label to the vertices $u_i, 1 \leq i \leq n$. Note that in this the vertex $u_{n-1}, u_n$ receive the label $n-1, n$. Assign the label as in case 1 to the vertices $v_i, 1 \leq i \leq n$. Note that in this way the vertex $v_{n-1}, v_n$ get the label $-n+1, -n$.

Case 4. $n \equiv 3 \pmod{4}$.
As in case 1 assign the label to the vertices $u_i, 1 \leq i \leq n$. Note that in this process the vertex $u_{n-1}, u_n$ receive the label $n-1, n$. Assign the label as in case 1 to the vertices $v_i, 1 \leq i \leq n$. Note that here the vertices $v_{n-1}, v_n$ get the label $-n, -n+1$.

The Table 5 given below establish that this vertex labeling $f$ is a pair difference cordial of $D_b(n, n)$.

Theorem 3.10. The dumbbell graph $D_b(n+1, n)$ is pair difference cordial for all values $n$.

Proof. The vertex set and the edge set of $D_b(n+1, n)$ is given in definition 2.2.

Case 1. $n \equiv 0 \pmod{4}$.

Subcase 1. $n > 4$. 
\[
\begin{array}{|c|c|c|}
\hline
\text{Nature of } n & \Delta f_i & \Delta f_c \\
\hline
n \equiv 0 \pmod{4} & n + 1 & n \\
\hline
n \equiv 1 \pmod{4} & n & n + 1 \\
\hline
n \equiv 2 \pmod{4} & n & n + 1 \\
\hline
n \equiv 3 \pmod{4} & n + 1 & n \\
\hline
\end{array}
\]

**Table 5**

Assign the labels 1, 2 respectively to the vertices \(u_1, u_2\) then assign the labels 4, 3 to the vertices \(u_3, u_4\). Secondly assign the labels 5, 6 to the vertices \(u_5, u_6\) then assign the labels 8, 7 to the vertices \(u_7, u_8\). Proceeding like this until we reach the vertex \(u_n\). Next assign the label 2 to the vertex \(u_{n+1}\). Now we consider the vertices \(v_i, 1 \leq i \leq n\). Assign the labels \(-1, -2\) to the vertices \(v_1, v_2\) then assign the labels \(-4, -3\) to the vertices \(v_3, v_4\). Secondly assign the labels \(-5, -6\) to the vertices \(v_5, v_6\) then assign the labels \(-8, -7\) to the vertices \(v_7, v_8\). Proceeding like this until we reach the vertex \(v_n\). Note that in this the vertex \(v_n\) receive the label \(-n + 1\).

**Subcase 2.** \(n = 4\).

As in case 1, assign the labels to the vertices \(u_i, 1 \leq i \leq 4\) and \(v_i, 1 \leq i \leq 4\). Finally assign the label 1 to the vertex \(u_5\).

**Case 2.** \(n \equiv 1 \pmod{4}\).

**Subcase 1.** \(n > 5\).

As in case 1, assign the labels to the vertices \(u_i, 1 \leq i \leq n + 1\). Next consider the vertices \(v_i, 1 \leq i \leq n\). Assign the labels \(-1, -2, -3\) to the vertices \(v_1, v_2, v_3\) then assign the labels \(-5, -4\) to the vertices \(v_4, v_5\). Secondly assign the labels \(-6, -7\) to the vertices \(v_6, v_7\) then assign the labels \(-8, -7\) to the vertices \(v_8, v_9\). Proceeding like this until we reach the vertex \(v_n\). Note that in this the vertex \(v_n\) receive the label \(-n + 1\).

**Subcase 2.** \(n = 5\).

As in case 1, assign the labels to the vertices \(u_i, 1 \leq i \leq 5\) and \(v_i, 1 \leq i \leq 5\). Finally assign the label 1 to the vertex \(u_5\).

**Case 3.** \(n \equiv 2 \pmod{4}\).

As in case 1, assign the labels to the vertices \(u_i, 1 \leq i \leq n + 1\) and \(v_i, 1 \leq i \leq n\).
Case 4. $n \equiv 2 \pmod{4}$.

Subcase 1. $n > 3$.
As in case 2, assign the labels to the vertices $u_i, 1 \leq i \leq n$ and $v_i, 1 \leq i \leq n$. Finally assign the label 1 to the vertex $u_{n+1}$.

Subcase 2. $n = 3$.
Assign the labels $-1, -2, -3$ to the vertices $v_1, v_2, v_3$. Now assign the labels 1, 2, 3 to the vertices $u_1, u_2, u_3$. Finally assign the label 1 to the vertex $u_4$.

□

Theorem 3.11. The dumbbell graph $Db(m,n)$ is pair difference cordial for all values $m > n + 1$.

Proof. Take the vertex set and edge set in definition 2.2.
There are four cases arises.

Case 1. $n \equiv 0 \pmod{4}$.
Assign the labels $-1, -2$ respectively to the vertices $v_1, v_2$ and assign the labels $-4, -3$ to the vertices $v_3, v_4$ respectively. Secondly assign the labels $-5, -6$ to the vertices $v_5, v_6$ respectively. Next assign the labels $-8, -7$ to the vertices $v_7, v_8$ respectively. Proceeding like this until we reach the vertex $v_n$. Note that in this the vertex $v_n$ receive the label $-n + 1$. Next consider the vertices $u_i, 1 \leq i \leq m$.
Assign the labels 1, 2 to the vertices $u_1, u_2$ respectively and assign the labels 4, 3 respectively to the vertices $u_3, u_4$. Now assign the labels 5, 6 to the vertices $u_5, u_6$ respectively and assign the labels 8, 7 respectively to the vertices $u_7, u_8$. Proceeding like this until we reach the vertex $u_n$. Finally consider the remaining $m - n$ vertices. There are four cases arises.

Subcase 1. $m \equiv 0 \pmod{4}$.
Assign the labels $n + 1, n + 2$ to the vertices $u_{n+1}, u_{n+2}$ respectively and assign the labels $-n - 1, -n - 2$ respectively to the vertices $u_{n+3}, u_{n+4}$. Secondly assign the labels $n + 3, n + 4$ to the vertices $u_{n+5}, u_{n+6}$ respectively. Next assign the labels $-n - 3, -n - 4$ respectively to the vertices $u_{n+7}, u_{n+8}$. Proceeding like this until we reach the vertex $u_m$.

Subcase 2. $m \equiv 1 \pmod{4}$.
As in subcase 1 assign the labels to the vertices $u_i, 1 \leq i \leq m - 1$ and assign the label $m - 1$ to the vertex $u_m$. 
Subcase 3. $m \equiv 2 \pmod{4}$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \leq i \leq m - 1$. Next assign the label $\frac{m+n}{2}$ to the vertex $u_m$.

Subcase 4. $m \equiv 3 \pmod{4}$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \leq i \leq m - 3$ and lastly assign the labels $-\frac{m+n}{2}, \frac{m+n}{2}, 2$ respectively to the vertices $u_{m-2}, u_{m-1}, u_m$.

The Table 6 given below establish that this vertex labeling $f$ is a pair difference cordial of $Db(m,n)$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$\Delta f_i$</th>
<th>$\Delta f_i^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \equiv 0 \pmod{4}$</td>
<td>$\frac{m+n}{2}$</td>
<td>$\frac{m+n+2}{2}$</td>
</tr>
<tr>
<td>$m \equiv 1 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
<tr>
<td>$m \equiv 2 \pmod{4}$</td>
<td>$\frac{m+n}{2}$</td>
<td>$\frac{m+n+2}{2}$</td>
</tr>
<tr>
<td>$m \equiv 3 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
</tbody>
</table>

Table 6

Case 2. $n \equiv 1 \pmod{4}$.

Assign the labels as in case 1 to the vertices $v_i, (1 \leq i \leq n)$. Here note that the vertex $v_n$ receive the label $-n + 1$.

Next consider the remaining $m - n$ vertices. There are four cases arises.

Subcase 1. $m \equiv 0 \pmod{4}$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, (1 \leq i \leq m - 3)$ and assign the labels $\frac{m+n-1}{2}, \frac{m+n-1}{2}, 2$ to the vertices $u_{m-2}, u_{m-1}, u_m$ respectively.

Subcase 2. $m \equiv 1 \pmod{4}$.

As in case 1, assign the labels to the vertices $u_i, 1 \leq i \leq m$.

Subcase 3. $m \equiv 2 \pmod{4}$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \leq i \leq m - 1$. Next assign the label 2 to the vertex $u_m$. 
Subcase 4. $m \equiv 3 \pmod{4}$.

Assign the labels as in subcase 1 to the vertices $u_i, 1 \leq i \leq m - 2$. Finally assign the labels $\frac{m+n}{2}, \frac{m+n}{2}$ respectively to the vertices $u_{m-1}, u_m$.

The Table 7 given below establish that this vertex labeling $f$ is a pair difference cordial of $Db(m,n)$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$\Delta_{f_1}$</th>
<th>$\Delta_{f_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \equiv 0 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
<tr>
<td>$m \equiv 1 \pmod{4}$</td>
<td>$\frac{m+n+2}{2}$</td>
<td>$\frac{m+n}{2}$</td>
</tr>
<tr>
<td>$m \equiv 2 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
<tr>
<td>$m \equiv 3 \pmod{4}$</td>
<td>$\frac{m+n}{2}$</td>
<td>$\frac{m+n+2}{2}$</td>
</tr>
</tbody>
</table>

Case 3. $n \equiv 2 \pmod{4}$.

Assign the labels as in case 1 to the vertices $v_i, (1 \leq i \leq n)$. Here note that the vertex $v_n$ received the label $-n$.

Finally we consider the remaining $m-n$ vertices. There are four cases arises.

Subcase 1. $m \equiv 0 \pmod{4}$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, (1 \leq i \leq m - 2)$ and assign the labels $-\frac{m+n}{2}, \frac{m+n}{2}$ respectively to the vertices $u_{m-1}, u_m$.

Subcase 2. $m \equiv 1 \pmod{4}$.

Assign the labels as in subcase 1 of case 1 to the vertices $u_i, (1 \leq i \leq m - 3)$ and assign the labels $\frac{m+n-1}{2}, -\frac{m+n-1}{2}, 2$ to the vertices $u_{m-2}, u_{m-1}, u_m$ respectively.

Subcase 3. $m \equiv 2 \pmod{4}$.

Assign the label as in subcase 1 to the vertices $u_i, (1 \leq i \leq m)$.

Subcase 4. $m \equiv 3 \pmod{4}$.

Assign the label as in subcase 1 to the vertices $u_i, (1 \leq i \leq m - 2)$ and assign the label $-\frac{m+n}{2}, \frac{m+n}{2}$ respectively to the vertices $u_{m-1}, u_m$.

The Table 8 given below establish that this vertex labeling $f$ is a pair difference cordial of $Db(m,n)$. 

Case 4. $n \equiv 3 \pmod{4}$.
Assign the labels as in case 1 to the vertices $v_i, 1 \leq i \leq n$ and $u_i, 1 \leq i \leq n$. Here note that the vertex $v_n$ received the label $-n$.

We now consider the remaining $m - n$ vertices. There are four cases arises.

Subcase 1. $m \equiv 0 \pmod{4}$.
Assign the labels as in subcase 1 of case 1 to the vertices $u_i, n + 1 \leq i \leq m - 1$ and assign the labels 2 to the vertex $u_m$.

Subcase 2. $m \equiv 1 \pmod{4}$.
Assign the labels as in subcase 1 of case 1 to the vertices $u_i, n + 1 \leq i \leq m - 2$ and assign the labels $\frac{m+n-1}{2}, -\frac{m+n-1}{2}$ to the vertices $u_{m-1}, u_m$ respectively.

Subcase 3. $m \equiv 2 \pmod{4}$.
Assign the labels as in subcase 1 of case 1 to the vertices $u_i, n + 1 \leq i \leq m - 3$. Finally assign the labels $\frac{m+n-1}{2}, -\frac{m+n-1}{2}, \frac{m+n-1}{2}$ respectively to the vertices $u_{m-2}, u_{m-1}, u_m$.

Subcase 4. $m \equiv 3 \pmod{4}$.
Assign the label as in subcase 1 to the vertices $u_i, 1 \leq i \leq m$.

The Table 9 given below establish that this vertex labeling $f$ is a pair difference cordial of $Db(m,n)$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$\Delta f_i$</th>
<th>$\Delta f'_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \equiv 0 \pmod{4}$</td>
<td>$\frac{m+n}{2}$</td>
<td>$\frac{m+n+2}{2}$</td>
</tr>
<tr>
<td>$m \equiv 1 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
<tr>
<td>$m \equiv 2 \pmod{4}$</td>
<td>$\frac{m+n+2}{2}$</td>
<td>$\frac{m+n}{2}$</td>
</tr>
<tr>
<td>$m \equiv 3 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
<tr>
<td>Nature of $n$</td>
<td>$\Delta f_1$</td>
<td>$\Delta f_c$</td>
</tr>
<tr>
<td>--------------</td>
<td>-------------</td>
<td>-------------</td>
</tr>
<tr>
<td>$m \equiv 0 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+2+1}{2}$</td>
</tr>
<tr>
<td>$m \equiv 1 \pmod{4}$</td>
<td>$\frac{m+n+2}{2}$</td>
<td>$\frac{m+n}{2}$</td>
</tr>
<tr>
<td>$m \equiv 2 \pmod{4}$</td>
<td>$\frac{m+n+1}{2}$</td>
<td>$\frac{m+n+1}{2}$</td>
</tr>
<tr>
<td>$m \equiv 3 \pmod{4}$</td>
<td>$\frac{m+n+2}{2}$</td>
<td>$\frac{m+n}{2}$</td>
</tr>
</tbody>
</table>

TABLE 9

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**


