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TIMELIKE – SPACELIKE INVOLUTE – EVOLUTE CURVE COUPLE ON DUAL LORENTZIAN SPACE

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Abstract. In this paper, Firstly we have defined the involute curves of the dual spacelike curve M_1 with a dual spacelike binormal in dual Lorentzian space ID_1^3. We have seen that the dual involute curve M_2 must be a dual timelike vector. Secondly, the relationship between the Frenet frames of couple of the timelike –spacelike involute – evolute dual curve has been found and finally some new characterizations related to the couple of the dual curve has been given.

Keywords: dual lorentzian space, dual involute – evolute curve couple, dual frenet frames.

Mathematics Subject Classification (2000): 53A04, 53

1. Introduction

The consept of the involute of a given curve is a well-known in 3-dimensional Euclidean space IR^3 ,[7,8,9,12,13]. Some basic notions of Lorentzian space are given ,[3,10,14,15]. M_1 is a timelike curve then the involute curve M_2 is a spacelike curve with a spacelike or timelike binormal. On the other hand, it has been investigated that the involute and evolute curves of the spacelike curve M_1 with a spacelike binormal in Minkowski 3-space and it has been seen that the involute curve M_2 is timelike, [4,5]. The involute curves of the spacelike curve M_1 with a timelike binormal is defined in Minkowski

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3-space IR_1^3, [2]. Lorentzian angle defined in,[11]. W.K. Clifford, introduced dual numbers as the set $ID=\left\{\hat{\lambda}=\lambda+\varepsilon\lambda^* \,\middle|\, \lambda,\lambda^*\in IR,\ \ \varepsilon^2=0 \text{ for } \varepsilon\neq 0\right\}$,[6].

Addition, product, division and absolute value operations are defined on ID like below, respectively:

$$\left(\lambda+\varepsilon\lambda^*\right)+\left(\beta+\varepsilon\beta^*\right)=\left(\lambda+\beta\right)+\varepsilon\left(\lambda^*+\beta^*\right),$$

$$\left(\lambda+\varepsilon\lambda^*\right)\left(\beta+\varepsilon\beta^*\right)=\lambda\beta+\varepsilon\left(\lambda\beta^*+\lambda^*\beta\right),$$

$$\frac{\lambda+\varepsilon\lambda^*}{\beta+\varepsilon\beta^*}=\frac{\lambda}{\beta}+\varepsilon\left(\frac{\lambda^*}{\beta}-\frac{\lambda\beta^*}{\beta^2}\right),$$

$$\left|\lambda+\varepsilon\lambda^*\right|=\left|\lambda\right|.$$

$ID^3=\left\{\vec{A}=\vec{a}+\varepsilon\vec{a}^* \,\middle|\, \vec{a},\vec{a}^*\in IR^3\right\}$. The elements of ID^3 are called dual vectors . On this set addition and scalar product operations are respectively

$$\begin{aligned}\oplus : ID^3\times ID^3 &\to ID^3\\ \left(\vec{A},\vec{B}\right) &\to \vec{A}\oplus\vec{B}=\vec{a}+\vec{b}+\varepsilon\left(\vec{a}^*+\vec{b}^*\right)\end{aligned}$$

$$\begin{aligned}\odot : ID\times ID^3 &\to ID^3\\ \left(\lambda,\vec{A}\right) &\to \lambda\odot\vec{A}=\left(\lambda+\varepsilon\lambda^*\right)\odot\left(\vec{a}+\varepsilon\vec{a}^*\right)=\lambda\vec{a}+\varepsilon\left(\lambda\vec{a}^*+\lambda^*\vec{a}\right).\end{aligned}$$

The set $\left(ID^3,\oplus\right)$ is a module over the ring $\left(ID,+,\cdot\right)$. $\left(ID-Modul\right)$.

The Lorentzian inner product of dual vectors $\vec{A},\ \vec{B}\in ID^3$ is defined by

$$\left\langle\vec{A},\vec{B}\right\rangle=\left\langle\vec{a},\vec{b}\right\rangle+\varepsilon\left(\left\langle\vec{a},\vec{b}^*\right\rangle+\left\langle\vec{a}^*,\vec{b}\right\rangle\right)$$

with the Lorentzian inner product $\vec{a}=\left(a_1,a_2,a_3\right)$ and $\vec{b}=\left(b_1,b_2,b_3\right)\in IR^3$

$$\left\langle\vec{a},\vec{b}\right\rangle=-a_1b_1+a_2b_2+a_3b_3.$$

Therefore, ID^3 with the Lorentzian inner product $\left\langle\vec{A},\vec{B}\right\rangle$ is called 3-dimensional dual Lorentzian space and denoted by of $ID_1^3=\left\{\vec{A}=\vec{a}+\varepsilon\vec{a}^* \,\middle|\, \vec{a},\vec{a}^*\in IR_1^3\right\}$.

A dual vector $\vec{A}=\vec{a}+\varepsilon\vec{a}^{*} \in ID_1^3$ is called

A dual space-like vector if $\vec{a}$ is spacelike vector ,

A dual time-like vector if $\vec{a}$ is timelike vector,

A dual null(light-like) vector if $\vec{a}$ is lightlike vector .

For $\vec{A}\neq 0$, the norm $\left\|\vec{A}\right\|$ of $\vec{A}=\vec{a}+\varepsilon\vec{a}^{*} \in ID_1^3$ is defined by

$$\left\|\vec{A}\right\|=\sqrt{\left|\left\langle\vec{A},\vec{A}\right\rangle\right|}=\left\|\vec{a}\right\|+\varepsilon\frac{\left\langle\vec{a},\vec{a}^{*}\right\rangle}{\left\|\vec{a}\right\|}\ ,\ \left\|\vec{a}\right\|\neq 0\ .$$

The dual Lorentzian cross-product of $\vec{A},\ \vec{B}\in ID_1^3$ is defined as

$$\vec{A}\wedge\vec{B}=\vec{a}\wedge\vec{b}+\varepsilon\left(\vec{a}\wedge\vec{b}^{*}+\vec{a}^{*}\wedge\vec{b}\right)$$

with the Lorentzian cross-product $\vec{a}$, $\vec{b}\in IR_1^3$

$$\vec{a}\wedge\vec{b}=\left(a_3b_2-a_2b_3,a_1b_3-a_3b_1,a_1b_2-a_2b_1\right),[17].$$

Dual Frenet trihedron of the differentiable curve M in dual space ID_1^3 and instantaneous dual rotation vector have given in ,[1,16].

The dual angle between $\vec{A}$ and $\vec{B}$ is $\Phi=\varphi+\varepsilon\varphi^{*}$, such that

$$\begin{cases}\sinh\Phi=\sinh\left(\varphi+\varepsilon\varphi^{*}\right)=\sinh\varphi+\varepsilon\varphi^{*}\cosh\varphi,\\ \cosh\Phi=\cosh\left(\varphi+\varepsilon\varphi^{*}\right)=\cosh\varphi+\varepsilon\varphi^{*}\sinh\varphi.\end{cases}$$

The dual Lorentzian sphere and the dual hyperbolic sphere of 1 radius in IR_1^3 are defined by

$$S_1^2=\left\{A=a+\varepsilon a_0 \,\middle|\, \left\|A\right\|=(1,0);a,a_0\in IR_1^3,\ \text{and } a \text{ is spacelike}\right\},$$

$$H_0^2=\left\{A=a+\varepsilon a_0 \,\middle|\, \left\|A\right\|=(1,0);a,a_0\in IR_1^3,\ \text{and } a \text{ is timelike}\right\}$$

respectively ,[15].

2. Preliminaries

Lemma 2.1 Let X and Y be nonzero Lorentz orthogonal vektors in ID_1^3. If X is timelike, then Y is spacelike, [11].

Lemma 2.2 Let X,Y be pozitive (negative) timelike vectors in ID_1^3. Then $\langle X,Y\rangle \le \|X\|\|Y\|$ with equality if and only if X and Y are linearly dependent, [11].

Lemma 2.3.

***i*)** Let X and Y be pozitive (negative) timelike vectors in ID_1^3. Then we have $\langle X,Y\rangle \le \|X\|\|Y\|$, there is a unique non negative dual number $\Phi(X,Y)$ such that

$$\langle X,Y\rangle = \|X\|\|Y\|\cosh\Phi(X,Y)$$

where $\Phi(X,Y)$ is theLorentzian timelike dual angle between X and Y.

***ii*)** Let X and Y be spacelike vectors in ID_1^3 that span a spacelike vector subspace. Then we have $|\langle X,Y\rangle| \le \|X\|\|Y\|$. Hence, there is a unique dual number $\Phi(X,Y)$ between 0 and π such that

$$\langle X,Y\rangle = \|X\|\|Y\|\cos\Phi(X,Y)$$

where $\Phi(X,Y)$ is the Lorentzian spacelike dual angle between X and Y.

***iii*)** Let X and Y be spacelike vectors in ID_1^3 that span a timelike vector subspace. Then wehave $|\langle X,Y\rangle| \ge \|X\|\|Y\|$. Hence, there is a unique positive dual number $\Phi(X,Y)$ such that

$$\langle X,Y\rangle = \|X\|\|Y\|\cosh\Phi(X,Y)$$

where $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y.

***iv*)** Let X be a spacelike vector and Y a positive timelike vector in ID_1^3. Then there is a unique non negative dual number $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y,such that

$$\langle X,Y\rangle = \|X\|\|Y\|\sinh\Phi(X,Y),[11].$$

Let $\{T,N,B\}$ be the dual Frenet trihedron of the differentiable curve M in the dual space ID_1^3 and $T=t+\varepsilon t^*$,$N=n+\varepsilon n^*$ and $B=b+\varepsilon b^*$ be the tangent, the principal normal and the binormal vector of M, respectively. Depending on the causal character of the curve M, we have an instantaneous dual rotation vector:

Let M be a unit speed timelike dual space curve with dual curvature $\kappa=k_1+\varepsilon k_1^*$ and dual torsion $\tau=k_2+\varepsilon k_2^*$.The Frenet vectors T, N and B of M are timelike vector, spacelike vectors, spacelike vector, respectively, such that

$$T\wedge N=-B \ , \ N\wedge B=T \ , \ B\wedge T=-N. \tag{2.1}$$

From here ,

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\\kappa&0&-\tau\\0&\tau&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix} \text{ ,[18].} \tag{2.2}$$

(2.2) leaves the real and dual components

$$\begin{cases}\begin{bmatrix}t'\\n'\\b'\end{bmatrix}=\begin{bmatrix}0&k_1&0\\k_1&0&-k_2\\0&k_2&0\end{bmatrix}\begin{bmatrix}t\\n\\b\end{bmatrix}\\ \\ \begin{bmatrix}t^{*\prime}\\n^{*\prime}\\b^{*\prime}\end{bmatrix}=\begin{bmatrix}0&k_1^*&0\\k_1^*&0&-k_2^*\\0&k_2^*&0\end{bmatrix}\begin{bmatrix}t\\n\\b\end{bmatrix}+\begin{bmatrix}0&k_1&0\\k_1&0&-k_2\\0&k_2&0\end{bmatrix}\begin{bmatrix}t^*\\n^*\\b^*\end{bmatrix}\end{cases}$$

The Frenet instantaneous rotation vector W of the timelike curve is given by

$$W=\tau T-\kappa B, \text{ [14].} \tag{2.3}$$

(2.3) leaves the real and dual components

$$\begin{cases}w=k_2t-k_1b\\ \\ w^*=k_2^*t+k_2t^*-k_1^*b-k_1b^*\end{cases}$$

Let $\Phi=\varphi+\varepsilon\varphi^*$ be a Lorentzian timelike dual angle between the spacelike binormal unit vector B and the Frenet instantaneous dual rotation vector W.Then $C=c+\varepsilon c^*$ is a unit dual vector in direction of W:

a) If $|\kappa|>|\tau|$, W is a spacelike vector. In this station, we can write

$$\begin{cases}\kappa=\|W\|\cosh\Phi \\ \tau=\|W\|\sinh\Phi\end{cases} \quad , \quad \|W\|^2=\langle W,W\rangle=\kappa^2-\tau^2 \tag{2.4}$$

and

$$C=\sinh\Phi T-\cosh\Phi B. \tag{2.5}$$

b) If $|\kappa|<|\tau|$, W is a timelike vector. In this station, we can write

$$\begin{cases}\kappa=\|W\|\sinh\Phi \\ \tau=\|W\|\cosh\Phi\end{cases} \quad , \quad \|W\|^2=-\langle W,W\rangle=-\left(\kappa^2-\tau^2\right) \tag{2.6}$$

and

$$C=\cosh\Phi T-\sinh\Phi B\ . \tag{2.7}$$

Let M be a unit speed dual spacelike space curve with spacelike binormal. The Frenet vectors T, N, B of M are spacelike vector, timelike vector, spacelike vector, respectively, such that

$$T\wedge N=-B\ ,\ N\wedge B=-T\ ,\ B\wedge T=N. \tag{2.8}$$

From here,

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\\kappa&0&\tau\\0&\tau&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix}, [18]. \tag{2.9}$$

(2.9) leaves the real and dual components

$$\begin{cases} \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ k_1^* & 0 & k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

and the Frenet instantaneous rotation vector for the spacelike curve is given by

$$W = -\tau T + \kappa B, [14]. \tag{2.10}$$

(2.10) leaves the real and dual components

$$\begin{cases} \overline{w} = -k_2 t + k_1 b, \\ \overline{w^*} = -k_2^* t - k_2 t^* + k_1^* b + k_1 b^* \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be a dual angle between the B and the W. If B and W spacelike vectors that span a spacelike vector subspace, we can write

$$\begin{cases} \kappa = \|W\| \cos\Phi \\ \tau = \|W\| \sin\Phi \end{cases} , \quad \|W\|^2 = \langle W, W \rangle = \kappa^2 + \tau^2 \tag{2.11}$$

and

$$C = -\sin\Phi T + \cos\Phi B \,. \tag{2.12}$$

Let M be a unit speed dual spacelike space curve. The Frenet vectors T , N and B of M are spacelike vector, timelike vector and spacelike vector, respectively, such that

$$T \wedge N = B \;,\; N \wedge B = -T \;,\; B \wedge T = -N. \tag{2.13}$$

From here

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, [18]. \tag{2.14}$$

(2.14) leaves the real and dual components

$$\begin{cases} \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ -k_1^* & 0 & k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

and the Frenet instantaneous dual rotation vector W of the spacelike curve is given by

$$W = \tau T - \kappa B, \text{ [14].} \tag{2.15}$$

(2.15) leaves the real and dual components

$$\begin{cases} w = k_2 t - k_1 b, \\ w^* = k_2^* t + k_2 t^* - k_1^* b - k_1 b^*. \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be a Lorentzian timelike dual angle between the B and W :

a) If $|\kappa| < |\tau|$, W is a spacelike vector. In this case, we can write

b)

$$\begin{cases} \kappa = \|W\| \sinh \Phi \\ \tau = \|W\| \cosh \Phi \end{cases}, \quad \|W\|^2 = \langle W, W \rangle = \tau^2 - \kappa^2 \tag{2.16}$$

and

$$C = \cosh \Phi T - \sinh \Phi B \ . \tag{2.17}$$

c) If $|\kappa| > |\tau|$, W is a timelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\| \cosh \Phi \\ \tau = \|W\| \sinh \Phi \end{cases}, \quad \|W\|^2 = -\langle W, W \rangle = -\left(\tau^2 - \kappa^2\right) \tag{2.18}$$

and

$$C = \sinh \Phi T - \cosh \Phi B \ . \tag{2.19}$$

3. Main Results

Definition 3.1. Let $M_1 : I \to ID_1^3 \quad M_1 = M_1(s)$ be the unit speed dual spacelike curve with spacelike binormal and $M_2 : I \to ID_1^3 \quad M_2 = M_2(s)$ be the unit speed dual timelike curve. If tangent vector of curve M_1 is ortogonal to tangent vector of M_2, M_1 is called evolute of curve M_2 and M_2 is called involute of M_1. Thus the dual involute – evolute curve couple is denoted by (M_2, M_1). Since the tangent vector of M_1 is spacelike, the tangent vector of M_2 must be timelike vector. So the tangent vector of M_2 must be timelike vector (M_2, M_1) is called "the timelike – spacelike involute – evolute dual curve couple".

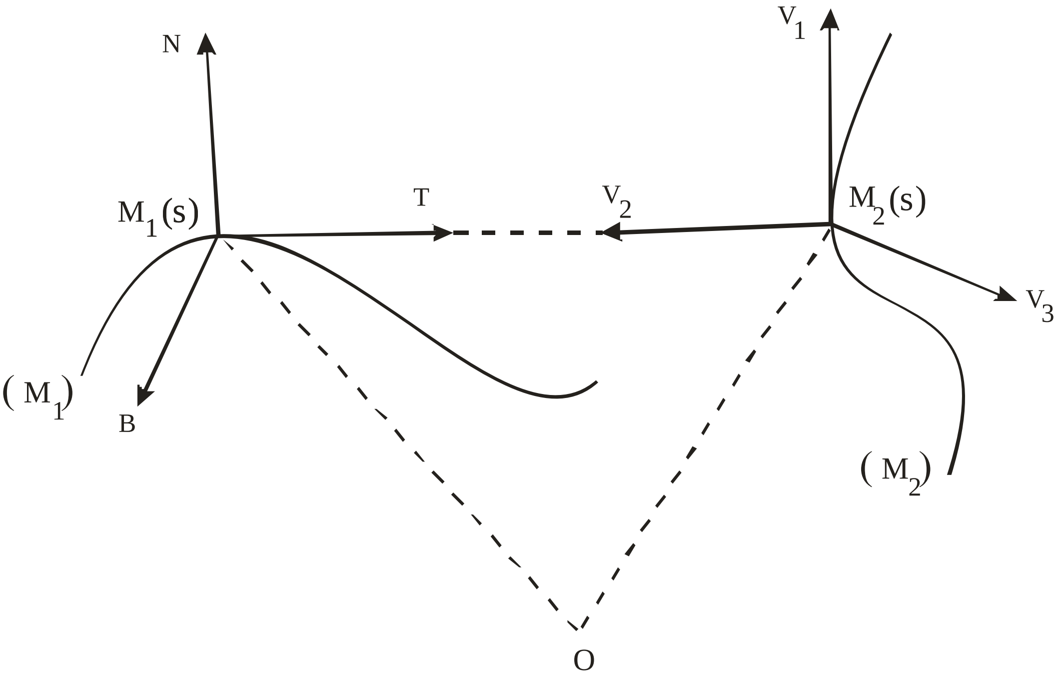

Fig. 2. Involute – evolute curve couple.

Theorem 3.1. Let (M_2, M_1) be the timelike – spacelike involute – evolute dual curve couple. Let $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively. The dual distance between M_1 and M_2 at the corresponding points is

$$d\left(M_1(s), M_2(s)\right) = |c_1 - s| + \varepsilon c_2, \quad c_1, \; c_2 = \text{constant}.$$

Proof. If M_2 is the dual involute of M_1, we can write from the fig. 2

$$M_2(s)=M_1(s)+\lambda T(s), \qquad \lambda=\lambda_1+\varepsilon\lambda_1^* \in ID \tag{3.1}$$

Differentiating (3.1) with respect to s we have

$$V_1\frac{ds^*}{ds}=(1+\lambda')T+\lambda\kappa N$$

where s and s^* are arc parameter of M_1 and M_2, respectively. Since the direction of T is orthogonal to the direction of V_1, we obtain

$$\lambda'=-1.$$

From here, it can be easily seen

$$\lambda=(c_1-s)+\varepsilon c_2 \tag{3.2}$$

Furthermore, the dual distance between the points $M_1(s)$ and $M_2(s)$

$$\begin{aligned} d\left(M_1(s),M_2(s)\right) &= \sqrt{\left|\left\langle \lambda T(s)\ \ \lambda T(s)\right\rangle\right|} \\ &= |\lambda_1|+\varepsilon\lambda_1^* \ . \end{aligned}$$

Since $\lambda_1=(c_1-s)$, $\lambda_1^*=c_2$, we have

$$d\left(M_1(s),M_2(s)\right)=|c_1-s|+\varepsilon c_2 \ . \tag{3.3}$$

Theorem 3.2. Let (M_2,M_1) be the timelike – spacelike involute – evolute dual curve couple. Let $\{T,N,B\}$ and $\{V_1,V_2,V_3\}$ be the dual Frenet frames of M_1 and M_2 ,respectively, Since the dual curvature of M_2 is $P=p+\varepsilon p^*$, we have

$$P^2=\frac{k_1^2+k_2^2}{(c_1-s)^2k_1^2}+\varepsilon\left[\frac{2k_2\left(k_1k_2^*-k_1^*k_2\right)}{(c_1-s)^2k_1^3}-\frac{2c_2\left(k_1^2+k_2^2\right)}{(c_1-s)^3k_1^2}\right].$$

where the dual curvature of M_1 is $\kappa=k_1+\varepsilon k_1^*$.

Proof. Differentiating (3.1), with respect to s, we get

$$\frac{dM_2}{ds^*}\frac{ds^*}{ds^?}=\frac{dM_1}{ds}+\frac{d\lambda}{ds}T+\lambda\frac{dT}{ds}$$

$$V_1\frac{ds^*}{ds}=\lambda\kappa N.$$

From here , we can write

$$V_1 = N \tag{3.4}$$

and

$$\frac{ds^*}{ds} = \lambda\kappa \, .$$

By differentiating the last equation and using (2.9), we obtain

$$\frac{dV_1}{ds^*}\frac{ds^*}{ds^?} = \frac{dN}{ds} = \kappa T + \tau B,$$

$$PV_2 = \frac{1}{\lambda\kappa}\left(\kappa T + \tau B\right)$$

From here,we have

$$P^2 = \frac{\left(\kappa^2 + \tau^2\right)}{\lambda^2\kappa^2} \tag{3.5}$$

From the fact that $P = p + \varepsilon p^*$, $\lambda = \lambda_1 + \varepsilon\lambda_1^*$, $\kappa = k_1 + \varepsilon k_1^*$ and $\tau = k_2 + \varepsilon k_2^*$ we get

$$P^2 = \frac{\left(k_1^2 + 2\varepsilon k_1 k_1^* + k_2^2 + 2\varepsilon k_2 k_2^*\right)}{\left(\lambda_1^2 + 2\varepsilon\lambda_1\lambda_1^*\right)\left(k_1^2 + 2\varepsilon k_1 k_1^*\right)}$$

$$= \frac{k_1^2 + k_2^2}{\lambda_1^2 k_1^2} + \varepsilon\left[\frac{2k_2\left(k_1 k_2^* - k_1^* k_2\right)}{\lambda_1^2 k_1^3} - \frac{2\lambda_1^*\left(k_1^2 + k_2^2\right)}{\lambda_1^3 k_1^2}\right].$$

From here, by using $\lambda_1 = (c_1 - s)$, $\lambda_2 = c_2$, we obtain

$$P^2 = \frac{k_1^2 + k_2^2}{\left(c_1 - s\right)^2 k_1^2} + \varepsilon\left[\frac{2k_2\left(k_1 k_2^* - k_1^* k_2\right)}{\left(c_1 - s\right)^2 k_1^3} - \frac{2c_2\left(k_1^2 + k_2^2\right)}{\left(c_1 - s\right)^3 k_1^2}\right]. \tag{3.6}$$

Theorem 3.3. Let (M_2, M_1) be the timelike – spacelike involute – evolute dual curve couple. Let $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively. The dual torsion $\tau = k_2 + \varepsilon k_2^*$ of M_1 and the dual torsion $Q = q + \varepsilon q^*$ of M_2 is the following equation

$$Q = \frac{k_2 k_1' - k_2' k_1}{\left(k_1^2 + k_2^2\right) k_1 \left|c_1 - s\right|} + \varepsilon\left[\frac{k_1\left(k_1' k_2^* - k_1 k_2^{*\prime}\right) + k_2\left(k_1 k_1^{*\prime} - k_1' k_1^*\right)}{\left(k_1^2 + k_2^2\right) k_1^2 \left|c_1 - s\right|} + \frac{\left(2k_1 k_1^* + 2k_2 k_2^*\right)\left(k_1 k_2' - k_1' k_2\right)}{\left(k_1^2 + k_2^2\right)^2 k_1 \left|c_1 - s\right|}\right].$$

Proof. By differentiating (3.1) three time with respect to s,we get

$$M_2' = \lambda\kappa N$$
$$M_2'' = \lambda\kappa^2 T + (\lambda\kappa' - \kappa) N + \lambda\kappa\tau B$$
$$M_2''' = (3\lambda\kappa\kappa' - 2\kappa^2) T + (\lambda\kappa^3 + \lambda\kappa\tau^2 - 2\kappa' + \lambda\kappa'') N + (-2\kappa\tau + 2\lambda\kappa'\tau + \lambda\kappa\tau') B$$

The vectorel product of M_2' and M_2'' are

$$M_2' \wedge M_2'' = -\lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B = \lambda^2\kappa^2(-\tau T + \kappa B) \tag{3.7}$$

From here, we obtain

$$\left\| M_2' \wedge M_2'' \right\|^2 = |\lambda|^4 |\kappa|^4 (\kappa^2 + \tau^2) \tag{3.8}$$

and

$$\det(M_2', M_2'', M_2''') = \lambda^3\kappa^3(\kappa'\tau - \kappa\tau'). \tag{3.9}$$

Substituting by (3.8) and (3.9) values into $Q = \dfrac{\det(M_2', M_2'', M_2''')}{\left\| M_2' \wedge M_2'' \right\|^2}$, we get

$$Q = \frac{(\kappa'\tau - \kappa\tau')}{|\lambda|\kappa(\kappa^2 + \tau^2)} \tag{3.10}$$

and then substituting $Q = q + \varepsilon q^*$, $\lambda = \lambda_1 + \varepsilon\lambda_1^*$, $\kappa = k_1 + \varepsilon k_1^*$ and $\tau = k_2 + \varepsilon k_2^*$ into the last equation, we have

$$Q = \frac{(k_1' + \varepsilon k_1^{*\prime})(k_2 + \varepsilon k_2^*) - (k_1 + \varepsilon k_1^*)(k_2' + \varepsilon k_2^{*\prime})}{|\lambda_1 + \varepsilon\lambda_1^*|(k_1 + \varepsilon k_1^*)((k_1^2 + k_2^2) + \varepsilon(2k_1k_1^* + 2k_2k_2^*))}$$

$$= \frac{(k_2k_1' - k_2'k_1) + \varepsilon(k_1'k_2^* - k_1k_2^{*\prime} + k_1^{*\prime}k_2 - k_1^*k_2')}{(|\lambda_1|k_1 + \varepsilon|\lambda_1|k_1^*)((k_1^2 + k_2^2) + \varepsilon(2k_1k_1^* + 2k_2k_2^*))}$$

$$= \frac{k_2k_1' - k_2'k_1}{|\lambda_1|k_1(k_1^2 + k_2^2)} + \varepsilon\left[\frac{k_1(k_1'k_2^* - k_1k_2^{*\prime}) + k_2(k_1k_1^{*\prime} - k_1'k_1^*)}{|\lambda_1|k_1^2(k_1^2 + k_2^2)} + \frac{(2k_1k_1^* + 2k_2k_2^*)(k_1k_2' - k_1'k_2)}{|\lambda_1|k_1(k_1^2 + k_2^2)^2}\right]$$

By the fact that $\lambda_1 = (c_1 - s)$, we get

$$Q = \frac{k_2 k_1' - k_2' k_1}{|c_1 - s| k_1 \left(k_1^2 + k_2^2\right)} + \varepsilon \left[\frac{k_1\left(k_1' k_2^* - k_1 k_2^{*\prime}\right) + k_2\left(k_1 k_1^{*\prime} - k_1' k_1^*\right)}{|c_1 - s| k_1^2 \left(k_1^2 + k_2^2\right)} + \frac{\left(2k_1 k_1^* + 2k_2 k_2^*\right)\left(k_1 k_2' - k_1' k_2\right)}{|c_1 - s| k_1 \left(k_1^2 + k_2^2\right)^2} \right] \quad (3.11)$$

Theorem 3.4. Let (M_2, M_1) be the timelike –spacelike involute – evolute dual curve couple. Let $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively and $\Phi = \varphi + \varepsilon\varphi^*$ be the Lorentzian dual spacelike angle between binormal vector B and W. For (M_2, M_1) dual curve couple, the following equations is obtained:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\Phi & 0 & -\sinh\Phi \\ -\sinh\Phi & 0 & \cosh\Phi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

leaves the real and dual components

$$\begin{cases} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\varphi & 0 & -\sin\varphi \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \\ \begin{bmatrix} v^*_1 \\ v^*_2 \\ v^*_3 \end{bmatrix} = \varphi^* \begin{bmatrix} 0 & 0 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ -\cos\varphi & 0 & -\sin\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -\cos\varphi & 0 & -\sin\varphi \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

Proof. From (2.11), (3.4) and (3.8), we have,

$$\left\| M_2' \wedge M_2'' \right\| = \lambda^2 \kappa^2 \|W\|. \quad (3.12)$$

By using (3.7) and (3.12) and from the fact that $V_3 = \dfrac{M_2' \wedge M_2''}{\left\| M_2' \wedge M_2'' \right\|}$ we obtain

$$V_3 = -\frac{\tau}{\|W\|} T + \frac{\kappa}{\|W\|} B,$$

substituning (2.11) into the last equation, we obtain

$$V_3 = -\sin\Phi T + \cos\Phi B. \quad (3.13)$$

Since $V_2 = -V_3 \wedge V_1$, it can be easily seen that

$$V_2 = -\cos\Phi T - \sin\Phi B. \tag{3.14}$$

Considering (3.4), (3.13) and (3.14) according to dual components, the following equations are obtained:

$$\begin{cases} V_1 = n + \varepsilon n^* \\ V_2 = (-\cos\varphi t - \sin\varphi b) + \varepsilon\left[(-\cos\varphi t^* - \sin\varphi b^*) + \varphi^*(\sin\varphi t - \cos\varphi b)\right] \\ V_3 = (-\sin\varphi t + \cos\varphi b) + \varepsilon\left[(-\sin\varphi t^* + \cos\varphi b^*) + \varphi^*(-\cos\varphi t - \sin\varphi b)\right] \end{cases} \tag{3.15}$$

written (3.15) in matrix form, the proof is completed.

Theorem 3.5. Let (M_2, M_1) be the timelike – spacelike involute – evolute dual curve couple, $W = w + \varepsilon w^*$ and $\overline{W} = \overline{w} + \varepsilon \overline{w}^*$ be the dual Frenet instantaneous rotation vectors of M_1 and M_2 respectively. Thus,

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(-\Phi' N + W\right)$$

Proof. From (2.3), we can write

$$\overline{W} = QV_1 - PV_3.$$

Using the (3.4), (3.5) , (3.10) and (3.13) the equations, we have

$$\overline{W} = \frac{(\kappa'\tau - \kappa\tau')}{|\lambda|\kappa(\kappa^2+\tau^2)} N - \frac{\sqrt{\kappa^2+\tau^2}}{|\lambda|\kappa}(-\sin\Phi T + \cos\Phi B). \tag{3.16}$$

Substituning (2.11) into the last equation, we obtain

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(\frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2} N - W\right)$$

and then, we get

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(-\Phi' N + W\right). \tag{3.17}$$

Considering (3.17) according to dual components and substituting $\lambda_1 = (c_1 - s)$ into (3.17), we leaves the real and dual components

$$\begin{cases} \overline{w} = \dfrac{-\varphi' n + w}{|c_1 - s| k_1}, \\ \overline{w^*} = \dfrac{-\varphi' n^* - \varphi^{*\prime} n + w^*}{|c_1 - s| k_1} - \dfrac{k_1^* (-\varphi' n + w)}{|c_1 - s| k_1^2}. \end{cases} \tag{3.18}$$

Theorem 3.6. Let (M_2, M_1) be the timelike – spacelike involute – evolute dual curve couple, $C = c + \varepsilon c^*$ and $\overline{C} = \overline{c} + \varepsilon \overline{c}^*$ be unit dual vector of W and $\overline{W}$, respectively. Thus,

$$\overline{C} = -\frac{-\Phi'}{\sqrt{|\kappa^2 + \tau^2 + \Phi'^2|}} N + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{|\kappa^2 + \tau^2 + \Phi'^2|}} C$$

Proof. From the fact that the unit dual vector of $\overline{W}$ is $\overline{C} = \dfrac{\overline{W}}{\|\overline{W}\|}$ we obtain

$$\overline{C} = \frac{-\Phi' N + W}{\sqrt{|\kappa^2 + \tau^2 + \Phi'^2|}}, \tag{3.19}$$

or

$$\overline{C} = -\frac{-\Phi'}{\sqrt{|\kappa^2 + \tau^2 + \Phi'^2|}} N + \frac{\sqrt{\kappa^2 + \tau^2}}{\sqrt{|\kappa^2 + \tau^2 + \Phi'^2|}} C \tag{3.20}$$

(3.20) leaves the real and dual components

$$\begin{cases} \overline{c} = \dfrac{1}{\sqrt{|k_1^2 + k_2^2 + \varphi'^2|}} \left(-\varphi' n + \sqrt{k_1^2 + k_2^2}\, c \right). \\ \overline{c^*} \dfrac{1}{\sqrt{|k_1^2 + k_2^2 + \varphi'^2|}} \left(-\varphi' n^* - \varphi^{*\prime} n + \sqrt{k_1^2 + k_2^2}\, c^* + \dfrac{k_1 k_1^* + k_2 k_2^*}{\sqrt{k_1^2 + k_2^2}} c \right) \end{cases} \tag{3.21}$$

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