TIMELIKE – SPACELIKE INVOLUTE – EVOLUTE CURVE COUPLE ON DUAL LORENTZIAN SPACE

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Abstract. In this paper, Firstly we have defined the involute curves of the dual spacelike curve $M_1$ with a dual spacelike binormal in dual Lorentzian space $ID^1$. We have seen that the dual involute curve $M_2$ must be a dual timelike vector. Secondly, the relationship between the Frenet frames of couple of the timelike –spacelike involute – evolute dual curve has been found and finally some new characterizations related to the couple of the dual curve has been given.

Keywords: dual lorentzian space, dual involute – evolute curve couple, dual frenet frames.

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1. Introduction

The concept of the involute of a given curve is a well-known in 3-dimensional Euclidean space $IR^3$, [7,8,9,12,13]. Some basic notions of Lorentzian space are given, [3,10,14,15]. $M_1$ is a timelike curve then the involute curve $M_2$ is a spacelike curve with a spacelike or timelike binormal. On the other hand, it has been investigated that the involute and evolute curves of the spacelike curve $M_1$ with a spacelike binormal in Minkowski 3-space and it has been seen that the involute curve $M_2$ is timelike, [4,5]. The involute curves of the spacelike curve $M_1$ with a timelike binormal is defined in Minkowski

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3-space $IR_3^3$, [2]. Lorentzian angle defined in,[11]. W.K. Clifford, introduced dual numbers as the set $ID = \left\{ \lambda = \lambda + \varepsilon \lambda^* \mid \lambda, \lambda^* \in IR, \varepsilon^2 = 0 \text{ for } \varepsilon \neq 0 \right\}$,[6].

Addition, product, division and absolute value operations are defined on $ID$ like below, respectively:

\[
(\lambda + \varepsilon \lambda^*) + (\beta + \varepsilon \beta^*) = (\lambda + \beta) + \varepsilon \left( \lambda^* + \beta^* \right),
\]

\[
(\lambda + \varepsilon \lambda^*) (\beta + \varepsilon \beta^*) = \lambda \beta + \varepsilon \left( \lambda \beta^* + \lambda^* \beta \right),
\]

\[
\frac{\lambda + \varepsilon \lambda^*}{\beta + \varepsilon \beta^*} = \frac{\lambda}{\beta} + \varepsilon \left( \frac{\lambda^*}{\beta} - \frac{\lambda \beta^*}{\beta^2} \right),
\]

\[
|\lambda + \varepsilon \lambda^*| = |\lambda|.
\]

$ID^3 = \left\{ \overline{A} = \overline{a} + \varepsilon \overline{a}^* \mid \overline{a}, \overline{a}^* \in IR^3 \right\}$. The elements of $ID^3$ are called dual vectors. On this set addition and scalar product operations are respectively

\[
\oplus: ID^3 \times ID^3 \rightarrow ID^3
\]

\[
(\overline{A}, \overline{B}) \rightarrow \overline{A} \oplus \overline{B} = \overline{a} + \overline{b} + \varepsilon \left( \overline{a}^* + \overline{b}^* \right)
\]

\[
\ominus: ID \times ID^3 \rightarrow ID^3
\]

\[
(\lambda, \overline{A}) \rightarrow \lambda \ominus \overline{A} = (\lambda + \varepsilon \lambda^*) \ominus \left( \overline{a} + \varepsilon \overline{a}^* \right) = \lambda \overline{a} + \varepsilon \left( \lambda \overline{a}^* + \lambda^* \overline{a} \right).
\]

The set $(ID^3, \oplus)$ is a module over the ring $(ID, +, \cdot)$. $(ID - Modul)$.

The Lorentzian inner product of dual vectors $\overline{A}, \overline{B} \in ID^3$ is defined by

\[
\langle \overline{A}, \overline{B} \rangle = \langle a, b \rangle + \varepsilon \left( \langle a, \overline{b}^* \rangle + \langle \overline{a}^*, b \rangle \right)
\]

with the Lorentzian inner product $\overline{a} = (a_1, a_2, a_3)$ and $\overline{b} = (b_1, b_2, b_3) \in IR^3$

\[
\langle \overline{a}, \overline{b} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

Therefore, $ID^3$ with the Lorentzian inner product $\langle \overline{A}, \overline{B} \rangle$ is called 3-dimensional dual Lorentzian space and denoted by of $ID^3 = \left\{ \overline{A} = \overline{a} + \varepsilon \overline{a}^* \mid \overline{a}, \overline{a}^* \in IR^3 \right\}$. 


A dual vector $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in ID_3^1$ is called

A dual space-like vector if $\overrightarrow{a}$ is spacelike vector ,

A dual time-like vector if $\overrightarrow{a}$ is timelike vector,

A dual null(light-like) vector if $\overrightarrow{a}$ is lightlike vector .

For $\overrightarrow{A} \neq 0$, the norm $\| \overrightarrow{A} \|$ of $\overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \in ID_3^1$ is defined by

$$\| \overrightarrow{A} \| = \sqrt{\langle \overrightarrow{A}, \overrightarrow{A} \rangle} = \| \overrightarrow{a} \| + \varepsilon \frac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle}{\| \overrightarrow{a} \|} , \quad \| \overrightarrow{a} \| \neq 0 .$$

The dual Lorentzian cross-product of $\overrightarrow{A}, \overrightarrow{B} \in ID_3^1$ is defined as

$$\overrightarrow{A} \wedge \overrightarrow{B} = \overrightarrow{a} \wedge \overrightarrow{b} + \varepsilon \left( \overrightarrow{a} \wedge \overrightarrow{b}^* + \overrightarrow{a}^* \wedge \overrightarrow{b} \right)$$

with the Lorentzian cross-product $\overrightarrow{a}, \overrightarrow{b} \in IR_3^1$

$$\overrightarrow{a} \wedge \overrightarrow{b} = (a_1b_2 - a_2b_1, a_2b_3 - a_3b_2, a_3b_1 - a_1b_3) . [17] .$$

Dual Frenet trihedron of the differentiable curve $M$ in dual space $ID_3^1$ and instantaneous dual rotation vector have given in ,[1,16].

The dual angle between $\overrightarrow{A}$ and $\overrightarrow{B}$ is $\Phi = \varphi + \varepsilon \varphi^*$, such that

$$\begin{cases}
\sinh \Phi = \sinh (\varphi + \varepsilon \varphi^*) = \sinh \varphi + \varepsilon \varphi^* \cosh \varphi , \\
\cosh \Phi = \cosh (\varphi + \varepsilon \varphi^*) = \cosh \varphi + \varepsilon \varphi^* \sinh \varphi .
\end{cases}$$

The dual Lorentzian sphere and the dual hyperbolic sphere of 1 radius in $IR_3^1$ are defined by

$$S_1^2 = \{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}_0 \mid \| \overrightarrow{A} \| = (1,0) ; \overrightarrow{a}, \overrightarrow{a}_0 \in IR_3^1 , \text{ and } \overrightarrow{a} \text{ is spacelike} \} ,$$

$$H_0^2 = \{ \overrightarrow{A} = \overrightarrow{a} + \varepsilon \overrightarrow{a}_0 \mid \| \overrightarrow{A} \| = (1,0) ; \overrightarrow{a}, \overrightarrow{a}_0 \in IR_3^1 , \text{ and } \overrightarrow{a} \text{ is timelike} \}$$

respectively ,[15].
2. Preliminaries

Lemma 2.1 Let $X$ and $Y$ be nonzero Lorentz orthogonal vectors in $I\mathbb{D}^3_1$. If $X$ is timelike, then $Y$ is spacelike, [11].

Lemma 2.2 Let $X, Y$ be positive (negative) timelike vectors in $I\mathbb{D}^3_1$. Then $\langle X, Y \rangle \leq \|X\|\|Y\|$ with equality if and only if $X$ and $Y$ are linearly dependent, [11].

Lemma 2.3.

i) Let $X$ and $Y$ be positive (negative) timelike vectors in $I\mathbb{D}^3_1$. Then we have $\langle X, Y \rangle \leq \|X\|\|Y\|$, there is a unique non negative dual number $\Phi(X, Y)$ such that

$$\langle X, Y \rangle = \|X\|\|Y\| \cosh \Phi(X, Y)$$

where $\Phi(X, Y)$ is the Lorentzian timelike dual angle between $X$ and $Y$.

ii) Let $X$ and $Y$ be spacelike vectors in $I\mathbb{D}^3_1$ that span a spacelike vector subspace. Then we have $|\langle X, Y \rangle| \leq \|X\|\|Y\|$. Hence, there is a unique dual number $\Phi(X, Y)$ between 0 and $\pi$ such that

$$\langle X, Y \rangle = \|X\|\|Y\| \cos \Phi(X, Y)$$

where $\Phi(X, Y)$ is the Lorentzian spacelike dual angle between $X$ and $Y$.

iii) Let $X$ and $Y$ be spacelike vectors in $I\mathbb{D}^3_1$ that span a timelike vector subspace. Then we have $|\langle X, Y \rangle| \geq \|X\|\|Y\|$. Hence, there is a unique positive dual number $\Phi(X, Y)$ such that

$$\langle X, Y \rangle = \|X\|\|Y\| \cosh \Phi(X, Y)$$

where $\Phi(X, Y)$ is the Lorentzian timelike dual angle between $X$ and $Y$.

iv) Let $X$ be a spacelike vector and $Y$ a positive timelike vector in $I\mathbb{D}^3_1$. Then there is a unique non negative dual number $\Phi(X, Y)$ is the Lorentzian timelike dual angle between $X$ and $Y$, such that

$$\langle X, Y \rangle = \|X\|\|Y\| \sinh \Phi(X, Y), \text{[11]}.$$
Let \( \{T, N, B\} \) be the dual Frenet trihedron of the differentiable curve \( M \) in the dual space \( ID^3 \) and \( T = t + \varepsilon t^* \), \( N = n + \varepsilon n^* \) and \( B = b + \varepsilon b^* \) be the tangent, the principal normal and the binormal vector of \( M \), respectively. Depending on the causal character of the curve \( M \), we have an instantaneous dual rotation vector:

Let \( M \) be a unit speed timelike dual space curve with dual curvature \( \kappa = k_1 + \varepsilon k_1^* \) and dual torsion \( \tau = k_2 + \varepsilon k_2^* \). The Frenet vectors \( T, N \) and \( B \) of \( M \) are timelike vector, spacelike vectors, spacelike vector, respectively, such that

\[
T \wedge N = -B, \quad N \wedge B = T, \quad B \wedge T = -N. \tag{2.1}
\]

From here,

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} = \begin{bmatrix}
0 & \kappa & 0 \\
\kappa & 0 & -\tau \\
0 & \tau & 0
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B
\end{bmatrix}, \tag{2.2}
\]

(2.2) leaves the real and dual components

\[
\begin{bmatrix}
t' \\
n' \\
b'
\end{bmatrix} = \begin{bmatrix}
0 & k_1 & 0 \\
k_1 & 0 & -k_2 \\
0 & k_2 & 0
\end{bmatrix} \begin{bmatrix}
t \\
n \\
b
\end{bmatrix}
\]

\[
\begin{bmatrix}
t'' \\
n'' \\
b''
\end{bmatrix} = \begin{bmatrix}
0 & k_1^* & 0 \\
k_1^* & 0 & -k_2^* \\
0 & k_2^* & 0
\end{bmatrix} \begin{bmatrix}
t \\
n \\
b
\end{bmatrix} + \begin{bmatrix}
0 & k_1 & 0 \\
k_1 & 0 & -k_2 \\
0 & k_2 & 0
\end{bmatrix} \begin{bmatrix}
t' \\
n' \\
b'
\end{bmatrix}
\]

The Frenet instantaneous rotation vector \( W \) of the timelike curve is given by

\[
W = \tau T - \kappa B, \tag{14} \tag{2.3}
\]

(2.3) leaves the real and dual components

\[
\begin{bmatrix}
w \\
w^*
\end{bmatrix} = \begin{bmatrix} k_2 t - k_i b \end{bmatrix} \begin{bmatrix} k_2 \end{bmatrix} + \begin{bmatrix} k_2^* t + k_i^* \end{bmatrix} \begin{bmatrix} k_i \end{bmatrix} - \begin{bmatrix} k_2 \end{bmatrix} \begin{bmatrix} k_i \end{bmatrix} \begin{bmatrix} b \end{bmatrix} - \begin{bmatrix} k_i^* \end{bmatrix} \begin{bmatrix} b^* \end{bmatrix}
\]
Let $\Phi = \varphi + \varepsilon \varphi^*$ be a Lorentzian timelike dual angle between the spacelike binormal unit vector $B$ and the Frenet instantaneous dual rotation vector $W$. Then $C = c + \varepsilon c^*$ is a unit dual vector in direction of $W$:

**a)** If $|\kappa| > |\tau|$, $W$ is a spacelike vector. In this station, we can write

\[
\begin{cases}
\kappa = \|W\| \cosh \Phi \\
\tau = \|W\| \sinh \Phi
\end{cases}
\]
and
\[
C = \sinh \Phi T - \cosh \Phi B.
\]  

(2.4)

**b)** If $|\kappa| < |\tau|$, $W$ is a timelike vector. In this station, we can write

\[
\begin{cases}
\kappa = \|W\| \sinh \Phi \\
\tau = \|W\| \cosh \Phi
\end{cases}, \quad \|W\|^2 = \langle W, W \rangle = \kappa^2 - \tau^2
\]
and
\[
C = \cosh \Phi T - \sinh \Phi B.
\]  

(2.5)

Let $M$ be a unit speed dual spacelike space curve with spacelike binormal. The Frenet vectors $T$, $N$, $B$ of $M$ are spacelike vector, timelike vector, spacelike vector, respectively, such that

\[
T \wedge N = -B, \quad N \wedge B = -T, \quad B \wedge T = N.
\]  

(2.6)

From here,

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
\kappa & 0 & \tau \\
0 & \tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}, \quad [18].
\]  

(2.7)

(2.9) leaves the real and dual components
and the Frenet instantaneous rotation vector for the spacelike curve is given by

\[ W = -\tau T + \kappa B, \]  \hspace{1cm} (2.10)

(2.10) leaves the real and dual components

\[ \vec{w} = -k^2t + k^2b, \]
\[ \vec{w'} = -k^2t - k^2t' + k^2b + k^2b' \]

Let \( \Phi = \phi + \epsilon \phi' \) be a dual angle between the \( B \) and the \( W \). If \( B \) and \( W \) spacelike vectors that span a spacelike vector subspace, we can write

\[
\begin{align*}
\kappa &= \|W\| \cos \Phi \\
\tau &= \|W\| \sin \Phi
\end{align*}
\]

and

\[ C = -\sin \Phi T + \cos \Phi B. \]  \hspace{1cm} (2.12)

Let \( M \) be a unit speed dual spacelike space curve. The Frenet vectors \( T, N \) and \( B \) of \( M \) are spacelike vector, timelike vector and spacelike vector, respectively, such that

\[ T \wedge N = B, \quad N \wedge B = -T, \quad B \wedge T = -N. \]  \hspace{1cm} (2.13)

From here

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-k & 0 & \tau \\
0 & -\kappa & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}, \]  \hspace{1cm} (2.14)

(2.14) leaves the real and dual components
and the Frenet instantaneous dual rotation vector $W$ of the spacelike curve is given by

\[ W = \tau T - \kappa B, \]  

\[ W = \tau T - \kappa B, \quad [14]. \]  

(2.15) leaves the real and dual components

\[ w = k_2 t - k_1 b, \]

\[ w^* = k_2^* t - k_1^* b - k_1 b^*. \]

Let $\Phi = \varphi + \epsilon \varphi^*$ be a Lorentzian timelike dual angle between the $B$ and $W$:

\( a \) If $|\kappa| < |\tau|$, $W$ is a spacelike vector. In this case, we can write

\[ \kappa = \|W\| \sinh \Phi \]

\[ \tau = \|W\| \cosh \Phi \]

and

\[ C = \cosh \Phi T - \sinh \Phi B. \]

\( b \) If $|\kappa| < |\tau|$, $W$ is a spacelike vector. In this case, we can write

\[ \kappa = \|W\| \cosh \Phi \]

\[ \tau = \|W\| \sinh \Phi \]

and

\[ C = \cosh \Phi T - \sinh \Phi B. \]

\( c \) If $|\kappa| > |\tau|$, $W$ is a timelike vector. In this case, we can write

\[ \kappa = \|W\| \cosh \Phi \]

\[ \tau = \|W\| \sinh \Phi \]

and

\[ C = \sinh \Phi T - \cosh \Phi B. \]
3. Main Results

Definition 3.1. Let $M_1 : I \to ID_1^3$, $M_1 = M_1(s)$ be the unit speed dual spacelike curve with spacelike binormal and $M_2 : I \to ID_1^3$, $M_2 = M_2(s)$ be the unit speed dual timelike curve. If tangent vector of curve $M_1$ is orthogonal to tangent vector of $M_2$, $M_1$ is called evolute of curve $M_2$ and $M_2$ is called involute of $M_1$. Thus the dual involute – evolute curve couple is denoted by $(M_2, M_1)$. Since the tangent vector of $M_1$ is spacelike, the tangent vector of $M_2$ must be timelike vector. So the tangent vector of $M_2$ must be timelike vector $(M_2, M_1)$ is called “the timelike – spacelike involute – evolute dual curve couple”.

Fig. 2. Involute – evolute curve couple.

Theorem 3.1. Let $(M_2, M_1)$ be the timelike – spacelike involute – evolute dual curve couple. Let $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of $M_1$ and $M_2$, respectively. The dual distance between $M_1$ and $M_2$ at the corresponding points is

$$d(M_1(s), M_2(s)) = |c_1 - s| + \varepsilon c_2$$

$c_2 = \text{constant}$.
**Proof.** If $M_2$ is the dual involute of $M_1$, we can write from the fig. 2

$$M_2(s) = M_1(s) + \lambda T(s),$$

where

$$\lambda = \lambda_1 + \varepsilon \lambda_1^* \in ID$$

Differentiating (3.1) with respect to $s$ we have

$$V_1 \frac{ds}{ds}^* = (1 + \lambda')T + \lambda \kappa N$$

where $s$ and $s^*$ are arc parameter of $M_1$ and $M_2$, respectively. Since the direction of $T$ is orthogonal to the direction of $V_1$, we obtain

$$\lambda' = -1.$$ From here, it can be easily seen

$$\dot{\lambda} = (c_1 - s) + \varepsilon c_2$$

(3.2)

Furthermore, the dual distance between the points $M_1(s)$ and $M_2(s)$

$$d(M_1(s), M_2(s)) = \sqrt{\left| (\lambda T(s) \lambda T(s)) \right|} = |\lambda| + \varepsilon \lambda^*.$$ Since $\lambda_1 = (c_1 - s)$, $\lambda_1^* = c_2$, we have

$$d(M_1(s), M_2(s)) = |c_1 - s| + \varepsilon c_2.$$ (3.3)

**Theorem 3.2.** Let $(M_2, M_1)$ be the timelike – spacelike involute – evolute dual curve couple.

Let $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of $M_1$ and $M_2$, respectively. Since the dual curvature of $M_2$ is $p = p + \varepsilon p^*$, we have

$$p^2 = \frac{k_1^2 + k_2^2}{(c_1 - s)^2 k_1^2} + \varepsilon \left[ \frac{2k_2 (k_1^2 k_2^* - k_1^* k_2)}{(c_1 - s)^2 k_1^2} - \frac{2c_2 (k_1^2 + k_3^2)}{(c_1 - s)^3 k_1^2} \right].$$

where the dual curvature of $M_1$ is $\kappa = k_1 + \varepsilon k_1^*$.

**Proof.** Differentiating (3.1), with respect to $s$, we get

$$\frac{dM_2}{ds} \frac{ds}{ds}^* = \frac{dM_1}{ds} + \dot{\lambda} T + \dot{\lambda}^* \frac{dT}{ds}$$

$$V_1 \frac{ds}{ds}^* = \lambda \kappa N.$$
From here, we can write

\[ V_1 = N \]  

(3.4)

and

\[ \frac{ds^*}{ds} = \lambda \kappa. \]

By differentiating the last equation and using (2.9), we obtain

\[ \frac{dV_1}{ds^*} \frac{ds^*}{ds} = \frac{dN}{ds} = \kappa T + \tau B, \]

\[ PV_2 = \frac{1}{\lambda \kappa} (\kappa T + \tau B) \]

From here, we have

\[ p^2 = \frac{(\kappa^2 + \tau^2)}{\lambda^2 \kappa^2} \]  

(3.5)

From the fact that \( P = p + \varepsilon p^* \), \( \lambda = \lambda_1 + \varepsilon \lambda_1^* \), \( \kappa = k_1 + \varepsilon k_1^* \) and \( \tau = k_2 + \varepsilon k_2^* \) we get

\[ p^2 = \left( \frac{k_1^2 + 2\varepsilon k_1 k_1^* + k_2^2 + 2\varepsilon k_2 k_2^*}{\lambda_1^2 + 2\varepsilon \lambda_1 \lambda_1^*} \right) \left( \frac{k_1^2 + 2\varepsilon k_1 k_1^*}{\lambda_1^2 k_1^2} \right) \]

\[ = \frac{k_1^2 + k_2^2}{\lambda_1^2 k_1^2} + \varepsilon \left[ \frac{2k_2 (k_1 k_1^* - k_2 k_2^*)}{\lambda_1^2 k_1^2} - \frac{2\lambda_1^* (k_1^2 + k_2^2)}{\lambda_1^3 k_1^2} \right]. \]

From here, by using \( \lambda_1 = (c_1 - s) \), \( \lambda_2 = c_2 \), we obtain

\[ p^2 = \frac{k_1^2 + k_2^2}{(c_1 - s)^2 k_1^2} + \varepsilon \left[ \frac{2k_2 (k_1 k_1^* - k_2 k_2^*)}{(c_1 - s)^3 k_1^3} - \frac{2c_2 (k_1^2 + k_2^2)}{(c_1 - s)^3 k_1^2} \right]. \]  

(3.6)

**Theorem 3.3.** Let \((M_2, \mathcal{D}_2)\) be the timelike – spacelike involute – evolute dual curve couple.

Let \( \{T, N, B\} \) and \( \{V_1, V_2, V_3\} \) be the dual Frenet frames of \( M_1 \) and \( M_2 \), respectively. The dual torsion \( \tau = k_2 + \varepsilon k_2^* \) of \( M_1 \) and the dual torsion \( Q = q + \varepsilon q^* \) of \( M_2 \) is the following equation

\[
Q = \frac{k_2 k_1' - k_1 k_2'}{(k_1^2 + k_2^2)k_1 |c_1 - s|} + \varepsilon \left[ \frac{k_1 (k_1' k_2 - k_2' k_1)}{(k_1^2 + k_2^2)k_1 |c_1 - s|} + \frac{(2k_1 k_1 + 2k_2 k_2^*) (k_1 k_1' - k_1' k_1)}{(k_1^2 + k_2^2)^3 k_1 |c_1 - s|} \right].
\]
**Proof.** By differentiating (3.1) three times with respect to \(s\), we get

\[
M_2' = \lambda \kappa N
\]
\[
M_2'' = \lambda \kappa^2 T + \left(\lambda \kappa' - \kappa\right) N + \lambda \kappa \tau B
\]
\[
M_2''' = \left(3\lambda \kappa \kappa' - 2\kappa^2\right) T + \left(\lambda \kappa^3 + \lambda \kappa \tau^2 - 2\kappa' + \lambda \kappa''\right) N + \left(-2\kappa \tau + 2\lambda \kappa' \tau + \lambda \kappa'\right) B
\]

The vector product of \(M_2'\) and \(M_2''\) are

\[
M_2' \wedge M_2'' = -\lambda^2 \kappa^2 \tau T + \lambda^2 \kappa^3 B = \lambda^2 \kappa^3 \left(-\tau T + \kappa B\right)
\]  

(3.7)

From here, we obtain

\[
\left\|M_2' \wedge M_2''\right\|^2 = |\lambda|^4 |k|^4 \left(\kappa^2 + \tau^2\right)
\]  

(3.8)

and

\[
\det(M_2', M_2'', M_2''') = \lambda^3 \kappa^3 \left(\kappa' \tau - \kappa \tau'\right).
\]  

(3.9)

Substituting by (3.8) and (3.9) values into \(Q = \frac{\det(M_2', M_2'', M_2''')}{\left\|M_2' \wedge M_2''\right\|^2}\), we get

\[
Q = \frac{\left(\kappa' \tau - \kappa \tau'\right)}{|\lambda| \kappa \left(\kappa^2 + \tau^2\right)}
\]  

(3.10)

and then substituting \(Q = q + \varepsilon q^*\), \(\lambda = \lambda_1 + \varepsilon \lambda_1^*\), \(\kappa = k_1 + \varepsilon k_1^*\) and \(\tau = k_2 + \varepsilon k_2^*\) into the last equation, we have

\[
Q = \frac{\left(k_1' + \varepsilon k_1''\right)\left(k_2 + \varepsilon k_2^*\right) - \left(k_1 + \varepsilon k_1^*\right)\left(k_2' + \varepsilon k_2''\right)}{|\lambda_1 + \varepsilon \lambda_1^*| \left(k_1 + \varepsilon k_1^*\right)\left((k_1^2 + k_2^2) + \varepsilon \left(2k_1 k_1^* + 2k_2 k_2^*\right)\right)}
\]

\[
= \frac{\left(k_2 k_1' - k_1' k_2\right) + \varepsilon \left(k_1' k_2' - k_1 k_2'' + k_1'' k_2 - k_1 k_2'\right)}{|\lambda_1| k_1 + \varepsilon |\lambda_1| k_1^*\left((k_1^2 + k_2^2) + \varepsilon \left(2k_1 k_1^* + 2k_2 k_2^*\right)\right)}
\]

\[
= \frac{k_2 k_1' - k_1' k_2}{\lambda_1 |k_1 (k_1^2 + k_2^2) + \varepsilon \left(k_1 \left(k_1 k_2' - k_2 k_1''\right) + k_2 \left(k_1 k_2'' - k_1 k_2'\right) + \frac{2k_1 k_1^* + 2k_2 k_2^*\left(k_1 k_2' - k_2 k_1''\right)}{|\lambda_1| k_1 (k_1^2 + k_2^2)}\right) + \frac{|\lambda_1| k_1 (k_1^2 + k_2^2)}{|\lambda_1| k_1 (k_1^2 + k_2^2)^2}}
\]
By the fact that \( \lambda_i = (c_i - s) \), we get
\[
Q = \frac{k_2 k'_2 - k'_1 k_1}{c_i - s} + k_2 \left( \frac{k_1 (k_1' k_2' - k_2 k_1'') + k_2 (k_2'' k_1' - k_1 k_2'')}{c_i - s} \right) + \left( \frac{2 k_2 k_1' + 2 k_2 k_1''}{c_i - s} \right) \left( k_2 k_1' - k_1 k_2' \right) \tag{3.11}
\]

**Theorem 3.4.** Let \((M_2, \mathcal{M}_1)\) be the timelike –spacelike involute – evolute dual curve couple. Let \(\{T, N, B\}\) and \(\{V_1, V_2, V_3\}\) be the dual Frenet frames of \(M_1\) and \(M_2\), respectively and \(\Phi = \varphi + \varepsilon \varphi^*\) be the Lorentzian dual spacelike angle between binormal vector \(B\) and \(W\). For \((M_2, \mathcal{M}_1)\) dual curve couple, the following equations is obtained:
\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 \\
-\cosh \Phi & 0 & -\sinh \Phi \\
-\sinh \Phi & 0 & \cosh \Phi
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\]
leaves the real and dual components
\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 \\
-\cos \varphi & 0 & -\sin \varphi \\
-\sin \varphi & 0 & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
t \\
n \\
b
\end{bmatrix}
\]
\[
\begin{bmatrix}
v_1^* \\
v_2^* \\
v_3^*
\end{bmatrix}
= \varphi^* 
\begin{bmatrix}
0 & 0 & 0 \\
\sin \varphi & 0 & -\cos \varphi \\
-\cos \varphi & 0 & -\sin \varphi
\end{bmatrix}
\begin{bmatrix}
t \\
n \\
b
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 1 & 0 \\
-\cos \varphi & 0 & -\sin \varphi \\
-\sin \varphi & 0 & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
t^* \\
n^* \\
b^*
\end{bmatrix}
\]

**Proof.** From (2.11), (3.4) and (3.8), we have,
\[
\left\| M_2' \wedge M_2'' \right\| = \lambda^2 \kappa^2 \left\| W \right\|. \tag{3.12}
\]
By using (3.7) and (3.12) and from the fact that \( V_3 = \frac{M_2' \wedge M_2''}{\left\| M_2' \wedge M_2'' \right\|} \) we obtain
\[
V_3 = -\frac{\tau}{\left\| W \right\|} T + \frac{\kappa}{\left\| W \right\|} B,
\]
substituting (2.11) into the last equation, we obtain
\[
V_3 = -\sin \Phi T + \cos \Phi B. \tag{3.13}
\]
Since \( V_2 = -V_1 \wedge V_1 \), it can be easily seen that

\[
V_2 = -\cos \Phi T - \sin \Phi B .
\]  
(3.14)

Considering (3.4), (3.13) and (3.14) according to dual components, the following equations are obtained:

\[
\begin{align*}
V_1 &= n + \epsilon n^* \\
V_2 &= (-\cos \phi t - \sin \phi b^*) + \epsilon \left( (-\cos \phi t^* - \sin \phi b^*) + \phi^* (\sin \phi t - \cos \phi b) \right) \\
V_3 &= (-\sin \phi t + \cos \phi b) + \epsilon \left( (-\sin \phi t^* + \cos \phi b^*) + \phi^* (-\cos \phi t - \sin \phi b) \right)
\end{align*}
\]  
(3.15)

written (3.15) in matrix form, the proof is completed.

**Theorem 3.5.** Let \((M_2, \bar{M}_1)\) be the timelike – spacelike involute – evolute dual curve couple, \(W = w + \epsilon w^*\) and \(\bar{W} = \bar{w} + \epsilon \bar{w}^*\) be the dual Frenet instantaneous rotation vectors of \(M_1\) and \(M_2\) respectively. Thus,

\[
\bar{W} = \frac{1}{|\lambda|} \kappa (-\Phi' N + W)
\]

**Proof.** From (2.3), we can write

\[\bar{W} = QV_1 - PV_3.\]

Using the (3.4), (3.5), (3.10) and (3.13) the equations, we have

\[
\bar{W} = \frac{\left( \kappa' \tau - \kappa \tau' \right)}{|\lambda| \kappa \left( \kappa^2 + \tau^2 \right)} N - \frac{\sqrt{\kappa^2 + \tau^2}}{|\lambda| \kappa} (-\sin \Phi T + \cos \Phi B). 
\]  
(3.16)

Substituting (2.11) into the last equation, we obtain

\[
\bar{W} = \frac{1}{|\lambda| \kappa} \frac{\kappa \tau' - \kappa' \tau}{\kappa^2 + \tau^2} N - W
\]

and then, we get

\[
\bar{W} = \frac{1}{|\lambda| \kappa} (-\Phi' N + W). 
\]  
(3.17)

Considering (3.17) according to dual components and substituting \(\lambda_1 = (c_1 - s)\) into (3.17), we leaves the real and dual components.
Theorem 3.6. Let \((M_2, h, I_1)\) be the timelike – spacelike involute – evolute dual curve couple, 
\[ C = c + \varepsilon c^* \quad \text{and} \quad \overline{C} = \overline{c} + \varepsilon \overline{c}^* \] 
be unit dual vector of \(W\) and \(\overline{W}\), respectively. Thus,
\[
\overline{C} = -\frac{-\Phi'}{\sqrt{|k^2 + \tau^2 + \Phi'^2|}} N + \frac{\sqrt{k^2 + \tau^2}}{\sqrt{|k^2 + \tau^2 + \Phi'^2|}} \overline{C}.
\]

**Proof.** From the fact that the unit dual vector of \(\overline{W}\) is \(\overline{C} = \frac{\overline{W}}{|\overline{W}|}\) we obtain
\[
\overline{C} = -\frac{-\Phi'}{\sqrt{|k^2 + \tau^2 + \Phi'^2|}} N + \frac{\sqrt{k^2 + \tau^2}}{\sqrt{|k^2 + \tau^2 + \Phi'^2|}} \overline{C}.
\]

(3.19)

or
\[
\overline{C} = -\frac{-\Phi'}{\sqrt{|k^2 + \tau^2 + \Phi'^2|}} N + \frac{\sqrt{k^2 + \tau^2}}{\sqrt{|k^2 + \tau^2 + \Phi'^2|}} \overline{C}.
\]

(3.20)

(3.20) leaves the real and dual components
\[
\left\{ \begin{array}{l}
\overline{c} = \frac{1}{\sqrt{k_1^2 + k_2^2 + \phi'^2}} \left( -\phi' n + \sqrt{k_1^2 + k_2^2} c \right), \\
\overline{c}^* = \frac{1}{\sqrt{k_1^2 + k_2^2 + \phi'^2}} \left( -\phi' n^* - \phi' n + \sqrt{k_1^2 + k_2^2} c^* + \frac{k_1 k_1^* + k_2 k_2^*}{\sqrt{k_1^2 + k_2^2}} c \right)
\end{array} \right.
\]

(3.21)

REFERENCES


