DISJOINT ELEMENTS AND SEMI SOLIDS IN RIESZ $I_{G}$–MODULE

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Abstract. Disjoint elements and semi solids in Riesz $I_{G}$-module are introduced and properties are studied.

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1. INTRODUCTION

Lattice ordered algebraic structures were discussed by Blyth [9] and Steinberg [8]. Based on group action dealt by Gallian [3] and Michel, Zhilinskii [5], representation theory was developed by Curtis, Reiner[2] and Steinberg [1]. This concept was studied in lattice structure which leads to the definition of lattice ordered G-modules by Ursala, Isaac [6] and Riesz $I_{G}$-module by Sowmya, Magie and Ursala [4]. Disjoint elements in Riesz spaces were studied by Luxemburg, Zaanen [10] and Gloden [7]. Solid space (Ideal) of a Riesz space which acts as a black hole was also introduced in [7, 10]. In this paper, the concepts of disjoint elements and semi solids are introduced in a Riesz $I_{G}$–module.

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2. Preliminaries

In this section, some basic definitions and results are reviewed.

Throughout this paper, \( e \) denotes the identity element in the group \( G \) with binary operation \( \ast \) and \( 0 \) denotes the identity element in the vector space \( E \) over the set of reals \( \mathbb{R} \).

Definition 2.1. [8] A partial order on a nonempty set \( L \) is a binary relation on \( L \) that is reflexive, anti-symmetric, and transitive. A partially ordered set or poset is a set in which a partial order is defined.

Definition 2.2. [8] A lattice \( L \) is a poset in which the infimum \( a \land b \) and supremum \( a \lor b \) exist for any two elements \( a \) and \( b \) in \( L \).

Definition 2.3. [9] Let \((G,\ast)\) be a group and \( \leq \) be a partial order on it. Then \( G \) is a lattice ordered group or an \( l \)-group if \((G,\leq)\) is a lattice and the binary operation in \( G \) is order preserving. That is, \( g \leq h \implies x \ast g \ast y \leq x \ast h \ast y \) for all \( x,y,g,h \in G \).

Definition 2.4. [9] An \( l \)-subgroup of \( G \) is a subgroup of \( G \) which is a sublattice of \( G \).

Definition 2.5. [9] Let \( G \) be a lattice-ordered group. The set \( G^+ = \{ g \in G : g \geq e \} \) is the positive cone of \( G \), whose elements are termed as positive elements of \( G \) and the set \( G^- = \{ g \in G : g \leq e \} \) is the negative cone of \( G \) which contains all negative elements of \( G \).

Definition 2.6. [9] Let \( G \) be a lattice-ordered group. Then for every \( g \in G \) the positive part of \( g \) is \( g^+ = g \lor e \in G^+ \), and the negative part is \( g^- = g \land e \in G^- \). The absolute value of \( g \) is \( |g| = g \lor g^{-1} = g^+ * (g^-)^{-1} \) and \( |g| \in G^+ \).

Definition 2.7. [7] A real vector space \( V \) which is a poset is called an ordered vector space if for \( x, y, z \in V \) and \( 0 \leq \alpha \in \mathbb{R} \), \( x \leq y \implies x + z \leq y + z \) and \( \alpha x \leq \alpha y \).

Definition 2.8. [7] An ordered vector space which is a lattice is a vector lattice or Riesz space.

Definition 2.9. [7] Let \( E \) be a Riesz space. Two elements \( x \) and \( y \) in \( E \) are said to be disjoint (denoted as \( x \perp y \)) if \( |x| \land |y| = 0 \).
Theorem 2.10. [7] Let $E$ be a Riesz space. For $x, y \in E$,

(i): If $x \perp y$, then $rx \perp y$ for every real number $r$.

(ii): If $x_1, x_2 \perp y$, then $x_1 + x_2 \perp y$.

(iii): If $x_0 = \sup \{x_i : i \in I\}$ and if $x_i \perp y$ for all $i$, then $x_0 \perp y$.

(iv): If $x \perp y$, then $|x + y| = |x| + |y|$.

Definition 2.11. [7] Let $E$ be a Riesz space. An ideal $A$ is a linear subspace of $E$ such that for $x \in A$ and $|y| \leq |x| \implies y \in A$.

Definition 2.12. [4] Let $G$ be an $l$-group. A Riesz space $E$ is called a Riesz $lG$-module if the group action $G$ on $E$ denoted by $g \circ x \in E$ for all $g \in G$ and $x \in E$ and has the following properties

(i): $e \circ x = x$

(ii): $(g \ast h) \circ x = g \circ (h \circ x)$

(iii): $g \circ (rx + sy) = r(g \circ x) + s(g \circ y)$

(iv): $|g| \circ (x \land y) = (|g| \circ x) \land (|g| \circ y)$

$|g| \circ (x \lor y) = (|g| \circ x) \lor (|g| \circ y)$

$(g \land h) \circ |x| = (g \circ |x|) \land (h \circ |x|)$

$(g \lor h) \circ |x| = (g \circ |x|) \lor (h \circ |x|)$ for all $g, h \in G$, $x, y \in E$, $r, s \in \mathbb{R}$.

Remark 2.13. [4] $g \circ 0 = 0$ for all $g \in G$.

Example 2.14. [4] $\mathbb{R}^2$ is a Riesz $lG$-module under the action of $\mathbb{R}^+$, the set of positive real numbers, where the group action is defined by $r \circ (x, y) = (rx, ry)$, for $r \in \mathbb{R}^+$ and $(x, y) \in \mathbb{R}^2$.

Definition 2.15. [4] Let $E$ be a Riesz $lG$-module. A vector sublattice (Riesz subspace) $F$ of $E$ is a Riesz $lG$-submodule or $RlG$-submodule of $E$ if $F$ itself is a Riesz $lG$-module under the same action of $G$ as that on $E$.

3. Main Results

Theorem 3.1. Let $E$ be a Riesz $lG$-module. Then $G^+$ maps $E^+$ into $E^+$. 
Proof. Let \( x, y \in E \) and \( \hat{g} \in G^+ \).

By condition (iv) in the definition of a *Riesz \( lG\)–module*, \( x \leq y \) shows that \( \hat{g} \circ x \leq \hat{g} \circ y \). Now, \( 0 \leq x \implies 0 = \hat{g} \circ 0 \leq \hat{g} \circ x \). Hence, \( \hat{g} \circ x \in E^+ \). \( \square \)

**Theorem 3.2.** \( G^+ \) sends a Riesz subspace (vector sublattice) to a Riesz subspace (vector sublattice).

Proof. Let \( E \) be a *Riesz \( lG\)–module* and \( K \) be a Riesz subspace (vector sublattice) of \( E \). Then for \( \hat{g} \in G^+ \), we show that \( K' = \{ \hat{g} \circ x : x \in K \} \) is a Riesz subspace (vector sublattice) of \( E \). First, note that \( K' \) is non empty, for, \( 0 = \hat{g} \circ 0 \in K' \). Let \( x, y \in K, \hat{g} \in G^+ \) and \( r \in \mathbb{R} \). Then \( x + y, rx, x \wedge y, x \vee y \in K \). Now \( \hat{g} \circ x + \hat{g} \circ y = \hat{g} \circ (x + y) \in K' \). Also, \( r(\hat{g} \circ x) = \hat{g} \circ (rx) \in K' \). \( \hat{g} \circ x \wedge \hat{g} \circ y = \hat{g} \circ (x \wedge y) \in K' \) and \( \hat{g} \circ x \vee \hat{g} \circ y = \hat{g} \circ (x \vee y) \in K' \). Thus \( K' \) is a Riesz subspace (vector sublattice) of \( E \). \( \square \)

**Definition 3.3.** A *Riesz \( lG\)–module* \( E \) is said to be *distributive \( RlG\)–module*, if \( \hat{g} \circ (x \wedge y) = \hat{g} \circ x \wedge \hat{g} \circ y \) and \( \hat{g} \circ (x \vee y) = \hat{g} \circ x \vee \hat{g} \circ y \) holds for all \( g \in G \).

**Example 3.4.** The real plane \( \mathbb{R}^2 \) is a distributive *RlG–module* under the action (as in Example 2.14) of the group \( \mathbb{R}^+ \).

**Theorem 3.5.** Let \( E \) be a distributive *RlG–module* and \( K \) be a Riesz subspace of \( E \). For \( g \in G, \) let \( K' = \{ g \circ x : x \in K \} \) is a Riesz subspace of \( E \).

Proof. Since \( E \) is a distributive *RlG–module*, from theorem 3.2 it follows that \( K' \) is a Riesz subspace of \( E \). \( \square \)

**Theorem 3.6.** For \( g \in G, x \in E, \ | g \circ x | = | g | \circ | x | \). Hence, for \( g \in G^+, g \circ | x | = | g \circ x | \).

Proof. \(| g | \circ | x | = | g | \circ (x \vee (-x)) = (| g | \circ x) \vee (| g | \circ (-x)) \) (by condition (iv) in the definition of a *Riesz \( lG\)–module*)

\[ = (| g | \circ x) \vee -(| g | \circ x) = | | g | \circ x | \]. The second result follows immediately. \( \square \)

**Theorem 3.7.** Let \( x \) and \( y \) be two disjoint elements of \( E \). Then \( g \circ x \) and \( g \circ y \) are disjoint for all \( g \in G^+ \).
Definition 3.8. Two elements $x$ and $y$ in a Riesz $lG-$module $E$ are said to be $Rlg-$disjoint denoted by $x \perp^{Rlg} y$ if $|g \circ x| \land |g \circ y| = 0$ for some $g \in G^+$. That is, if $g \circ x$ and $g \circ y$ are disjoint for some $g \in G^+$. If $x$ and $y$ are $Rlg-$disjoint for all $g \in G^+$, then they are called $Rlg-$disjoint.

Remark 3.9. In a Riesz $lG-$module $E$, the identity element $0$ is $Rlg-$disjoint to all other elements in $E$.

Remark 3.10. If $x$ and $y$ are disjoint $(x \perp y)$, then they are $Rlg-$disjoint.

Theorem 3.11. Let $E$ be a Riesz $lG-$module. Let $g \in G^+$. If $x$ and $y$ are $Rlg-$disjoint, then $|g \circ (x+y)| = |g \circ x| + |g \circ y|$.

Proof. If $x$ and $y$ are $Rlg-$disjoint, then $|g \circ x| \land |g \circ y| = 0$ for $g \in G^+$. That is, $g \circ x$ and $g \circ y$ are disjoint. Therefore, $|g \circ x + g \circ y| = |g \circ x| + |g \circ y|$. Hence, $|g \circ (x+y)| = |g \circ x| + |g \circ y|$.

Theorem 3.12. Let $x, y \in E$ and fix $g \in G^+$. Let $y^{\perp_{Rlg}} = \{x : x \perp^{Rlg} y\}$ denotes the set of all elements of $E$ which are $Rlg-$disjoint to $y$. Then $y^{\perp_{Rlg}}$ is a linear subspace of $E$.

Proof. Note that $y^{\perp_{Rlg}}$ is nonempty as $0 \in y^{\perp_{Rlg}}$. Let $x, z \in y^{\perp_{Rlg}}$ and $g \in G^+$. Then $|g \circ x| \land |g \circ y| = 0$ and $|g \circ z| \land |g \circ y| = 0$. That is, $g \circ x$ and $g \circ z$ are disjoint to $g \circ y$. Then, $(g \circ x + g \circ z) \perp g \circ y$. Therefore, $g \circ (x+z) \perp g \circ y$. Hence $x+z \in y^{\perp_{Rlg}}$.

Let $r \in \mathbb{R}$. Now $x \perp y$ implies $rx \perp y$. Since, $x$ and $y$ are $Rlg-$disjoint, $g \circ x$ is disjoint to $g \circ y$ which in turn shows that $r(g \circ x) \perp (g \circ y)$. But, $r(g \circ x) = g \circ (rx)$. Hence, $rx \in y^{\perp_{Rlg}}$.

Theorem 3.13. Let $E$ be a Riesz $lG-$module and $y \in E$. For $g \in G^+$, the set of distinct nonzero elements which are pairwise $Rlg-$disjoint is linearly independent.
Theorem 3.19. Intersection of any two \( R_lG \) denotes the set of all elements of \( E \) that are \( R_lG \)

Let \( E \) be a \( Riesz \) \( lG \) of \( E \). Then \( x_1 \perp R_lG \) \( x_1 \). That is, \( |g \circ x_1| \wedge |g \circ x_1| = 0 \). Hence, \( |g \circ x_1| = 0 \). That is, \( g \circ x_1 = 0 \implies x_1 = 0 \) which contradicts the choice of elements. \( \square \)

The positive cone \( G^+ \) maps \( E^+ \) onto \( E^+ \) (3.1). This made us to define the following.

Definition 3.14. \( z \in E^+ \implies g \circ z \in E^+ \) for all \( g \in G \), then \( G \) is said to be \( RlG - strict \) on \( z \). The \( l\)-group \( G \) is said to be \( RlG - strict \) on \( E \), if \( G \) is \( RlG - strict \) on \( x \) for every \( x \in E^+ \).

Theorem 3.15. Let \( E \) be a \( Riesz \) \( lG \) module and \( x,y \in E \). Then \( G \) is \( RlG - strict \) on \( E \) if and only if \( x \leq y \iff g \circ x \leq g \circ y \) for all \( g \in G \).

Theorem 3.16. Let \( E \) be a \( Riesz \) \( lG \) module and \( I \) is an ideal of \( E \). Let \( g \in G^+ \). Suppose that \( G \) is \( RlG - strict \) on \( E \). Then \( I' = \{ g \circ x : x \in I \} \) is an ideal of \( E \).

\( \square \)

Definition 3.17. Let \( E \) be a \( Riesz \) \( lG \) module and \( S \) be a vector subspace of \( E \). Then \( S \) is called a \( RlG \) semi solid in \( E \) if for any \( g \in G^+ \), \( x \in S, y \in E \), \( |g \circ y| \leq |g \circ x| \implies y \in S \).

Theorem 3.18. Let \( E \) be a \( Riesz \) \( lG \) module and \( D \) be a nonempty subset of \( E^+ \). Let \( D^{\perp R_lG} = \{ x : x \perp R_lG y \text{ for all } y \in D \} \). Then \( D^{\perp R_lG} \) is a \( RlG \) semi solid in \( E \). The set \( D^{\perp R_lG} \) denotes the set of all elements of \( E \) that are \( RlG \) disjoint to every \( y \in D \).

\( \square \)

Theorem 3.19. Intersection of any two \( RlG \) semi solids is again an \( RlG \) semi solid.
Proof. Let $E$ be a $Riesz\ lG -$ module and $I_1, I_2$ be two $RlG -$semi solids in $E$. Then $I_1 \cap I_2$ is a vector subspace of $E$. Let $z \in E$. Suppose $x \in I_1 \cap I_2$, and $|g \circ z| \leq |g \circ x|$. Since, $x \in I_1 : |g \circ z| \leq |g \circ x| \implies z \in I_1$. Since, $x \in I_2 : |g \circ z| \leq |g \circ x| \implies z \in I_2$. Therefore, $z \in I_1 \cap I_2$. □

Definition 3.20. Let $D$ be a nonempty subset of $E$. The intersection of $RlG -$semi solids in $E$ containing $D$ is an $RlG -$semi solid in $E$ and contains $D$. It is called an $RlG -$semi solid generated by $D$. If $D$ contains only one element, then it is called a principal $RlG -$semi solid.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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