NEIGHBORHOOD DEGREES OF m-BIPOLAR FUZZY GRAPH

RAMAKRISHNA MANKENA¹, T.V. PRADEEP KUMAR², CH. RAMPRASAD³*, K. V. RANGA RAO⁴,
T. SRINIVASA RAO⁵

¹Acharya Nagarjuna University and Department of Mathematics, Malla Reddy College of Engineering, Hyderabad, India
²Department of Mathematics, University College of Engineering, Acharya Nagarjuna University, Nagarjuna Nagar
³Department of Mathematics, Vasireddy Venkatadri Institute of Technology, Namburu, 522 508, India
⁴Department of CSE, V. F. S. T. R., Vadlamudi, 522 213, India
⁵Department of Mathematics, K L E F, Vaddeswaram, 522 502, India

Abstract: In this article, neighborhood, open and closed neighborhood degrees of the vertices in an m-bipolar fuzzy graph (m-BPFG) are discussed. Also, strongly regular and biregular m-BPFG are defined with some basic theorems and examples.

Keywords: m-bipolar fuzzy graph; strongly regular m-BPFG; biregular m-BPFG.

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1. INTRODUCTION

Fuzzy sets are initiated for the parameters to solve problems related to vague and uncertain in
real life situations are demonstrated by Zadeh [15] in 1965. The limitations of traditional model were overcome by the introduction of bipolar fuzzy concept in 1994 by Zhang [16, 17]. This was further improved by Chen et al. [7] to m-polar fuzzy set theory.

Free body diagrams using set of nodes connected by lines representing pairs are good problem solving tools in non-deterministic real life situations. Thus, Kaufmann [11] was first set up the thought of fuzzy graph is extracted from Zadeh fuzzy relation. Rosenfeld [12] gave the concept of fuzzy vertex, fuzzy edges and fuzzy cycle etc. Akram et al. [1-5] played a crucial role in studying some major properties of bipolar fuzzy graphs, interval-valued fuzzy graphs and m-polar fuzzy graphs which paved way for the decision making in resolving social problems with fuzzy environment. Later Rashmanlou et al. [14] studied the categorical properties of bipolar fuzzy graphs. Ghorai and Pal [8-10] studied the concept of m-polar fuzzy graphs and studied some of its properties. Ramprasad et al. [13] gave the idea of product m-polar fuzzy line and intersection graphs. Bera and pal [6] introduced the concept of m-polar interval-valued fuzzy graph and studied some algebraic properties like density, regularity and irregularity etc. on m-PIVFG.

This paper attempts to develop theory to analyze parameters combining concepts from m-polar fuzzy graphs and bipolar fuzzy graphs as a unique effort. The resultant graph is turned m-BPFG and studied properties on it.

2. PRELIMINARIES
Every vertex and edge of an m-polar fuzzy graph has $m$ elements and those elements are fixed. But these elements may be bipolar. By this arrangement, m-BPFG has been initiated.

Before defining m-bipolar fuzzy graph, we suppose the following:

For a supposed set $V$, classify an equivalence relation $\leftrightarrow$ on $V \times V - \{(u, u): u \in V\}$ as follows: $(u_1, v_1) \leftrightarrow (u_2, v_2) \leftrightarrow$ either $(u_1, v_1) = (u_2, v_2)$ or $u_1 = v_2, v_1 = u_2$. The quotient set got in this way is represented by $\overline{V^2}$. 
Throughout this research paper, we assume $G^*$ as a crisp graph $G^* = (V, E)$.

**Definition 2.1.** An m-bipolar fuzzy graph (m-BPFG) of $G^*$ is a pair $G=(V, S, T)$ where $S=\left[p_j \circ \psi^p_j, p_j \circ \psi^n_j\right]_{j=1}^{m}$, $p_j \circ \psi^p_j : V \to [0, 1]$ and $p_j \circ \psi^n_j : V \to [-1, 0]$ is an m-BPFS on $V$; and $T=\left[p_j \circ \psi^+_j, p_j \circ \psi^-_j\right]_{j=1}^{m}$, $p_j \circ \psi^+_j : \overline{V^2} \to [0, 1]$ and $p_j \circ \psi^-_j : \overline{V^2} \to [-1, 0]$ is an m-BPFS in $\overline{V^2}$ such that $p_j \circ \psi^+_j(k, l) \leq \min\{p_j \circ \psi^p_j(k), p_j \circ \psi^p_j(l)\}$, $p_j \circ \psi^-_j(k, l) \geq \max\{p_j \circ \psi^n_j(k), p_j \circ \psi^n_j(l)\}$ for all $(k, l) \in \overline{V^2}$, $j=1, 2, \ldots, m$ and $p_j \circ \psi^-_j(k, l) = p_j \circ \psi^n_j(k, l) = 0$ for all $(k, l) \in \overline{V^2} - E$.

**Definition 2.2.** An m-BPFG $G=(V, S, T)$ of $G^*$ is complete if for every $s, t \in V$ and $j=1, 2, \ldots, m$ satisfying $p_j \circ \psi^+_j(s, t) = \min\{p_j \circ \psi^p_j(s), p_j \circ \psi^p_j(t)\}$, $p_j \circ \psi^-_j(s, t) = \max\{p_j \circ \psi^n_j(s), p_j \circ \psi^n_j(t)\}$.

**Definition 2.3.** An m-BPFG $G=(V, S, T)$ of $G^*$ is strong if for every $(s, t) \in E$ and $j=1, 2, \ldots, m$ satisfying $p_j \circ \psi^+_j(s, t) = \min\{p_j \circ \psi^p_j(s), p_j \circ \psi^p_j(t)\}$, $p_j \circ \psi^-_j(s, t) = \max\{p_j \circ \psi^n_j(s), p_j \circ \psi^n_j(t)\}$.

**Definition 2.4.** Let $G=(V, S, T)$ be an m-BPFG of $G^*$. The complement of $G$ is an m-BPFG $\overline{G} = (V, \overline{S}, \overline{T})$ of $G^* = (V, \overline{V^2})$ such that $\overline{S} = S$ and $\overline{T}$ is defined by $p_j \circ \psi^+_j(s, t) = p_j \circ \psi^p_j(s, t)$, $p_j \circ \psi^-_j(s, t) = p_j \circ \psi^n_j(s, t)$ for every $(s, t) \in \overline{V^2}$ and $j=1, 2, \ldots, m$.

### 3. Regularity on m-BPFG

In this section, neighborhood degree of a vertex, open and closed neighborhood degree of vertices are defined and studied some of its properties.
Definition 3.1. The neighborhood degree of a vertex \( r \in V \) in an m-BPFG \( G = (V, S, T) \) is defined as
\[
d_N(r) = \left( \left[ p_j \circ d^p_S(r), p_j \circ d^m_N(r) \right] \right)_{j=1}^m = \left( \sum_{t \in N(r)} p_j \circ \psi_S^p(t), \sum_{t \in N(r)} p_j \circ \psi_S^m(t) \right)_{j=1}^m.
\]

Definition 3.2. The open neighborhood degree of a vertex \( r \in V \) in an m-BPFG \( G = (V, S, T) \) is defined as
\[
d_G(r) = \left( \left[ p_j \circ d^p_G(r), p_j \circ d^m_G(r) \right] \right)_{j=1}^m = \left( \sum_{(r,s) \in E} p_j \circ \psi_T^p(r,s), \sum_{(r,s) \in E} p_j \circ \psi_T^m(r,s) \right)_{j=1}^m.
\]

Definition 3.3. The closed neighborhood degree of a vertex \( r \in V \) in an m-BPFG \( G = (V, S, T) \) is defined as
\[
d_G[r] = \left( \left[ p_j \circ d^p_G[r], p_j \circ d^m_G[r] \right] \right)_{j=1}^m
\]
\[
= \left( \sum_{(r,s) \in E} p_j \circ \psi_T^p(r,s), \sum_{(r,s) \in E} p_j \circ \psi_T^m(r,s) \right)_{j=1}^m + \left( \left[ p_j \circ \psi_S^p(r), p_j \circ \psi_S^m(r) \right] \right)_{j=1}^m.
\]

Definition 3.4. An m-BPFG \( G = (V, S, T) \) of \( G^* \) is said to be \( \left[ \eta^p_j, \eta^m_j \right] \) - regular if all the vertices in \( G \) have same open neighborhood degrees \( \left[ \eta^p_j, \eta^m_j \right] \).

Definition 3.5. An m-BPFG \( G = (V, S, T) \) of \( G^* \) is said to be \( \left[ \gamma^p_j, \gamma^m_j \right] \) - totally regular if all the vertices in \( G \) have same closed neighborhood degrees \( \left[ \gamma^p_j, \gamma^m_j \right] \).

Definition 3.6. Let \( G = (V, S, T) \) be an m-BPFG of \( G^* \)

Then the order of \( G \) is
\[
O(G) = \left( \left[ p_j \circ O^p(G), p_j \circ O^m(G) \right] \right)_{j=1}^m = \left( \sum_{r \in V} p_j \circ \psi_S^p(r), \sum_{r \in V} p_j \circ \psi_S^m(r) \right)_{j=1}^m,
\]
and the size of \( G \) is
\[
S(G) = \left( \left[ p_j \circ S^p(G), p_j \circ S^m(G) \right] \right)_{j=1}^m = \left( \sum_{(r,s) \in E} p_j \circ \psi_T^p(r,s), \sum_{(r,s) \in E} p_j \circ \psi_T^m(r,s) \right)_{j=1}^m.
\]
Proposition 3.1. Let $G = (V, S, T)$ be a $\left[\eta_j^p, \eta_j^n\right]_{j=1}^m$-regular m-BPFG of $G^*$. Then $S(G) = \frac{n}{2}\left[\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right]$ where $|V| = n$.

Proof. Suppose $G$ is a $\left[\eta_j^p, \eta_j^n\right]_{j=1}^m$-regular m-BPFG. Then $d_G(r) = \left[\eta_j^p, \eta_j^n\right]_{j=1}^m$ for all $r \in V$.

This implies that $\left[\sum_{(r,s) \in E} p_j \cdot \psi_j^p(r,s), \sum_{(r,s) \in E} p_j \cdot \psi_j^p(r,s)\right]_{j=1}^m = \left[\eta_j^p, \eta_j^n\right]_{j=1}^m$ for all $r \in V$.

Hence $S(G) = \frac{n}{2}\left[\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right]$. \qed

Proposition 3.2. Let $G = (V, S, T)$ be a $\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$-totally regular m-BPFG of $G^*$. Then $2S(G) + O(G) = n\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$ where $|V| = n$.

Proof. Suppose $G$ is a $\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$-totally regular m-BPFG. Then $d_G[r] = \left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$ for all $r \in V$. This implies that $d_G(r) + \left[\sum_{(r,s) \in E} p_j \cdot \psi_j^p(r,s), \sum_{(r,s) \in E} p_j \cdot \psi_j^p(r,s)\right]_{j=1}^m = \left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$ for all $r \in V$.

Therefore $\sum_{r \in V} d_G(r) = \sum_{r \in V} \left[\sum_{(r,s) \in E} p_j \cdot \psi_j^p(r,s), \sum_{(r,s) \in E} p_j \cdot \psi_j^p(r,s)\right]_{j=1}^m = \sum_{r \in V} \left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$.

i.e. $2S(G) + O(G) = n\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$. \qed

Proposition 3.3. Let $G = (V, S, T)$ be a $\left[\eta_j^p, \eta_j^n\right]_{j=1}^m$-regular and $\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$-totally regular m-BPFG of $G^*$. Then $O(G) = n\left[\gamma_j^p - \eta_j^p, \gamma_j^n - \eta_j^n\right]_{j=1}^m$ where $|V| = n$.

Proof. From Proposition 3.2, we get $2S(G) + O(G) = n\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m$.

i.e. $O(G) = n\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m - 2S(G) = n\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m - 2\frac{n}{2}\left[\eta_j^p, \eta_j^n\right]_{j=1}^m = n\left[\gamma_j^p - \eta_j^p, \gamma_j^n - \eta_j^n\right]_{j=1}^m$. \qed
Theorem 3.1. Let $G = (V, S, T)$ be an m-BPFG of $G^*$. Then $S = \left\{ p_j \circ \psi_j^p, p_j \circ \psi_j^s \right\}_{j=1}^m$ is a constant function if and only if the subsequent conditions are equivalent.

(i) $G$ is $\left[ \eta_j^p, \eta_j^s \right]_{j=1}^m$-regular m-BPFG,

(ii) $G$ is $\left[ \gamma_j^p, \gamma_j^s \right]_{j=1}^m$-totally regular m-BPFG.

Proof. Suppose $S = \left[ p_j \circ \psi_j^p, p_j \circ \psi_j^s \right]_{j=1}^m$ is a constant function.

Then $\left[ p_j \circ \psi_j^p (r), p_j \circ \psi_j^s (r) \right]_{j=1}^m = \left[ \tau_j^p, \tau_j^s \right]_{j=1}^m \forall r \in V$, where $\tau_j^p \in [0, 1]$, $\tau_j^s \in [-1, 0]$ for all $j = 1, 2, \ldots, m$. Let $G$ be a $\left[ \eta_j^p, \eta_j^s \right]_{j=1}^m$-regular m-BPFG.

Then for all $r \in V$, $d_G(r) = \left[ \eta_j^p, \eta_j^s \right]_{j=1}^m$.

Hence, $G$ is $\left[ \eta_j^p + \tau_j^p, \eta_j^s + \tau_j^s \right]_{j=1}^m$-totally regular m-BPFG.

Conversely, suppose that conditions (i) and (ii) are equivalent. Now we have to prove that $\left[ p_j \circ \psi_j^p, p_j \circ \psi_j^s \right]_{j=1}^m$ is a constant function.

In a contrary way, we suppose that $\left[ p_j \circ \psi_j^p, p_j \circ \psi_j^s \right]_{j=1}^m$ is not a constant function.

Then $\left[ p_j \circ \psi_j^p (r), p_j \circ \psi_j^s (r) \right]_{j=1}^m \neq \left[ p_j \circ \psi_j^p (s), p_j \circ \psi_j^s (s) \right]_{j=1}^m$ for at least one pair of vertices $r, s \in V$. 


Let $G$ be a $\left\{ [\eta_j^p, \eta_j^n]_{j=1}^m \right\}$-regular m-BPFG. Then $d_G(r_i) = d_G(s_i) = \left\{ [\eta_j^p, \eta_j^n]_{j=1}^m \right\}$.

So for all $r_i, s_i \in V$,

$$d_G[r_i] = d_G(r_i) + \left\{ [p_j \circ \psi_S^p(r_i), p_j \circ \psi_S^n(r_i)]_{j=1}^m \right\} = \left\{ [\eta_j^p + p_j \circ \psi_S^p(r_i), \eta_j^n + p_j \circ \psi_S^n(r_i)]_{j=1}^m \right\},$$

$$d_G[s_i] = d_G(s_i) + \left\{ [p_j \circ \psi_S^p(s_i), p_j \circ \psi_S^n(s_i)]_{j=1}^m \right\} = \left\{ [\eta_j^p + p_j \circ \psi_S^p(s_i), \eta_j^n + p_j \circ \psi_S^n(s_i)]_{j=1}^m \right\}$$

and $d_G[r_i] \neq d_G[s_i]$ since $\left\{ [p_j \circ \psi_S^p(r_i), p_j \circ \psi_S^n(r_i)]_{j=1}^m \right\} \neq \left\{ [p_j \circ \psi_S^p(s_i), p_j \circ \psi_S^n(s_i)]_{j=1}^m \right\}$.

Thus, $G$ is not a totally regular m-BPFG. This contradicts our assumption. Hence $\left\{ [p_j \circ \psi_S^p, p_j \circ \psi_S^n]_{j=1}^m \right\}$ is a constant function.

Similarly, $\left\{ [p_j \circ \psi_S^p, p_j \circ \psi_S^n]_{j=1}^m \right\}$ is a constant function for totally regular m-BPFG. □

**Proposition 3.4.** Let $G = (V, S, T)$ be an m-BPFG of $G^*$ and $G$ is both regular and totally regular. Then $S = \left\{ [p_j \circ \psi_S^p, p_j \circ \psi_S^n]_{j=1}^m \right\}$ is constant.

**Proof.** Let $G$ be a $\left\{ [\eta_j^p, \eta_j^n]_{j=1}^m \right\}$-regular and $\left\{ [\gamma_j^p, \gamma_j^n]_{j=1}^m \right\}$-totally regular m-BPFG. Then

$$d_G[r] = d_G(r) + \left\{ [p_j \circ \psi_S^p(r), p_j \circ \psi_S^n(r)]_{j=1}^m \right\} = \left\{ [\gamma_j^p - \eta_j^p, \gamma_j^n - \eta_j^n]_{j=1}^m \right\}$$

for all $r \in V$. This shows that $S = \left\{ [p_j \circ \psi_S^p, p_j \circ \psi_S^n]_{j=1}^m \right\}$ is constant. □

**Example 3.1.** The converse of the above proposition need not be true. This can be proved with an example given below. The open and closed neighborhood degree of the vertices for the 2-BPFG $G$ of $G^*$ shown in Figure 1. are $d_G(A) = \langle [1.3, -1.1], [0.2, -0.3] \rangle$,

$$d_G(B) = \langle [1.1, -0.8], [0.25, -0.45] \rangle, \quad d_G(C) = \langle [1.4, -1.1], [0.25, -0.35] \rangle,$$

$$d_G[A] = \langle [2.2, -1.9], [0.4, -0.6] \rangle, \quad d_G[B] = \langle [2.0, -1.6], [0.45, -0.75] \rangle,$$

$$d_G[C] = \langle [2.3, -1.9], [0.45, -0.65] \rangle.$$
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Figure 1. $S = \left\{ p_j \circ \psi_S^n, p_j \circ \psi_S^n \right\}_{j=1}^m$ is constant

but $G$ is neither regular nor totally regular $m$-BPFG

Hence, it shows that $S$ is constant but $G$ is neither regular and nor totally regular $m$-BPFG.

Theorem 3.2. Let $G = (V, S, T)$ be an $m$-BPFG of an odd cycle of $G^*$. Then $G$ is regular $m$-BPFG if and only if $T = \left\{ p_j \circ \psi_T^n, p_j \circ \psi_T^n \right\}_{j=1}^m$ is constant.

Proof. Suppose $G$ is a $\left( \eta, \eta \right)_{j=1}^m$-regular $m$-BPFG. Let $t_1$, $t_2$, $t_3$, $\ldots$, $t_{2n+1}$ be the edges of $G^*$ such that $t_i = (r_{i-1}, r_i) \in E$, $r_0, r_i \in V$, $i = 1, 2, \ldots, 2n+1$ and $r_0 = r_{2n+1}$.

Let $\left\{ p_j \circ \psi_T^n(t_i), p_j \circ \psi_T^n(t_i) \right\}_{j=1}^m = \left\{ a^p_j, a^p_j \right\}_{j=1}^m$ where $a^p_j \in [0, 1]$, $a^n_j \in [-1, 0]$ for all $j = 1, 2, \ldots, m$. Since $G$ is $\left( \eta, \eta \right)_{j=1}^m$-regular, we have $d_G(r_i) = \left\{ \eta^p_j, \eta^n_j \right\}_{j=1}^m$.

This means,

$$d_G(r_i) = \left\{ p_j \circ \psi_T^n(t_i), p_j \circ \psi_T^n(t_i) \right\}_{j=1}^m = \left\{ p_j \circ \psi_T^n(t_1), p_j \circ \psi_T^n(t_1) \right\}_{j=1}^m = \left\{ \eta^p_j, \eta^n_j \right\}_{j=1}^m,$$

i.e. $\left\{ p_j \circ \psi_T^n(t_2), p_j \circ \psi_T^n(t_2) \right\}_{j=1}^m = \left\{ \eta^p_j, \eta^n_j \right\}_{j=1}^m - \left\{ p_j \circ \psi_T^n(t_1), p_j \circ \psi_T^n(t_1) \right\}_{j=1}^m$.

Again, $d_G(r_2) = \left\{ p_j \circ \psi_T^n(t_1), p_j \circ \psi_T^n(t_1) \right\}_{j=1}^m + \left\{ p_j \circ \psi_T^n(t_3), p_j \circ \psi_T^n(t_3) \right\}_{j=1}^m$

$$= \left\{ \eta^p_j, \eta^n_j \right\}_{j=1}^m.$$
Therefore, \( \left\{ p_j \circ \psi_T^p(t), p_j \circ \psi_T^n(t) \right\}_{j=1}^m = \left\{ \left[ a_j^p, a_j^n \right]_{j=1}^m \right\} \) if \( i \) is odd

\[ \left\{ \left[ \eta_j^p - a_j^p, \eta_j^n - a_j^n \right]_{j=1}^m \right\} \] if \( i \) is even

Hence, \( \left\{ p_j \circ \psi_T^p(t), p_j \circ \psi_T^n(t) \right\}_{j=1}^m = \left\{ p_j \circ \psi_T^p(t_{2n+1}), p_j \circ \psi_T^n(t_{2n+1}) \right\}_{j=1}^m = \left\{ a_j^p, a_j^n \right\}_{j=1}^m \).

Since \( t_1 \) and \( t_{2n+1} \) are incident on the vertex \( r_6 \) and \( d_G(r_6) = \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \), we have

\[ \left\{ p_j \circ \psi_T^p(t_1), p_j \circ \psi_T^n(t_1) \right\}_{j=1}^m + \left\{ p_j \circ \psi_T^p(t_{2n+1}), p_j \circ \psi_T^n(t_{2n+1}) \right\}_{j=1}^m = \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \]

i.e. \( \left\{ 2a_j^p, 2a_j^n \right\}_{j=1}^m = \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \), \( \left\{ a_j^p, a_j^n \right\}_{j=1}^m = \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \) for all \( i = 1, 2, \ldots, 2n+1 \).

Hence, \( T = \left\{ p_j \circ \psi_T^p, p_j \circ \psi_T^n \right\}_{j=1}^m \) is constant.

Conversely, let \( \left\{ p_j \circ \psi_T^p, p_j \circ \psi_T^n \right\}_{j=1}^m \) be a constant function.

Let \( \left\{ p_j \circ \psi_T^p(r, s), p_j \circ \psi_T^n(r, s) \right\}_{j=1}^m = \left\{ a_j^p, a_j^n \right\}_{j=1}^m \), for all \( (r, s) \in E \) where

\( a_j^p \in [0, 1], a_j^n \in [-1, 0] \) for all \( j = 1, 2, \ldots, m \).

Then \( d_G(r) = \left\{ \sum_{j \in E} p_j \circ \psi_T^p(r, s), \sum_{j \in E} p_j \circ \psi_T^n(r, s) \right\}_{j=1}^m = \left\{ 2a_j^p, 2a_j^n \right\}_{j=1}^m \) for all \( r \in V \).

Consequently, \( G \) is a \( \left\{ 2a_j^p, 2a_j^n \right\}_{j=1}^m \) -regular m-BPFG. \( \square \)

### 4. Strongly Regular Bipolar Fuzzy Graph

In this section, we initiated the concept of strongly regular and biregular m-BPFGS.

**Definition 4.1.** A finite m-BPFG \( G = (V, S, T) \) is said to be strongly regular m-BPFG if

(i) \( G \) is \( \eta = \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \)-regular m-BPFG,
(ii) The sum of the positive membership values and negative membership values of the common neighborhood vertices of any pair of adjacent vertices and non-adjacent vertices of $G$ has the same weight and is denoted by $\lambda = \left\{ \left[ \lambda_{j}^{p}, \lambda_{j}^{n} \right]_{j=1}^{m} \right\}$, $\delta = \left\{ \left[ \delta_{j}^{p}, \delta_{j}^{n} \right]_{j=1}^{m} \right\}$ respectively.

A strongly regular m-BPFG $G$ is denoted by $G = (n, \eta, \lambda, \delta)$ where $n = |V|$.

Example 4.1.

Let us consider the 2-BPFG $G = (V, S, T)$ of $G^{*} = (V, E)$ shown in Figure 2.

Here, $n = 4$, $\eta = \left\{ [1.5, -1.2], [1.8, -0.9] \right\}$, $\lambda = \left\{ [1.1, -1.0], [1.4, -1.2] \right\}$ and $\delta = \left\{ [0, 0], [0, 0] \right\}$. Hence $G$ is a strongly regular 2-BPFG.

Definition 4.2. An m-BPFG $G = (V, S, T)$ of $G^{*}$ is said to be a biregular m-BPFG if $G$ is $\eta = \left\{ \left[ \eta_{j}^{p}, \eta_{j}^{n} \right]_{j=1}^{m} \right\}$-regular m-BPFG and $V$ can be partitioned into $V_{1} \cup V_{2}$ such that each vertex in $V_{1}$ has the same neighborhood degree $M = \left\{ \left[ M_{j}^{p}, M_{j}^{n} \right]_{j=1}^{m} \right\}$ and each vertex in $V_{2}$ has the same neighborhood degree $N = \left\{ \left[ N_{j}^{p}, N_{j}^{n} \right]_{j=1}^{m} \right\}$, where $M$ and $N$ are constants.
Example 4.2.

Let us consider the 2-BPFG $G = (V, S, T)$ of $G^* = (V, E)$ shown in Figure 3.

Here $n = 8$, $\eta = \{1.7, -1, 0.9, -1.4\}$, $V_1 = \{A, C, X, Z\}$, $V_2 = \{B, D, W, Y\}$, $M = \{[2.4, -1.8], [1.2, -1.5]\}$ and $N = \{[1.8, -2.7], [1.2, -1.5]\}$. Hence $G$ is a biregular 2-BPFG.

**Theorem 4.1.** Let $G = (V, S, T)$ be a complete m-BPFG of $G^*$ in which $S$ and $T$ are constant functions. Then $G$ is strongly regular m-BPFG.

**Proof.** Let $G = (V, S, T)$ be a complete bipolar fuzzy graph where $V = \{t_1, t_2, ..., t_n\}$.

Let $S(t_k) = \left[\begin{array}{c} a_j^p \\ a_j^n \end{array}\right]_{j=1}^m$ for all $t_k \in V$ and $T(t_p, t_l) = \left[\begin{array}{c} b_j^p \\ b_j^n \end{array}\right]_{j=1}^m$ for all $(t_p, t_l) \in E$ where $a_j^p$, $a_j^n$, $b_j^p$, $b_j^n$ are constants. Since $G$ is complete, we have $G$ is $\lambda = \left[\begin{array}{c} (n-1)b_j^p \\ (n-1)b_j^n \end{array}\right]_{j=1}^m$-regular m-BPFG. Again $G$ is complete, therefore the sum of the positive membership values and negative membership values of the common neighborhood vertices of any pair of adjacent vertices has the same weight $\lambda = \left[\begin{array}{c} (n-2)a_j^p \\ (n-2)a_j^n \end{array}\right]_{j=1}^m$ and the sum of the positive membership values and negative membership values of the common neighborhood vertices of any pair of non adjacent vertices has the same weight.
\[ \delta = \{[0, 0], [0, 0], \ldots, [0, 0]\} \]. So \( G \) is strongly regular m-BPFG. \( \Box \)

**Theorem 4.2.** If \( G = (V, S, T) \) is a strongly regular m-BPFG which is strong, then \( \overline{G} \) is a \( \eta = \left\langle \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle \)-regular.

**Proof.** Let \( G = (V, S, T) \) be a strongly regular m-BPFG. Then by definition, \( G \) is \( \eta = \left\langle \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle \)-regular. Since \( G \) is strong and for all \( j = 1, 2, \ldots, m \), we have

\[
\begin{align*}
p_j \circ \psi^p_T(t_k, t_i) &= \begin{cases} 0 & \text{for all } (t_k, t_i) \in E \\ \{ p_j \circ \psi^p_S(t_k) \wedge p_j \circ \psi^p_S(t_i) \} & \text{for all } (t_k, t_i) \notin E \end{cases} \\
p_j \circ \psi^n_T(t_k, t_i) &= \begin{cases} 0 & \text{for all } (t_k, t_i) \in E \\ \{ p_j \circ \psi^n_S(t_k) \vee p_j \circ \psi^n_S(t_i) \} & \text{for all } (t_k, t_i) \notin E \end{cases}
\end{align*}
\]

Since \( G \) is strong, we have the degree of a vertex \( t_k \) in \( \overline{G} \) is

\[
d_{\overline{G}}(t_k) = \left\langle \left[ p_j \circ d^p_G(t_k), p_j \circ d^n_G(t_k) \right]_{j=1}^m \right\rangle
\]

where

\[
p_j \circ d^p_G(t_k) = \sum_{(t_k, t_i) \in E} p_j \circ \psi^p_T(t_k, t_i) = \sum_{(t_k, t_i) \notin E} \{ p_j \circ \psi^p_S(t_k) \wedge p_j \circ \psi^p_S(t_i) \} = \eta_j^p,
\]

\[
p_j \circ d^n_G(t_k) = \sum_{(t_k, t_i) \in E} p_j \circ \psi^n_T(t_k, t_i) = \sum_{(t_k, t_i) \notin E} \{ p_j \circ \psi^n_S(t_k) \vee p_j \circ \psi^n_S(t_i) \} = \eta_j^n,
\]

\( \forall t_k \in V, j = 1, 2, \ldots, m. \)

Hence \( d_{\overline{G}}(t_k) = \left\langle \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle \) \( \forall t_k \in V. \) So \( \overline{G} \) is \( \eta = \left\langle \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle \)-regular m-BPFG. \( \Box \)

**Theorem 4.3.** Let \( G = (V, S, T) \) be a strong m-BPFG. Then, \( G \) is a strongly regular if and only if \( \overline{G} \) is a strongly regular.

**Proof.** Suppose that \( G = (V, S, T) \) is a strongly regular m-BPFG. Then \( G \) is \( \left\langle \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle \)-regular and the adjacent vertices and the non-adjacent vertices have the same common neighborhood weight \( \left\langle \left[ \lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\rangle \) and \( \left\langle \left[ \delta_j^p, \delta_j^n \right]_{j=1}^m \right\rangle \) respectively. Now we have to prove that \( \overline{G} \) is strongly regular m-BPFG. If \( G \) is strongly regular m-BPFG and which is strong then
\( \bar{G} \) is \( \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \)-regular m-BPFG by Theorem 4.2. Next, let \( F_1 \) and \( F_2 \) be the set of all adjacent vertices and non-adjacent vertices of \( G \); \( \overline{F_1} \) and \( \overline{F_2} \) denote set of all adjacent vertices and non-adjacent vertices of \( \bar{G} \).

i.e. \( F_1 = \{(t_k, t_l) \mid (t_k, t_l) \in E\} \), where \( t_k \) and \( t_l \) have same common neighborhood weight \( \lambda = \left\{ \left[ \lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\} \) and \( F_2 = \{(t_k, t_l) \mid (t_k, t_l) \notin E\} \) where \( t_k \) and \( t_l \) have same common neighborhood weight \( \delta = \left\{ \left[ \delta_j^p, \delta_j^n \right]_{j=1}^m \right\} \). Then, \( \overline{F_1} = \{(t_k, t_l) \mid (t_k, t_l) \in \overline{E}\} \) where \( t_k \) and \( t_l \) have same common neighborhood weight \( \delta = \left\{ \left[ \delta_j^p, \delta_j^n \right]_{j=1}^m \right\} \) and \( \overline{F_2} = \{(t_k, t_l) \mid (t_k, t_l) \notin \overline{E}\} \), where \( t_k \) and \( t_l \) have a same common neighborhood weight \( \lambda = \left\{ \left[ \lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\} \). This implies \( \bar{G} \) is a strongly regular. Similarly, we can prove \( G \) is strongly regular if \( \bar{G} \) is strongly regular. \( \square \)

**Theorem 4.4.** A strongly regular m-BPFG \( G = (V, S, T) \) is a biregular m-BPFG if the adjacent vertices have the same common neighborhood weight \( \lambda = \left\{ \left[ \lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\} \neq \{[0, 0], [0, 0], \ldots, [0, 0]\} \) and the non-adjacent vertices have the same common neighborhood weight \( \delta = \left\{ \left[ \delta_j^p, \delta_j^n \right]_{j=1}^m \right\} \neq \{[0, 0], [0, 0], \ldots, [0, 0]\} \).

**Proof.** Let \( G = (V, S, T) \) be a strongly regular m-BPFG. Then we have \( d_G(t_k) = \left\{ \left[ \eta_j^p, \eta_j^n \right]_{j=1}^m \right\} \) for all \( t_k \in V \). Let \( F_1 \) be the set of all non-adjacent vertices of \( G \). Then \( F_1 \) is a non empty subset of \( V \) since non adjacent vertices have the same common neighborhood weight \( \delta = \left\{ \left[ \delta_j^p, \delta_j^n \right]_{j=1}^m \right\} \neq \{[0, 0], [0, 0], \ldots, [0, 0]\} \).

So, \( F_1 = \{t_k, t_l \mid t_k \neq l, t_k, t_l \in V\} \). Then the vertex partition of \( G \) is \( V_1 = \{t_k \mid t_k \in F_1\} \) and \( V_2 = \{t_l \mid t_l \in F_1\} \). Hence, \( G \) is a biregular m-BPFG. \( \square \)
CONCLUSIONS

In this paper, we proved some properties of open and closed neighborhood degree of the vertices in an m-BPFG. Also, strongly regular and biregular m-BPFGs are described with sustaining illustrations and theorems. In future, we intend our investigations to the other properties of m-BPFG and extend them to solve different decision making problems in fuzzy environment.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES


