(1 + ϑ)-CONSTACYCLIC CODES OVER \( \mathbb{Z}_8 + ϑ\mathbb{Z}_8 \)

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Abstract. In this paper, (1 + ϑ)-constacyclic codes of arbitrary length \( m \) over a non-chain finite local frobenious ring \( \mathbb{Z}_8 + ϑ\mathbb{Z}_8 \) are introduced. A new Gray map is constructed from \( \mathbb{Z}_8 + ϑ\mathbb{Z}_8 \) to \( \mathbb{Z}^8_8 \) and proved that the \( \mathbb{Z}_8 \)-Gray image of (1 + ϑ)-constacyclic codes having prescribed length \( m \) over the ring \( \mathbb{Z}_8 + ϑ\mathbb{Z}_8 \) is a cyclic code of length \( 8m \) over the ring \( \mathbb{Z}_8 \). Moreover, it has been obtained that the binary image of the (1 + ϑ)-constacyclic code of length \( m \) over \( \mathbb{Z}_8 + ϑ\mathbb{Z}_8 \) is a distance invariant binary quasi-cyclic code of length 32\( m \) with index 16.

Keywords: constacyclic code; gray map; distance invariant; cyclic code; quasi-cyclic code.

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1. BACKGROUND

Many optimal binary linear codes have been studied from codes over several new classes of rings via some Gray map. Over the ring \( F_2 + uF_2 + vF_2 + uvF_2 \), linear codes are discussed in [1], self dual codes in [2], cyclic codes in [3] and (1 + u)-constacyclic codes are described in [4] alongwith the construction of many optimal binary linear codes. More generally, cyclic codes over the ring \( R_8 \) were investigated in [12]. The rings mentioned above are not finite chain rings, however have rich algebraic structures and produce binary codes with large automorphism

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groups and new binary self-dual codes. This demonstrates that the linear codes over such non-chain rings have been received increasing attention to the authors (see [10]-[12], [14]). More recently, linear codes over the non-chain ring \( Z_4 + uZ_4 \), where \( u^2 = 0 \), has been explored in [6]. Also, linear codes over the non-chain ring \( Z_8 + uZ_8 \), with \( u^2 = 0 \), were obtained in [14]. \((1 + u)\)-constacyclic codes over \( Z_4 + uZ_4 \) and a class of constacyclic codes over \( F_p + uF_p \) and its gray image were studied in [7] and [13] respectively. Motivated by the work over the ring presented in [7] and [14], we focus on the construction of the constacyclic codes over the ring \( Z_8 + \vartheta Z_8 \), with \( \vartheta^2 = 0 \) and intent to establish some good binary codes from such codes.

2. The Ring \( Z_8 + \vartheta Z_8 \)

Throughout this paper, the ring \( Z_8 + \vartheta Z_8 \) with \( \vartheta^2 = 0 \) is denoted by \( R \). An arbitrary element \( a + \vartheta b \) is a unit in \( R \) if and only if \( a \) is a unit in \( Z_8 \). The ring \( R \) is a local Frobenius ring and a finite non-chain ring having total of 12 ideals defined as

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Ideals</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>( \mathcal{I}_0 = {0} )</td>
</tr>
<tr>
<td>2</td>
<td>( \mathcal{I}_1 = Z_8 + \vartheta Z_8 )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{I}_2 = {a + \vartheta b : a, b \in {0, 2, 4, 6}} )</td>
</tr>
<tr>
<td>4</td>
<td>( \mathcal{I}_4 = {0, 4, 4\vartheta, 4 + 4\vartheta} )</td>
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<tr>
<td>5</td>
<td>( \mathcal{I}_6 = {a \vartheta : a \in Z_8} )</td>
</tr>
<tr>
<td>6</td>
<td>( \mathcal{I}_{2\vartheta} = {a \vartheta : a \in {0, 2, 4, 6}} )</td>
</tr>
<tr>
<td>7</td>
<td>( \mathcal{I}_{4\vartheta} = {0, 4\vartheta} )</td>
</tr>
<tr>
<td>8</td>
<td>( \mathcal{I}_{2+\vartheta} = {0, 2\vartheta, 4\vartheta, 6\vartheta, 2 + \vartheta, 2 + 3\vartheta, 2 \pm 5\vartheta, 2 \pm 7\vartheta, 4, 4 + 2\vartheta, 4 + 4\vartheta, 4 + 6\vartheta, 6 + \vartheta, 6 + \vartheta, 6 + 3\vartheta, 6 + 5\vartheta, 6 + 7\vartheta} )</td>
</tr>
<tr>
<td>9</td>
<td>( \mathcal{I}_{4+\vartheta} = {0, 2\vartheta, 4\vartheta, 6\vartheta, 4 + \vartheta, 4 + 3\vartheta, 4 + 5\vartheta, 4 + 7\vartheta} )</td>
</tr>
<tr>
<td>10</td>
<td>( \mathcal{I}_{4+2\vartheta} = {0, 4\vartheta, 4 + 2\vartheta, 4 + 6\vartheta} )</td>
</tr>
<tr>
<td>11</td>
<td>( \mathcal{I}_{4+\vartheta} = {a + b \vartheta : a \in {0, 4}, b \in Z_8} )</td>
</tr>
<tr>
<td>12</td>
<td>( \mathcal{I}_{4+2\vartheta} = {0, 4, 2\vartheta, 4\vartheta, 6\vartheta, 4 + 2\vartheta, 4 + 4\vartheta, 4 + 6\vartheta} )</td>
</tr>
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A non-empty subset \( \hat{A} \) over \( R^m \) of length \( m \) is said to be a linear code, if it is an \( R \)-submodule of \( R^m \).
Now, defining the mappings $\varpi$, $\gamma$ and $\zeta$ from $R^m$ to $R^m$ as follows:

\[
\varpi(c_0,c_1,\ldots,c_{m-1}) = (c_{m-1},c_0,c_1,\ldots,c_{m-2}),
\]
\[
\gamma(c_0,c_1,\ldots,c_{m-1}) = (-c_{m-1},c_0,c_1,\ldots,c_{m-2}),
\]
\[
\zeta(c_0,c_1,\ldots,c_{m-1}) = (\zeta c_{m-1},c_0,c_1,\ldots,c_{m-2}).
\]

Here, the defined mappings $\varpi$, $\gamma$ and $\zeta$ are known as the cyclic, negacyclic and constacyclic shift respectively. Moreover, $\hat{A}$ is a cyclic code, negacyclic code and $\zeta$-constacyclic code if $\varpi(\hat{A}) = \hat{A}$, $\gamma(\hat{A}) = \hat{A}$ and $\zeta(\hat{A}) = \hat{A}$ respectively.

The polynomial representation of the codeword $c = (c_0,c_1,\ldots,c_{m-1})$ is $c(x) = c_0 + c_1 x + \ldots + c_{m-1} x^{m-1}$ and $xc(x)$ corresponds to a $\zeta$-constacyclic shift of $c(x)$ in the ring $R[x]/<x^m - \zeta>$. Thus, $\zeta$-constacyclic codes of length $m$ over $R$ can be identified as ideals in the ring $R[x]/<x^m - \zeta>$. Thus, we have the following proposition.

**Proposition 2.1.** A subset $C$ of $R^m$ is a linear cyclic code of length $m$ if and only if $C$ is an ideal of $S_m = R[x]/<x^m - 1>$. A subset $C$ of $R^m$ is a linear $(1+\varpi)$-constacyclic code of length $m$ over $R$ if and only if $C$ is an ideal of $B_m = R[x]/<x^m - 1 - \varpi>$.

A unique set of generators for cyclic codes over $Z_8$ are discussed in the next lemma.

**Lemma 2.2.** Let $C$ be a cyclic code of length $m$ over $Z_8$. Then,

1. If $m$ is odd then, $C = <g(x),4a(x)> = <g(x) + 4a(x)>$, where $g(x)$, $a(x)$ are binary polynomials with $a(x)|g(x)|(x^m - 1) \mod 2$.
2. If $m$ is even, then
   
   (i): If $g(x) = a(x)$ then, $C = <g(x),4a(x)> = <g(x) + 4a(x)>$, where $g(x),a(x)$ are the binary polynomials with $g(x)|(x^m - 1) \mod 2$, and $g(x)|p(x)(x^m - 1)/g(x)$.
   
   (ii): $C = <g(x) + 4p(x),4a(x)>$, where $g(x),a(x)$ and $p(x)$ are the binary polynomials with $a(x)|g(x)|(x^m - 1) \mod 2$, $a(x)|p(x)(x^m - 1)/g(x)$ and $\deg(g(x)) > \deg(a(x)) > \deg(p(x))$. 

For a linear code $C$ of length $m$ over $\mathcal{R}$, the two linear codes: Torsion code, $\text{Tor}(C)$ and Residue code, $\text{Res}(C)$ of length $m$ over $\mathbb{Z}_8$ are defined as:

$$\text{Tor}(C) = \{x \in \mathbb{Z}_8^m \mid \vartheta x \in C\},$$
$$\text{Res}(C) = \{x \in \mathbb{Z}_8^m \mid \exists y \in \mathbb{Z}_8^m : x + \vartheta y \in C\}.$$  

The homomorphism $\varphi : \mathcal{R} \to \mathbb{Z}_8$ as $\varphi(a + \vartheta b) = a$, extends naturally to a ring homomorphism $\varphi : \mathcal{R}^m \to \frac{\mathbb{Z}_8[x]}{(x^m - 1)}$ defined as

$$\varphi(c_0 + c_1x + ... + c_{m-1}x^{m-1}) = \varphi(c_0) + \varphi(c_1)x + ... + \varphi(c_{m-1})x^{m-1}.$$  

Acting $\varphi$ on $C$ over $\mathcal{R}$, define a ring homomorphism $\varphi : C \to \text{Res}(C)$ as $\varphi(a + \vartheta b) = a$, where $a, b \in \mathbb{Z}_8$ with $\text{Ker}\varphi \cong \text{Tor}(C)$ and $\varphi(C) = \text{Res}(C)$.

By the application of first isomorphism theorem of finite groups, $|C| = |\text{Tor}(C)||\text{Res}(C)|$. Also, the image of $C$ under the map $\varphi$ is a cyclic code of length $m$ over $\mathbb{Z}_8$.

Combining the above result with lemma 2.2, the set of generators for cyclic code of length $m$ over $\mathcal{R}$ can be obtained as provided in following theorem.

**Theorem 2.3.** Let $C$ be a $(1 + \vartheta)$-constacyclic code of length $m$ over $\mathcal{R}$. Then

1. If $m$ is odd then, $C = \langle g_1(x), 4a_1(x) + \vartheta b(x), \vartheta(g_2(x) + 4a_2(x)) \rangle$, where $b(x)$ is a polynomial in $\mathbb{Z}_8[x]$ and for $i = 1, 2$, $g_i(x), a_i(x)$ are the binary polynomials with $a_i(x) \mid g_i(x) \mid (x^m - 1) \mod 2$.

2. If $m$ is even then,
   
   (i): If $g_i(x) = a_i(x)$ then, $C = \langle g_1(x) + 4p_1(x) + \vartheta d_1, \vartheta(g_2(x) + 4p_2(x)) \rangle$, where $b(x)$ is a polynomial in $\mathbb{Z}_8[x]$, and for $i = 1, 2$, $g_i(x), a_i(x)$ are the binary polynomial with $g_i(x) \mid (x^m - 1) \mod 2$, and $g_i(x) \mid p_i(x) \frac{(x^m - 1)}{g_i(x)}$.

   (ii): $C = \langle g_1(x) + 4p_1(x) + \vartheta e_1(x), 4a_1(x) + \vartheta e_2(x), \vartheta g_2(x) + 4\vartheta p_2(x), 4a_2(x) \rangle$, where $g(x), a(x)$ and $p(x)$ are the binary polynomials with $a(x) \mid g(x) \mid (x^m - 1) \mod 2$, $a(x) \mid p(x) \frac{(x^m - 1)}{g(x)}$ and $\deg(g(x)) > \deg(a(x)) > \deg(p(x))$. 

3. Gray Maps

Gray images of \((1 + \vartheta)\)-constacyclic codes over \(\mathcal{R}\)

The gray map \(\rho_1\) from \(\mathbb{Z}_8^m\) to \(\mathbb{Z}_2^{4m}\) defined as

\[
\rho_1(z) = (q + r, r, p + r, q + r),
\]

where \(z = p + 2q + 4r\) with \(p, q, r \in \mathbb{Z}_2\), is a distance preserving map from \(\mathbb{Z}_8^m\) (Lee distance) to \(\mathbb{Z}_2^{4m}\) (Hamming distance) and can be extended to \(\mathbb{Z}_8^m\) as:

\[
\rho_1 : \mathbb{Z}_8^m \rightarrow \mathbb{Z}_2^{4m},
\]

\[
\rho_1(z_0, z_1, \ldots, z_{m-1}) = (q_0 + r_0, \ldots, q_{m-1} + r_{m-1}, r_0, \ldots, r_{m-1}, p_0 + r_0, \ldots, p_{m-1} + r_{m-1}, q_0 + r_0, \ldots, q_{m-1} + r_{m-1}).
\]

Now, defining a new gray map \(\rho_2\) from \(\mathcal{R}^m\) to \(\mathbb{Z}_8^{5n}\) as

\[
\rho_2(c) = (b + 7a, b + 6a, b + 5a, b + 4a, b + 3a, b + 2a, b + a, b),
\]

where \(c = a + ub\) and \(a, b \in \mathbb{Z}_8\) and can also be extended from \(\mathcal{R}^m\) to \(\mathbb{Z}_8\) as

\[
\rho_2(c_0, c_1, \ldots, c_{m-1}) = (b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-1}).
\]

where \(c_i = a_i + \vartheta b_i\) and \(a_i, b_i \in \mathbb{Z}_8\).

It is well known that the homogeneous weight has many applications for codes over finite rings and provides a good metric for the underlying ring in constructing superior codes. Next, a homogeneous weight on \(\mathcal{R}\) is defined after defining of the homogeneous weight on arbitrary finite ring \(\mathcal{K}\).

**Definition 3.1.** A real valued function \(w\) on the finite ring \(\mathcal{K}\) is called a left homogeneous weight if \(w(0) = 0\) and the following holds:

1. For all \(x, y \in \mathcal{K}\), \(\mathcal{K} x = \mathcal{K} y\) implies \(w(x) = w(y)\).
(2) There exists a real number $\gamma$ such that

$$\sum_{y \in \mathcal{K}(x)} w(y) = \gamma |\mathcal{K}| x$$

for all $x \in \mathcal{K} \setminus \{0\}$.

The Right homogeneous weight can be defined in a similar manner and if weight is both left homogeneous and right homogeneous, it is known as a homogeneous weight. For any element $c = a + \vartheta b \in \mathcal{R}$; the homogeneous weight denoted by $w_{\text{hom}}(c)$, as $w_L(b + 7a, b + 6a, ..., b + a, b)$.

By simple calculations the weight of any element $c = a + \vartheta b \in \mathcal{R}$ is:

$$w_{\text{hom}}(x) = \begin{cases} 
0 & \text{if } c = 0, \\
8 & \text{if } c = \vartheta, 7\vartheta, \\
24 & \text{if } c = 3\vartheta, 5\vartheta, \\
32 & \text{if } c = 4\vartheta, \\
16 & \text{if otherwise}.
\end{cases}$$

It is easy to verify that, the above defined weight meets the conditions of the Definition 3.1, hence it is actually a homogeneous weight on $\mathcal{R}$. The homogeneous distance of a linear code $C$ over $\mathcal{R}$, denoted by $d_{\text{hom}}(C)$, is defined as the minimum homogeneous weight of the non-zero codewords of $C$.

The map $\rho_2$ is a distance preserving map from $\mathcal{R}^m$(homogeneous distance) to $Z_{8}^{8m}$(Lee distance). Thus, we have the following three distance preserving maps:

$$\rho_1 : (Z_{8}^{m}, \text{Lee Distance}) \rightarrow (Z_{2}^{4m}, \text{Hamming Distance})$$

$$\rho_2 : (\mathcal{R}^m, \text{Homogeneous Distance}) \rightarrow (Z_{8}^{8m}, \text{Lee Distance})$$

$$\rho = \rho_1 \rho_2 : (\mathcal{R}^m, \text{Homogeneous Distance}) \rightarrow (Z_{2}^{32m}, \text{Hamming Distance})$$

4. $(1 + \vartheta)$-Constacyclic Codes

The following theorem defined a result on the above defined map $\rho_2$.

**Theorem 4.1.** If $\zeta$ is a $(1 + \vartheta)$ - constacyclic shift on $\mathcal{R}^m$, $\varpi$ is a cyclic shift on $Z_{8}^{8m}$ and $\rho_2$ be a map defined as above, then $\rho_2 \zeta = \varpi \rho_2$. 
Proof. If \( c = (c_0, c_1, \ldots, c_{m-1}) \in \mathbb{R}^m \) where \( c_i = a_i + \vartheta b_i \) and \( a_i, b_i \in \mathbb{Z}_8 \) for \( 0 \leq i \leq m - 1 \). The definition of the map \( \rho_2 \), implies

\[
\rho_2(c) = (b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-1}),
\]

and

\[
\vartheta \rho_2(c) = (b_{m-1}, b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-2}).
\]

On the other hand,

\[
\zeta(c) = ((1 + \vartheta)c_{m-1}, c_0, c_1, \ldots, c_{m-2})
\]

\[
= ((1 + \vartheta)(a_{m-1} + \vartheta b_{m-1}), a_0 + \vartheta b_0, a_1 + \vartheta b_1, \ldots, a_{m-2} + \vartheta b_{m-2}),
\]

and therefore,

\[
\rho_2 \zeta(c) = (b_{m-1} + a_{m-1} + 7a_{m-1}, b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-1}).
\]

Hence, the result follows. \( \square \)

**Theorem 4.2.** A linear code \( \mathcal{C} \) of length \( m \) over \( \mathbb{R} \) is a \( (1 + \vartheta) \)-constacyclic code if and only if \( \rho_2(\mathcal{C}) \) is a cyclic code of length \( 8m \) over \( \mathbb{Z}_8 \).

**Proof.** If \( \mathcal{C} \) is a \( (1 + \vartheta) \)-constacyclic code, then Theorem 4.1 implies

\[
\vartheta(\rho_2(\mathcal{C})) = \rho_2(\zeta(\mathcal{C})) = \rho_2(\mathcal{C}).
\]
Hence, $\rho_2(\mathcal{C})$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$. Further, if $\rho_2(c)$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$, then use Theorem 4.1 to obtain

$$\rho_2(\zeta(\mathcal{C})) = \sigma(\rho_2(\mathcal{C})) = \rho_2(\mathcal{C})$$

Since, $\rho_2$ is an injective mapping, therefore $\zeta(\mathcal{C}) = \mathcal{C}$ and hence, the result holds. \qed

The following corollary is an immediate consequence of above theorem.

**Corollary 4.3.** The image of $(1 + \vartheta)$-constacyclic code of length $m$ over $\mathcal{R}$ under the map $\rho_2$ is a distance invariant cyclic code of length $8m$ over $\mathbb{Z}_8$.

If $\sigma$ is a cyclic shift, then for a positive integer $s$, the quasi-shift $\sigma_s$ is given by

$$\sigma_s(a^{(1)}|a^{(2)}|...|a^{(s)}) = (\sigma(a^{(1)})|\sigma(a^{(2)})|...|\sigma(a^{(s)})),$$

where $a^{(1)}, a^{(2)}, ..., a^{(s)} \in F_2^{(2m)}$ and “|” represents the usual vector concatenation. A binary quasi-cyclic code $\mathcal{C}$ of index $s$ and length $2ms$ is a subset of $(\mathbb{Z}_2^{2m})^s$ such that $\sigma_s(\mathcal{C}) = \mathcal{C}$.

**Lemma 4.4.** If $\zeta$ is a $(1 + \vartheta)$-constacyclic shift on $\mathcal{R}^m$ and $\rho$ be a mapping defined as above, then $\rho \zeta = \sigma_{16} \rho$.

**Proof.** For $r = (r_0, r_1, ..., r_{m-1}) \in \mathcal{R}^m$, where $r_i = a_i + 2b_i + 4c_i + \vartheta d_i + 2\vartheta e_i + 4\vartheta f_i, a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{Z}_2$, for $0 \leq i \leq m - 1$. Then,
\[ \rho(r) = (c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, b_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + e_0 + f_0, \ldots, \\ a_{m-1} + e_{m-1} + f_{m-1}, a_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + e_0 + f_0, \ldots, b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + e_0 + f_0, \ldots, \\ a_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + f_0, \ldots, a_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, b_0 + c_0 + f_0, \ldots, b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, b_0 + c_0 + f_0, \ldots, \\ b_{m-1} + a_0 + d_0 + f_0, \ldots, b_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, a_0 + b_0 + d_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, \\ b_0 + d_0 + f_0, \ldots, b_{m-1} + d_{m-1} + f_{m-1}, a_0 + c_0 + d_0 + f_0, \ldots, a_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, b_0 + d_0 + f_0, \ldots, \\ a_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, \\ c_0 + e_0 + f_0, \ldots, b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, e_0 + f_0, \ldots, e_{m-1} + f_{m-1}) \]
and therefore,

\[ \mathcal{O}_{16}(r) = (b_{m-1} + e_{m-1} + f_{m-1}, c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, \\
\quad b_{m-2} + e_{m-2} + f_{m-2}, a_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} \\
\quad + b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + e_0 + f_0, \ldots, a_{m-2} + e_{m-2} + f_{m-2}, a_{m-1} \\
\quad + b_{m-1} + e_{m-1} + f_{m-1}, a_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, \\
\quad a_0 + b_0 + e_0 + f_0, \ldots, a_{m-2} + b_{m-2} + e_{m-2} + f_{m-2}, e_{m-1} + f_{m-1}, b_0 + c_0 + e_0 + f_0, \ldots, \\
\quad + f_0, \ldots, b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, e_0 + f_0, \ldots, e_{m-2} + f_{m-2}, a_{m-1} + b_{m-1} \\
\quad + f_{m-1}, a_0 + b_0 + c_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + f_{m-1}, a_0 + b_0 + f_0, \ldots, \\
\quad a_{m-2} + b_{m-2} + f_{m-2}, a_{m-1} + f_{m-1}, a_0 + c_0 + f_0, \ldots, a_{m-1} + c_{m-1} + f_{m-1}, \\
\quad a_0 + f_0, \ldots, a_{m-2} + f_{m-2}, b_{m-1} + f_{m-1}, b_0 + c_0 + f_0, \ldots, b_{m-1} + c_{m-1} + f_{m-1}, \\
\quad b_0 + f_0, \ldots, b_{m-2} + f_{m-2}, f_{m-1}, c_0 + f_0, \ldots, c_{m-1} + f_{m-1}, f_0, \ldots, f_{m-2}, a_{m-1} \\
\quad + b_{m-1} + d_{m-1} + f_{m-1}, b_0 + c_0 + d_0 + f_0, \ldots, b_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, \\
\quad a_0 + b_0 + d_0 + f_0, \ldots, a_{m-2} + b_{m-2} + d_{m-2} + f_{m-2}, a_{m-1} + d_{m-1} + f_{m-1}, c_0 \\
\quad + d_0 + f_0, \ldots, c_{m-1} + d_{m-1} + f_{m-1}, a_0 + d_0 + f_0, \ldots, a_{m-2} + d_{m-2} + f_{m-2}, \\
\quad b_{m-1} + d_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + d_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + d_{m-1} \\
\quad + f_{m-1}, b_0 + d_0 + f_0, \ldots, b_{m-2} + d_{m-2} + f_{m-2}, d_{m-1} + f_{m-1}, a_0 + c_0 + d_0 \\
\quad + f_0, \ldots, a_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, d_0 + f_0, \ldots, d_{m-2} + f_{m-2}, b_{m-1} + e_{m-1} \\
\quad + f_{m-1}, c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, b_{m-2} + e_{m-2} \\
\quad + f_{m-2}, a_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} \\
\quad + e_{m-1} + f_{m-1}, a_0 + e_0 + f_0, \ldots, a_{m-2} + e_{m-2} + f_{m-2}, a_{m-1} + b_{m-1} + e_{m-1} \\
\quad + f_{m-1}, a_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + e_0 + f_0, \\
\quad \ldots, a_{m-2} + b_{m-2} + e_{m-2} + f_{m-2}, e_{m-1} + f_{m-1}, b_0 + c_0 + e_0 + f_0, \ldots, b_{m-1} \\
\quad + c_{m-1} + e_{m-1} + f_{m-1}, e_0 + f_0, \ldots, e_{m-2} + f_{m-2}) \]
On the other hand,
\[
\zeta(r) = ((1 + \vartheta) r_{m-1}, r_0, r_1, \ldots, r_{m-2})
\]
\[
= ((1 + \vartheta)(a_{m-1} + 2b_{m-1} + 4c_{m-1} + \vartheta d_{m-1} + 2\vartheta e_{m-1} + 4\vartheta f_{m-1}), r_0, \ldots, r_{m-2})
\]
\[
= (a_{m-1} + 2b_{m-1} + 4c_{m-1} + \vartheta (a_{m-1} + d_{m-1}) + 2\vartheta (b_{m-1} + e_{m-1}) + 4\vartheta (c_{m-1} + f_{m-1}), r_0, \ldots, r_{m-2})
\]
and therefore,
\[
\rho(\zeta(r)) = (b_{m-1} + e_{m-1} + f_{m-1}, c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots,
\]
\[
b_{m-2} + e_{m-2} + f_{m-2}, c_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1}
\]
\[
+ b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + e_0 + f_0, \ldots, a_{m-2} + e_{m-2} + f_{m-2}, a_{m-1}
\]
\[
+ b_{m-1} + e_{m-1} + f_{m-1}, a_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + f_0, \ldots,
\]
\[
a_{m-2} + b_{m-2} + f_{m-2}, a_{m-1} + e_{m-1} + f_{m-1}, a_0 + c_0 + f_0, \ldots, a_{m-1} + c_{m-1} + f_{m-1}, a_0 + b_0 + f_0, \ldots,
\]
\[
f_0, \ldots, a_{m-2} + f_{m-2}, b_{m-1} + f_{m-1}, b_0 + c_0 + f_0, \ldots, b_{m-1} + c_{m-1} + f_{m-1}, b_0 + f_0, \ldots,
\]
\[
+ b_{m-1} + c_{m-1} + f_{m-1}, b_0 + c_0 + d_0 + f_0, \ldots, b_{m-1} + c_{m-1} + d_{m-1} + f_{m-1},
\]
\[
a_0 + b_0 + d_0 + f_0, \ldots, a_{m-2} + b_{m-2} + d_{m-2} + f_{m-2}, a_{m-1} + d_{m-1} + f_{m-1}, c_0
\]
\[
+ d_0 + f_0, \ldots, c_{m-1} + d_{m-1} + f_{m-1}, a_0 + d_0 + f_0, \ldots, a_{m-2} + d_{m-2} + f_{m-2},
\]
\[
b_{m-1} + d_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + d_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + d_{m-1}
\]
\[
+ f_{m-1}, b_0 + d_0 + f_0, \ldots, b_{m-2} + d_{m-2} + f_{m-2}, d_{m-1} + f_{m-1}, a_0 + c_0 + d_0 + f_0, \ldots,
\]
\[
\ldots, a_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, d_0 + f_0, \ldots, d_{m-2} + f_{m-2}, b_{m-1} + e_{m-1} + f_{m-1},
\]
\[
c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, b_{m-2} + e_{m-2} + f_{m-2},
\]
\[
a_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + e_{m-1}
\]
\[
+ f_{m-1}, a_0 + e_0 + f_0, \ldots, a_{m-2} + e_{m-2} + f_{m-2}, a_{m-1} + b_{m-1} + e_{m-1} + f_{m-1},
\]
\[
a_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + e_0 + f_0, \ldots, a_{m-2}
\]
\[
+ b_{m-2} + e_{m-2} + f_{m-2}, e_{m-1} + f_{m-1}, b_0 + c_0 + e_0 + f_0, \ldots, b_{m-1} + c_{m-1}
\]
\[
+ e_{m-1} + f_{m-1}, e_0 + f_0, \ldots, e_{m-2} + f_{m-2})
\]

Hence the result. \[\square\]
**Theorem 4.5.** A linear code $C$ of length $m$ over $\mathbb{R}$ is a $(1+\vartheta)$-constacyclic code if and only if $\rho(C)$ is a binary quasi-cyclic code of length $32m$ with index 16.

**Proof.** If $C$ is a $(1+\vartheta)$-constacyclic code, then use of Theorem 4.4 gives,

$$\mathcal{O}_{16}(\rho(C)) = \rho(\zeta(C)) = \rho(C),$$

which implies $\rho(C)$ is a binary quasi-cyclic code of length $32m$ with index 16, and again applying Theorem 4.4 to obtain

$$\rho(\zeta(C)) = \mathcal{O}_{16}(\rho(C)) = \rho(C).$$

Further, $\rho$ is an injective mapping and therefore, $\zeta(C) = C$. $\square$

From Theorem 4.5 and the definition of the map $\rho$, the following result holds immediately.

**Corollary 4.6.** The image of a $(1+\vartheta)$-constacyclic code of length $m$ over $\mathbb{R}$ under the map $\rho$ is a distance invariant binary quasi-cyclic code of length $32m$ with index 16.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

**References**