CHARACTERIZATIONS OF UNIQUE MAXIMAL SUBMODULE AND STRONG \( m \)-SYSTEM

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Abstract. In this paper we have introduced the concept of strong \( m \)-system in modules over non-commutative rings. Using the strong \( m \)-system, the existence of unique maximal submodule is proved. In fact we have shown that if \( I \) is a submodule and \( S \) is a strong \( m \)-system with \( I \cap S = \emptyset \) then there exists a unique maximal submodule \( P \) with \( I \subseteq P \) such that \( P \cap S = \emptyset \). We have also obtained a characterization for unique maximal submodule.

Keywords: unique maximal submodule; strong \( m \)-system.

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1. INTRODUCTION

Throughout this paper \( R \) stands for a ring with identity and \( M \) stands for a unital left \( R \)-module. Prime submodules over the rings which are not necessarily commutative have been studied in a number of paper, for example [5],[1]. The notion of prime submodules was first introduced by J. Dauns in [2]. A proper submodule \( P \) of \( M \) is called a prime submodule, if for any ideal \( A \) of \( R \) and for any submodule \( N \) of \( M \), \( AN \subseteq P \) implies either \( N \subseteq P \) or \( AM \subseteq P \).

David Ssevviiri[[4],Proposition 4.1.1] has shown that as in ring theory, a prime submodule \( P \) of \( M \), is defined equivalently as, for any \( a \) in \( R \) and for any \( m \) in \( M \), \( aRm \subseteq P \) implies \( m \in P \) or

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\( aM \subseteq P \). So as in ring theory an \( m \)-system in a module should have been defined as follows. A nonempty subset \( S \) of \( M \) is said to be an \( m \)-system if for any \( m \) in \( S \) and for any \( a \) in \( R \) with \( ak \in S \) for some \( k \) in \( M \) then there exists an \( r \) in \( R \) such that \( arm \in S \). One can easily prove that a submodule \( P \) of \( M \) is prime submodule if and only if \( C(P) \), the compliment of \( P \) is an \( m \)-system.

But David Ssevviiri[4] has defined \( m \)-system in modules by taking sum of two submodules instead of taking two elements. Ofcourse they have shown that a submodule is prime if and only if it’s compliment is an \( m \)-system. Following David Ssevviiri[4] we have the definition of an \( m \)-system for modules as follows. A subset \( S \subseteq M \setminus \{0\} \) of \( M \) is an \( m \)-system if for any submodules \( K, L \) and if \( (K + L) \cap S \neq \emptyset \) and \( (K + AM) \cap S \neq \emptyset \) then \( (K + AL) \cap S \neq \emptyset \). In this paper, we have introduced the notion of strong \( m \)-system in modules.

We have shown that a strong \( m \)-system in modules is always an \( m \)-system but the converse need not to be true. We have given an example of an \( m \)-system which is not a strong \( m \)-system. Using strong \( m \)-system we have proved the existence of a unique maximal submodule.

We have shown that if \( I \) is a submodule and \( S \) is a strong \( m \)-system with \( I \cap S = \emptyset \) then there exists a unique maximal submodule \( P \) with \( I \subset P \) such that \( P \cap S = \emptyset \). If \( A, B \) are the submodules of \( M \), one can easily check that \( (A : B) = \{ r \in R : rB \subseteq A \} \) is an ideal of \( R \). For any \( a \in R \), \( \langle a \rangle \) denotes the ideal generated by \( a \).

2. Preliminaries

If \( R \) is any ring with unity, we say that \( M \) is a unital left \( R \)-module if, for any \( r \in R \) and \( m \in M \), an element \( rm \in M \) is defined such that the following conditions hold for all \( m, n \in M \) and \( r, s \in R \):

\[
\begin{align*}
  r(m+n) &= rm + rn \\
  (r+s)m &= rm + sm \\
  (rs)m &= r(sm) \\
  1m &= m
\end{align*}
\]

A subset \( N \) of \( M \) is called an \( R \)-submodule if the following conditions are satisfied:

- \( N \) is a subgroup of the (additive, abelian) group \( M \).
- \( rn \) is in \( N \) for all \( r \in R \) and \( n \in N \).
A proper submodule $P$ of $M$ is called a prime submodule, if for any ideal $A$ of $R$ and for any submodule $N$ of $M$, $AN \subseteq P$ implies either $N \subseteq P$ or $AM \subseteq P$.

The definition of an $m$-system for modules as follows[4]. A subset $S \subseteq M \setminus \{0\}$ of $M$ is an $m$-system if for any submodules $K, L$ and if $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$ then $(K + AL) \cap S \neq \emptyset$.

3. **Main Results**

**Theorem 3.1.** For a proper submodule $P$ of $M$, the axioms that follow are equivalent

(i) $P$ is the unique maximal submodule.

(ii) For any ideal $A$ of $R$ and for submodules $K \neq M, N$ of $M$ such that $K \cap N = \{0\}$ and if $A(K + N) \subseteq P$, then either $K + N \subseteq P$ or $K + AM \subseteq P$.

(iii) For all $a \in R$ and for a submodule $K \neq M$ and for all $m$ in $M$ such that $K \cap Rm = \{0\}$ and if $a(K + Rm) \subseteq P$, then either $K + Rm \subseteq P$ or $K + aM \subseteq P$.

(iv) For all $a \in R$, submodule $K \neq M$ and for all $m$ in $M$ such that $K \cap Rm = \{0\}$ and if $< a > (K + Rm) \subseteq P$, then either $K + Rm \subseteq P$ or $K + < a > M \subseteq P$.

(v) For every left ideal $A \subseteq R$, submodule $K \neq M$ and for all $m$ in $M$

such that $K \cap Rm = \{0\}$ and if $A(K + Rm) \subseteq P$, then either $K + Rm \subseteq P$ or $K + AM \subseteq P$.

(vi) For every right ideal $B \subseteq R$, submodule $K \neq M$ and for all $m$ in $M$ such that $K \cap Rm = \{0\}$ and if $B(K + Rm) \subseteq P$, then either $K + Rm \subseteq P$ or $K + BM \subseteq P$.

**Proof** (i) $\implies$ (ii) Let $A$ be an ideal of $R$ and let $K \neq M, N$ be the submodules of $M$ such that $K \cap N = \{0\}$ and $A(K + N) \subseteq P$. As $P$ is the unique maximal submodule, we have $K \subseteq P$. Suppose $K + N \nsubseteq P$. Then $(K + N) + P = M$. Let $m \in M$. Then there exists $k \in K, n \in N$ and $p \in P$ such that $m = (k + n) + p$. Let $a \in A$ and $k_1 \in K$. Then $k_1 + am = k_1 + a(k + n) + ap \in P$ since $k_1 \in K \subseteq P, a(k_1 + n) \in A(K + N) \subseteq P$ and $ap \in AP \subseteq P$. Hence $K + AM \subseteq P$. 

(ii) $\Rightarrow$ (i) First let us show that $P$ is a maximal submodule. Suppose there exists a submodule $K$ such that $P \subseteq K \subseteq M$ with $K \neq M$. If $A = \{0\}$ and $N = \{0\}$ then $A(K + N) \subseteq P$ and hence by assumption $K \subseteq P$. Thus $P = K$.

Suppose $L$ is any other maximal submodule of $M$. By taking $A = \{0\}$ and $N = \{0\}$ we have $\{0\}(L + \{0\}) \subseteq P$. Hence by (ii) $L \subseteq P$. Hence $P$ is the unique maximal submodule.

(ii) $\Rightarrow$ (iii) Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ be such that $K \cap Rm = \{0\}$. Suppose $a(K + Rm) \subseteq P$. Now, $Ra(K + Rm) \subseteq Ra(K + Rm) \subseteq RP \subseteq P$. Hence $(K + Rm) \subseteq P$ or $K + (Ra)M \subseteq P$. Thus $(K + Rm) \subseteq P$ or $K + aM \subseteq P$.

(iii) $\Rightarrow$ (ii) Let $A$ be an ideal of $R$. Suppose $K \neq M, N$ are submodules of $M$ in such a way that $K \cap N = \{0\}$ and $A(K + N) \subseteq P$. Let $a \in A$ and $n \in N$. Clearly $K \cap Rn = \{0\}$ and we have $a(K + Rn) \subseteq A(K + N) \subseteq P$. This implies that $K + Rn \subseteq P$ or $K + aM \subseteq P$. Hence $K + N \subseteq P$ or $K + AM \subseteq P$.

(iii) $\Rightarrow$ (iv) Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ such that $K \cap Rm = \{0\}$. Suppose $< a > (K + Rm) \subseteq P$. Then for all $x \in < a >, x(K + Rm) \subseteq < a > (K + Rm) \subseteq P$ and from (iii), $K + Rm \subseteq P$ or $K + xM \subseteq P$. This is true for all $x$ in $< a >$. Hence (iv) holds.

(iii) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (vi) are similar to the above proof.

(iv) $\Rightarrow$ (iii) Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ be such that $K \cap Rm = \{0\}$. Suppose $a(K + Rm) \subseteq P$. Then $a \in (P : K + Rm)$. As $P$ and $K + Rm$ are submodules of $M$, $(P : K + Rm)$ is an ideal of $R$. It follows that $< a > (P : K + Rm)$ and hence $< a > (K + Rm) \subseteq P$. Hence $K + Rm \subseteq P$ or $K + < a > M \subseteq P$. Hence it is clear that $K + Rm \subseteq P$ or $K + aM \subseteq P$.

(v) $\Rightarrow$ (iii) Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ be such that $K \cap Rm = \{0\}$. Suppose $a(K + Rm) \subseteq P$. Then $a \in (P : K + Rm)$. It follows that $< a > (P : K + Rm)$ since
Suppose $A$ then

\begin{align*}
\text{(vi) } & \implies \text{(iii) Similar to the above proof.}
\end{align*}

This completes the proof.

4. Unique Maximal Submodules in Terms of Strong $m$-System

A nonempty set $S \subseteq M \setminus \{0\}$ is said to be a $m$-system in the sense of David Ssevviiri[4] if, for each ideal $A \subseteq R$ and for all submodules $K$, $L$ of $M$, if $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$ then $(K + AL) \cap S \neq \emptyset$.

Now we define the notion of strong $m$-system.

**Definition 4.1.** A nonempty set $S \subseteq M \setminus \{0\}$ is called strong $m$-system if for each ideal $A \subseteq R$ and for all submodules $K$, $L$ of $M$ such that $K \cap L = \{0\}$ and if $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$ then $A(K + L) \cap S \neq \emptyset$.

For let us show that every strong $m$-system is an $m$-system. Let $S$ be a strongly $m$-system. Suppose $A$ is an ideal of $R$ and $K$, $L$ are submodules of $M$ such that $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$. If $K \cap L = \{0\}$, then $A(K + L) \cap S \neq \emptyset$. Since $A(K + L) \subseteq (K + AL)$, we have $(K + AL) \cap S \neq \emptyset$.

If $K \cap L \neq \{0\}$, let $K' = K + L$ and $L' = \{0\}$. Then $K' \cap L' = \{0\}$, $(K' + L') \cap S \neq \emptyset$ and $(K' + AM) \cap S \neq \emptyset$. Since $S$ is a strong $m$-system $A(K' + L') \cap S \neq \emptyset$. Since $A(K + L) \subseteq K + AL$, we have $(K + AL) \cap S \neq \emptyset$ and it follows that $S$ is an $m$-system.

Hence every strong $m$-system is an $m$-system. But the converse need not to be true. The following example shows that a $m$-system need not be a strong $m$-system.

Let $R = \mathbb{Z}_6$ be the ring and let the $R$-module be $R_R$. Then the subset $S = \{1, 3, 5\}$ is an $m$-system but not a strong $m$-system. Since $A(K + L) \cap S = \emptyset$ where $A = \{0, 2, 4\}$ is an ideal of $R = \mathbb{Z}_6$, $K = \{0, 3\}$ and $L = \{0, 2, 4\}$ are submodules of $M = R_R$ with $K \cap L = \{0\}$ and $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$. 

$(P : K + Rm)$ is an ideal of $R$. Then $< a > (K + Rm) \subseteq P$.

As $< a >$ is an ideal of $R$, $< a >$ is a left ideal also. Hence $K + Rm \subseteq P$ or $K + < a > M \subseteq P$.

Hence $K + Rm \subseteq P$ or $K + aM \subseteq P$. 

$(vi) \implies (iii)$ Similar to the above proof.

This completes the proof.
In the case of prime submodules, a submodule $P$ of $M$ is prime and $M \setminus P$ is an $m$-system are both equivalent. Now, let us extend the result to unique maximal submodules.

**Theorem 4.2.** Let $M$ be an $R$-module and let $P$ be a submodule of $M$. Then $P$ is a unique maximal submodule if and only if $M \setminus P$ is a strong $m$-system.

**Proof** Let $P$ be a unique maximal submodule. Let $S = M \setminus P$. Let $A$ be an ideal of $R$. Let $K$, $L$ be the submodules of $M$ with $K \cap L = \{0\}$ and $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$. If $A(K + L) \cap S = \emptyset$, then $A(K + L) \subseteq P$. By Theorem 3.1, we have $K + L \subseteq P$ or $K + AM \subseteq P$. Hence $(K + L) \cap S = \emptyset$ or $(K + AM) \cap S = \emptyset$, leads a contradiction. Thus $A(K + L) \cap S \neq \emptyset$ and hence $S$ is a strong $m$-system.

Conversely, let $S = M \setminus P$ be a strong $m$-system. Suppose $A(K + L) \subseteq P$ where $A$ is an ideal of $R$ and $K \neq M$, $L$ are the submodules of $M$ be such that $K \cap L = \{0\}$. If $K + L \not\subseteq P$ and $K + AM \not\subseteq P$, then $(K + L) \cap S \neq \emptyset$ and $(K + AM) \cap S \neq \emptyset$. As $S$ is a strong $m$-system, we have $A(K + L) \cap S \neq \emptyset$. Then $A(K + L) \not\subseteq P$, leads a contradiction. Hence $P$ is a unique maximal submodule.

The well known fact is, if $S \subseteq M$ is an $m$-system and if $P$ is a submodule of $M$ such that $P \cap S = \emptyset$ is maximal in concert with this property, then $P$ is prime submodule. A similar result does hold for strong $m$-system.

**Theorem 4.3.** Let $S \subseteq M$ be non-void strong $m$-system in $M$ and $I$, a submodule of $M$ with $I \cap S = \emptyset$. Then $I$ is contained in a unique maximal submodule $P$ with $P \cap S = \emptyset$.

**Proof** Let $\mathcal{A} = \{ J : J$ is a submodule of $M$ with $I \subseteq J$ and $J \cap S = \emptyset \}$. Clearly $I \in \mathcal{A}$. Then by Zorn’s lemma, $\mathcal{A}$ contains a maximal element say $P$ with $P \cap S = \emptyset$. Now to claim that $P$ is unique maximal submodule of $M$.

If $A(K + L) \subseteq P$ where $A$ is an ideal of $R$ and $K \neq M$, $L$ are the submodules of $M$ in such a manner that $K \cap L = \{0\}$. Suppose $K + L \not\subseteq P$ and $K + AM \not\subseteq P$. Then by the maximality of $P$, we have $P + (K + L)$ and $P + (K + AM)$ are submodules and $P + (K + L) \cap S \neq \emptyset$ and $P + (K + AM) \cap S \neq \emptyset$. Now, let $L' = \{0\}$, the zero submodule of $M$. Then $(P + K + L') \cap S \neq \emptyset$. Since $(P + K + AM) \cap S \neq \emptyset$, this implies $(P + K + L + AM) \cap S \neq \emptyset$. Let $P + K + L = K'$. Then $(K' + L') \cap S \neq \emptyset$ and $(K' + AM) \cap S \neq \emptyset$ with $K' \cap L' = \{0\}$. Since $S$ is a strong $m$-system,
A(K' + L') \cap S \neq \emptyset. Thus A((P + K) + L) \cap S \neq \emptyset.

Since A(P + (K + L)) \subseteq AP + A(K + L) \subseteq P, a contradiction to the fact that P \cap S = \emptyset.

Hence P is a unique maximal submodule of M containing I.

CONCLUSION

In this paper, the properties of unique maximal submodules related with m-system is studied. We have established the concept of strong m-system in modules over non-commutative rings and the existence of unique maximal submodule using the strong m-system. As a future work we will extend the results in semimodules over semirings.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES


