ON $\hat{D}$-HOMEOMORPHISM IN TOPOLOGICAL SPACES

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Abstract. In this paper we introduce and investigate new class of maps called $\hat{D}$-homeomorphism, $\hat{D}$-quotient map and several characterization and some of their properties. Also we investigate its relationship with other types of functions.

Keywords: $\hat{D}$-open set; $\hat{D}$-closed set; $\hat{D}$-homeomorphism; $\hat{D}$-quotient map.

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1. INTRODUCTION

The notion homeomorphism plays a very important role in topology. By definition a homeomorphism between two topological spaces $X$ and $Y$ is a bijective map $f : (X, \tau) \rightarrow (Y, \sigma)$ when both $f$ and $f^{-1}$ are continuous map. K. Dass and G. Suresh [10] introduced $\hat{D}$-closed set in topological spaces. K. Dass and G. Suresh [3] introduced $\hat{D}$-continuous map, in topological spaces. In this paper we introduce the concept of $\hat{D}$-open maps, quasi $\hat{D}$-open maps and strongly $\hat{D}$-open maps in topological spaces and also $\hat{D}$-homeomorphism, strongly $\hat{D}$-homeomorphism and $\hat{D}$-quotient map are obtained.

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2. Preliminaries

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned and \( f : (X, \tau) \rightarrow (Y, \sigma) \) (or simply \( f : X \rightarrow Y \)) denotes a function \( f \) of a space \((X, \tau)\) into a space \((Y, \sigma)\). Let \( A \) be a subset of a space \( X \). The closure, the interior and complement of \( A \) are denoted by \( \text{cl}(A) \), \( \text{int}(A) \) and \( A^c \) respectively.

**Definition 2.1.** A subset \( A \) of a topological space \((X, \tau)\) is called

1) a pre-open set \([5]\) if \( A \subset \text{int}(\text{cl}(A)) \) and a pre-closed set if \( \text{cl}(\text{int}(A)) \subset A \),
2) a semi-open set \([2]\) if \( A \subset \text{cl}(\text{int}(A)) \) and a semi-closed set if \( \text{int}(\text{cl}(A)) \subset A \),
3) a semi-pre-open set \([7]\) (\( \beta \)-open \([1]\)) if \( A \subset \text{cl}(\text{int}(\text{cl}(A))) \) and a semi-preclosed set (\( = \beta \)-closed) if \( \text{int}(\text{cl}(\text{int}(A))) \subset A \).

**Definition 2.2.** Let \((X, \tau)\) be a topological space and \( A \subset X \)

1) an \( \omega \)-closed set \([8]\) (\( = \tilde{g} \)-closed \([9]\)) if \( \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is semi-open in \((X, \tau)\),
2) a \( D \)-closed set \([4]\) if \( \text{pcl}(A) \subset \text{int}(U) \) whenever \( A \subset U \) and \( U \) is \( \omega \)-open in \((X, \tau)\).

Complements of the above mentioned sets are called their respectively open sets.

**Definition 2.3.** A subset \( A \) of \((X, \tau)\) is called an \( \hat{D} \)-closed \([10]\) set if \( \text{spcl}(A) \subset U \) whenever \( A \subset U \) and \( U \) is \( \hat{D} \)-open in \((X, \tau)\). The class of all \( \hat{D} \)-closed sets in \((X, \tau)\) is denoted by \( \hat{Dc}(\tau) \).

That is, \( \hat{Dc}(\tau) = \{ A \subset X : A \text{ is } \hat{D} \text{-closed in } (X, \tau) \} \).

**Definition 2.4.** Let \((X, \tau)\) be a topological space and \( A \subset X \)

1) semi-pre interior of \( A \) denoted by \( \text{spint}(A) \) is the union of all semi-pre open subsets of \( A \)
2) semi-pre closure of \( A \) denoted by \( \text{spcl}(A) \) is the intersection of all semi-pre closed subsets of \( A \)

**Definition 2.5.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \hat{D} \)-continuous \([3]\) if \( f^{-1}(H) \) is \( \hat{D} \)-closed in \((X, \tau)\) for every closed set \( H \) in \( Y \).

**Definition 2.6.** A map \( f : X \rightarrow Y \) is called \( \hat{D} \)-irresolute \([6]\) if \( f^{-1}(F) \) is \( \hat{D} \)-closed in \( X \) for every \( \hat{D} \)-closed set \( F \) of \( Y \).
Proposition 2.7. [6] If \( f : X \to Y \) is \( \hat{D} \)-irresolute, then \( f \) is \( \hat{D} \)-continuous but not conversely.

Proposition 2.8. Let \( f : X \to Y \) and \( g : Y \to Z \) be any two maps. Then

(a) \( g \circ f \) is \( \hat{D} \)-irresolute if both \( f \) and \( g \) are \( \hat{D} \)-irresolute.

(b) \( g \circ f \) is \( \hat{D} \)-continuous if \( g \) is \( \hat{D} \)-continuous and \( f \) is \( \hat{D} \)-irresolute.

Proposition 2.9. Let \( X \) be a topological space, \( Y \) be a \( T_{\hat{D}} \)-space and \( f : X \to Y \) be a map. Then the following are equivalent:

(i) \( f \) is \( \hat{D} \)-irresolute,

(ii) \( f \) is \( \hat{D} \)-continuous.

3. \( \hat{D} \)-HOMEOMORPHISM

Definition 3.1. A map \( f : X \to Y \) is said to be an \( \hat{D} \)-open map if the image \( f(A) \) is \( \hat{D} \)-open in \( Y \) for each open set \( A \) in \( X \).

Example 3.2. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, Y\} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be an identity map. Here \( \hat{D}o(\sigma) = P(X) - \{c\} \). Then \( f \) is an \( \hat{D} \)-open map.

Theorem 3.3. A surjective map \( f : X \to Y \) is \( \hat{D} \)-open if and only if for any subset \( S \) of \( Y \) and for any closed \( F \) containing \( f^{-1}(S) \), there exists an \( \hat{D} \)-closed set \( K \) of \( Y \) containing \( S \) such that \( f^{-1}(K) \subset F \).

Theorem 3.4. For any bijection \( f : X \to Y \), the following conditions are equivalent.

i) \( f^{-1} : Y \to X \) is \( \hat{D} \)-continuous.

ii) \( f \) is an \( \hat{D} \)-open map.

iii) \( f \) is an \( \hat{D} \)-closed map.

Proof. (i) \( \implies \) (ii): Let \( U \) be an open set of \( X \). By assumption \( (f^{-1})^{-1}(U) = f(U) \) is \( \hat{D} \)-open in \( Y \) and so \( f \) is \( \hat{D} \)-open.

(ii) \( \implies \) (iii): Let \( F \) be a closed set of \( X \). Then \( F^c \) is open in \( X \). By (ii), \( f(F^c) \) is \( \hat{D} \)-open in \( Y \) and therefore \( f(F^c) = (f(F))^c \) is \( \hat{D} \)-open in \( Y \). Thus \( f(F) \) is \( \hat{D} \)-closed in \( Y \) implies \( f \) is \( \hat{D} \)-closed.
(iii) \implies (i): Let \( F \) be a closed set of \( X \). By (iii), \( f(F) \) is \( \hat{D} \)-closed in \( Y \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is \( \hat{D} \)-continuous. \hfill \Box 

**Definition 3.5.** A map \( f : X \to Y \) is said to be strongly \( \hat{D} \)-open if the image of every \( \hat{D} \)-open set in \( X \) is \( \hat{D} \)-open in \( Y \).

**Definition 3.6.** A map \( f : X \to Y \) is said to be quasi-\( \hat{D} \)-open if the image every \( \hat{D} \)-open set in \( X \) is open in \( Y \).

**Theorem 3.7.** A surjective map \( f : X \to Y \) is quasi-\( \hat{D} \)-open if and only if for any subset \( B \) of \( Y \) and any \( \hat{D} \)-closed set \( F \) of \( X \) containing \( f^{-1}(B) \), there exists a closed set \( G \) of \( Y \) containing \( B \) such that \( f^{-1}(G) \subseteq F \).

**Proof.** Suppose \( f \) is quasi-\( \hat{D} \)-open. Let \( B \subseteq Y \) and \( F \) be an \( \hat{D} \)-closed set of \( X \) containing \( f^{-1}(B) \). Now, put \( G = (f(F))^c \). Then \( G \) is a closed set of \( Y \) containing \( B \) such that \( f^{-1}(G) \subseteq F \).

Conversely, let \( U \) be an \( \hat{D} \)-open set of \( X \) and put \( B = (f(U))^c \). Then \( U^c \) is an \( \hat{D} \)-closed set in \( X \) containing \( f^{-1}(B) \). By hypothesis, there exists a closed set \( F \) of \( Y \) such that \( B \subseteq F \) and \( f^{-1}(F) \subseteq U^c \). Hence, we obtain \( f(U) \subseteq F^c \). On the other hand it follows that \( B \subseteq F \), \( F^c \subseteq B^c = f(U) \). Thus we obtain \( f(u) = F^c \) which is open in \( Y \) and hence \( f \) is quasi-\( \hat{D} \)-open map. \hfill \Box 

**Remark 3.8.** From the above definitions we obtain the following implications.

 quasi-\( \hat{D} \)-open strongly-\( \hat{D} \)-open \(\implies\) \(\hat{D} \)-open. However the reverse implications are not true by the following examples.

**Example 3.9.** Let \( X = \{p, q, r\} \), \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\} \) and \( \sigma = \{\phi, \{p, q\}, Y\} \). Clearly identity map \( f : (X, \tau) \to (Y, \sigma) \) is strongly \( \hat{D} \)-open map but not quasi \( \hat{D} \)-open map, since \( \{q\} \) is \( \hat{D} \)-open in \( X \) but \( f(\{q\}) = \{q\} \) is not open in \( Y \).

**Example 3.10.** Let \( X = \{p, q, r\} \), \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{p, q\}, X\} \) and \( \sigma = \{\phi, \{r\}, \{q, r\}, X\} \). Clearly identity map \( f : (X, \tau) \to (Y, \sigma) \) is \( \hat{D} \)-open map but not strongly \( \hat{D} \)-open, Since \( \{q\} \) is \( \hat{D} \)-open in \( X \) but \( f(\{q\}) = \{q\} \) is not \( \hat{D} \)-open in \( Y \).

**Theorem 3.11.** For any bijection \( f : X \to Y \), the following conditions are equivalent:
i) $f^{-1} : Y \to X$ is $\hat{D}$-irresolute,

ii) $f$ is a strongly $\hat{D}$-open map,

iii) $f$ is a strongly $\hat{D}$-closed map.

Proof. Similar to that of Theorem 3.4. □

**Definition 3.12.** A bijection $f : X \to Y$ is called $\hat{D}$-homeomorphisms if $f$ is both $\hat{D}$-continuous and $\hat{D}$-open.

**Proposition 3.13.** Every homeomorphism is an $\hat{D}$-homeomorphism but not conversely.

Proof. Follows from Definitions. □

**Example 3.14.** Let $X = \{p,q,r\}$ and $Y = \{p,q,r\}$, $\tau = \{\phi, \{p\}, \{q\}, \{p,q\}, X\}$ and $\sigma = \{\phi, \{p\}, \{q\}, Y\}$. Clearly identity map $f : (X, \tau) \to (Y, \sigma)$ is $\hat{D}$-homeomorphisms but not homeomorphisms, Since $f(\{q\}) = \{q\}$ is open in $X$ but $f(\{p\}) = \{q\}$ is not open in $Y$, hence $f$ is not an open map.

**Theorem 3.15.** Let $f : X \to Y$ be a bijective, $\hat{D}$-continuous map. Then the following conditions are equivalent:

i) $f$ is an $\hat{D}$-open map,

ii) $f$ is an $\hat{D}$-homeomorphism,

iii) $f$ is an $\hat{D}$-closed map.

Proof. (i) $\implies$ (ii) : Obvious from definition.

(ii) $\implies$ (iii) : Suppose $f$ is an $\hat{D}$-open map and let $F$ be a closed set in $X$. Then $F^c$ is open in $X$, hence $f(F^c) = (f(F))^c$ is $\hat{D}$-open in $Y$ implies $f$ is a closed map Converse follows by the same technique. □

**Remark 3.16.** The composition of two $\hat{D}$-homeomorphisms need not be an $\hat{D}$-homeomorphisms as seen from the following example.

**Example 3.17.** Let $X = Y = Z = \{p,q,r\}$, $\tau = \{\phi, \{p\}, X\}$, $\sigma = \{\phi, \{p,q\}, Y\}$ and $\eta = \{\phi, \{p\}, \{q\}, \{p,q\}, Z\}$. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two identity map. Then $f$ and
\(g\) are \(\hat{D}\) - homeomorphisms. Let \(A = \{p, r\}\) be a closed in \(Z\). Then \((g \circ f)^{-1}(A) = f^{-1}g^{-1}(A) = \{p, r\}\) which is not \(\hat{D}\)-closed in \((X, \tau)\). Therefore composition \(g \circ f : (X, \tau) \rightarrow (Z, \eta)\) is not an \(\hat{D}\) - homeomorphisms.

**Definition 3.18.** A bijection \(f : X \rightarrow Y\) is said to be strongly - \(\hat{D}\) - homeomorphisms if both \(f\) and \(f^{-1}\) and \(\hat{D}\) - irresolute.

**Example 3.19.** Let \(X = \{p, q, r\}\) and \(Y = \{p, q, r\}\), \(\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}\) and \(\sigma = \{\phi, \{p, q\}\}, Y\). Let \(f : (X, \tau) \rightarrow (Y, \phi)\) be an identity map. Then \(f\) is strongly \(\hat{D}\) - homeomorphism.

We denote the family of all \(\hat{D}\) - homeomorphisms (resp, strongly \(\hat{D}\) - homeomorphism) of a topological space \(X\) onto itself by \(\hat{D} - h(X)\) (resp. \(S\hat{D} - h(X)\)).

**Proposition 3.20.** Every strongly \(\hat{D}\) - homeomorphism is an \(\hat{D}\) - homeomorphism but not conversely. In otherwards for any space \(X\), \(S\hat{D} - h(X) \subset \hat{D} - h(X)\).

**Proof.** Since every \(\hat{D}\) - irresolute map is \(\hat{D}\) - continuous and also from remark 3.8, we get the proof. \(\Box\)

**Example 3.21.** Let \(X = \{p, q, r\}\) and \(Y = \{p, q, r\}\), \(\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}\) and \(\sigma = \{\phi, \{p, q\}\}, \{p, q\}, Y\). Clearly identity map \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(\hat{D}\) - homeomorphisms but not strongly \(\hat{D}\) - homeomorphisms, Since \(\{r\}\) is \(\hat{D}\) - open in \(Y\) but \(f^{-1}(\{r\}) = \{r\}\) is not \(\hat{D}\) - open in \(X\). Hence \(f\) is \(\hat{D}\) - irresolute and so \(f\) is not strongly \(\hat{D}\) - homeomorphisms.

**Proposition 3.22.** If \(f : X \rightarrow Y\) and \(f : Y \rightarrow Z\) are two strongly \(\hat{D}\) - homeomorphisms then their composition \(g \circ f : X \rightarrow Z\) is also a strongly \(\hat{D}\) - homeomorphism.

**Proof.** Let \(U\) be an \(\hat{D}\) - open set in \(Z\). Now \((g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)\) where \(V = g^{-1}(U)\). By hypothesis, \(V\) is \(\hat{D}\) - open in \(Y\) and so again by hypothesis \(f^{-1}(V)\) is \(\hat{D}\) - open in \(X\). Thus \(g \circ f\) is \(\hat{D}\) - irresolute. Also for an \(\hat{D}\) - open set \(G\) in \(X\), we have \((g \circ f)(G) = g(D)\) where \(D = f(G)\), by hypothesis \(f(G)\) is \(\hat{D}\) - open in \(Y\) and so again by hypothesis, \(g(f(G))\) is \(\hat{D}\) - open in \(Z\). Thus \((g \circ f)^{-1}\) is \(\hat{D}\) - irresolute. Hence \((g \circ f)\) is strongly \(\hat{D}\) - homeomorphism. \(\Box\)
Proposition 3.23. The set $\tilde{S}D - h(X)$ is a group under the composition of maps.

Proof. Define a binary operation $\circ : \tilde{S}D - h(X) \times \tilde{S}D - h(X) \to \tilde{S}D - h(X)$ by $f \circ g = g \circ f$ for all $f$ and $g$ in $\tilde{S}D - h(X)$ and $\circ$ is the usual operation of composition of maps. Then by Proposition 3.22, $g \circ f \in \tilde{S}D - h(X)$. We know that the composition of maps is associative and the identity map $i : X \to X$ belonging to $\tilde{S}D - h(X)$ serves as the identity element. If $f \in \tilde{S}D - h(X)$, then $f^{-1} \in \tilde{S}D - h(X)$ such that $f \circ f^{-1} = f^{-1} \circ f = i$ and so inverse exist for each element of $\tilde{S}D - h(X)$. Therefore, $(\tilde{S}D - h(X), \circ)$ is a group under the composition of maps. □

Theorem 3.24. Let $f : X \to Y$ be an $\tilde{S}D$ - homeomorphism. Then $f$ induces an isomorphism from the group $\tilde{S}D - h(X)$ onto the group $\tilde{S}D - h(Y)$.

Proof. Using the map $f$, we define a map $\psi_f : \tilde{S}D - h(X) \to \tilde{S}D - h(Y)$ by $\psi_f(h) = f \circ h \circ f^{-1}$ for each $h \in \tilde{S}D - h(X)$. Then $\psi_f$ is a bijection, further for $h_1, h_2 \in \tilde{S}D - h(X)$. $\psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \psi_f(h_1) \circ \psi_f(h_2)$. Therefore, $\psi_f$ is a homomorphism and so it induces an isomorphism induced by $f$. □

Theorem 3.25. $\tilde{S}D$ - homeomorphism is an equivalence relation on the collection of all topological spaces.

Proof. Reflexivity and symmetry are immediate and transitivity follows from Proposition 3.22. □

4. $\tilde{D}$ - Quotient Map

Definition 4.1. A surjective map $f : X \to Y$ is said to be an $\tilde{D}$ - quotient map if $f$ is $\tilde{D}$ - continuous and $f^{-1}(V)$ is open in $X$ implies $V$ is $\tilde{D}$ - open in $Y$.

The following proposition is an easy consequence from the definitions.

Proposition 4.2. Every quotient map is $\tilde{D}$ - quotient but not conversely.

Proof. The proof follows from the Definitions. □

Example 4.3. Let $X = \{p, q, r\}$ and $Y = \{p, q, r\}$, $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}$ and $\sigma = \{\phi, \{p, q\}\}$,
Clearly identity map \( f : (X, \tau) \to (Y, \sigma) \) is an \( \hat{D} \) - quotient map but not a quotient map, since \( \{q\} \) is open in \( X \) but \( f^{-1}(\{q\}) = \{q\} \) is not open in \( Y \).

**Proposition 4.4.** If a map \( f : X \to Y \) is surjective, \( \hat{D} \) - continuous and \( \hat{D} \) - open, then \( f \) is an \( \hat{D} \) - quotient map.

**Proof.** We only need to prove that \( f^{-1}(V) \) is open in \( X \) implies \( V \) is an \( \hat{D} \) - open set in \( Y \). Let \( f^{-1}(V) \) be open in \( X \). Then \( f(f^{-1}(V)) \) is an \( \hat{D} \) - open set, since \( f \) is \( \hat{D} \) - open. Hence, \( V \) is an \( \hat{D} \) - open set, as \( f \) is surjective and \( f(f^{-1}(V)) = V \). Thus \( f \) is an \( \hat{D} \) - quotient map. \( \square \)

**Proposition 4.5.** If a map \( f : X \to Y \) is a homeomorphism, then \( f \) is a quotient map but not conversely.

**Proof.** Clearly follows from the definition. \( \square \)

**Example 4.6.** Let \( X = \{p, q, r\} \) and \( Y = \{p, q, r\} \), \( \tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\} \) and \( \sigma = \{\phi, \{p, q\}\} \). Clearly identity map \( f : (X, \tau) \to (Y, \sigma) \) is an \( \hat{D} \) - quotient map but not homeomorphism, since \( \{q\} \) is open in \( X \) but \( f^{-1}(\{q\}) = \{q\} \) is not open in \( Y \).

**Proposition 4.7.** Let \( f : X \to Y \) be an open surjective, \( \hat{D} \) - irresolute map and \( g : Y \to Z \) be an \( \hat{D} \) - quotient map. Then the composition \( g \circ f : X \to Z \) is an \( \hat{D} \) - quotient map.

**Proof.** Let \( V \) be any open set in \( Z \). Then \( g^{-1}(V) \) is an \( \hat{D} \) - open set, since \( g \) is an \( \hat{D} \) - quotient map. Since \( f \) is \( \hat{D} \) - irresolute, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is an \( \hat{D} \) - open in \( X \), which implies \( (g \circ f)^{-1}(V) \) is an \( \hat{D} \) - open set. This shows that \( g \circ f \) is \( \hat{D} \) - continuous. Also, assume that \( (g \circ f)^{-1}(V) \) is open in \( X \) for \( V \subset Z \), that is, \( f^{-1}(g^{-1}(V)) \) is open set in \( X \). Since \( f \) is open \( f(f^{-1}(V)) \) is open in \( Y \). It follows that \( g^{-1}(V) \) is open in \( Y \), because \( f \) is surjective. Since \( g \) is a \( \hat{D} \) - quotient map, \( V \) is an \( \hat{D} \) - open set. Thus \( g \circ f : X \to Z \) is an \( \hat{D} \) - quotient map. \( \square \)

**Proposition 4.8.** Let \( h : X \to Y \) is an \( \hat{D} \) - quotient map and \( g : X \to Z \) is a continuous map where \( Z \) is a space that is constant on each set \( h^{-1}(\{y\}) \) for each \( y \in Y \), then \( g \) induces an \( \hat{D} \) - continuous an \( \hat{D} \) - continuous map \( f : Y \to Z \) such that \( f \circ h = g \).
Proposition 4.15. Every completely $\hat{D}$-quotient map $f : X \to Y$ is said to be a strongly $\hat{D}$-quotient map if $f$ is $\hat{D}$-continuous and $f^{-1}(V)$ is $\hat{D}$-open in $X$ implies $V$ is $\hat{D}$-open in $Y$.

Definition 4.9. A surjective map $f : X \to Y$ is said to be a completely $\hat{D}$-quotient map if $f$ is $\hat{D}$-continuous and $f^{-1}(V)$ is $\hat{D}$-open in $X$ implies $V$ is $\hat{D}$-open in $Y$.

Proposition 4.10. Every strongly $\hat{D}$-quotient map is an $\hat{D}$-quotient map.

Proof. Let $f : X \to Y$ be a strongly $\hat{D}$-quotient map. Let $f^{-1}(V)$ be an open in $X$. Then $f^{-1}(V)$ be an $\hat{D}$-open in $X$. Since $f$ is a strongly $\hat{D}$-quotient map, $V$ is $\hat{D}$-open in $Y$. This shows that $f$ is an $\hat{D}$-quotient map. □

Remark 4.11. The converse of the above proposition need not be true in general as shown in the following example.

Example 4.12. Let $X = \{p,q,r\}$ and $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q,r\},X\}$ and $\sigma = \{\phi,\{p\},\{q\}\}$, $\{p,q\},Y\}$. Clearly identity map $f : (X,\tau) \to (Y,\sigma)$ is an $\hat{D}$-quotient map but not strongly $\hat{D}$-quotient map, since $\{q\}$ is $\hat{D}$-open in $X$ but $f^{-1}(\{r\}) = \{r\}$ is not $\hat{D}$-open in $Y$.

Definition 4.13. Let $f : X \to Y$ be a surjective map. Then $f$ is called a completely $\hat{D}$-quotient map if $f$ is $\hat{D}$-irresolute and $f^{-1}(V)$ is $\hat{D}$-open in $X$ implies $U$ is open in $Y$.

Theorem 4.14. Let $f : X \to Y$ be a surjective map. Strongly $\hat{D}$-open and $\hat{D}$-irresolute map and $g : Y \to Z$ be a completely $\hat{D}$-quotient map. Then $g \circ f$ is a completely $\hat{D}$-quotient map.

Proof. Since $f$ and $g$ are $\hat{D}$-irresolute, $g \circ f$ is $\hat{D}$-irresolute, by Proposition 2.8. Suppose $(g \circ f)^{-1}(V)$ is an $\hat{D}$-open in $X$ for $V \subset Z$, that is, $f^{-1}(g^{-1}(V))$ is an $\hat{D}$-open in $X$. Since $f$ is surjective and strongly $\hat{D}$-open, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is $\hat{D}$-open in $Y$. Also $g$ is completely $\hat{D}$-quotient implies $V$ is open in $Z$. Thus $g \circ f$ is a completely $\hat{D}$-quotient map. □

Proposition 4.15. Every completely $\hat{D}$-quotient map is strongly $\hat{D}$-quotient map.
Proof. Let $f : X \rightarrow Y$ be a completely $\hat{D}$-quotient map. By Proposition 2.7, $f$ is $\hat{D}$- irresolute implies $f$ is $\hat{D}$- continuous. Hence the proof follows. \hfill \Box

Remark 4.16. The converse of the above proposition need not be true in general as shown in the following example.

Example 4.17. Let $X = \{p, q, r\}$ and $Y = \{p, q, r\}$, $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}$ and $\sigma = \{\phi, \{p\}, \{q, r\}, Y\}$. Clearly identity map $f : (X, \tau) \rightarrow (Y, \sigma)$ is an strongly $\hat{D}$- quotient map but not a completely $\hat{D}$- quotient map, since $\{r\}$ is $\hat{D}$- open in $X$ but $f^{-1}(\{r\}) = \{r\}$ is not $\hat{D}$- open in $X$, implies that $f$ is not $\hat{D}$- irresolute.

Theorem 4.18. Let $f : X \rightarrow Y$ be a surjective map and both $X$ and $Y$ be $T_{\hat{D}}$- spaces. Then the following are equivalent.

(i) $f$ is a completely $\hat{D}$- quotient map;

(ii) $f$ is a strongly $\hat{D}$- quotient map;

(iii) $f$ is a $\hat{D}$- quotient map;

Proof. (i) $\implies$ (ii) : Follows by Proposition 4.15.

(ii) $\implies$ (iii) : Follows by Proposition 4.10.

(iii) $\implies$ (i) : Since $Y$ is a $T_{\hat{D}}$- space, $f$ is $\hat{D}$- irresolute, by Proposition 2.9. Suppose $f^{-1}(V)$ is $\hat{D}$- open in $X$. Since $X$ is a $T_{\hat{D}}$, $f^{-1}(V)$ is open in $X$. By (iii), $V$ is $\hat{D}$- open in $Y$. Since $Y$ is a $T_{\hat{D}}$- space, $V$ is open in $Y$. Hence, we get (i). \hfill \Box

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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