QUALITATIVE RESULTS FOR FINITE SYSTEM OF R-L FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCE

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Abstract. Qualitative results such as existence and uniqueness of finite system of Riemann-Liouville (R-L) fractional differential equations with initial time difference are obtained. Monotone technique coupled with method of lower and upper solutions is developed to obtain existence and uniqueness of solutions of finite system of R-L fractional differential equations with initial time difference.

Keywords: existence and uniqueness; initial time difference; fractional differential equations; monotone method.

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1. INTRODUCTION

Due to wide applications of fractional calculus in sciences, engineering, nature and social sciences numerous methods of solving fractional differential equations are developed [1, 8, 9]. V.Lakshmikantham et.al [6] obtained local and global existence results for solutions of Riemann-Liouville fractional differential equations. The Caputo fractional differential equation with periodic boundary conditions have been studied in [3, 4] and developed monotone method.

Existence and uniqueness of solution of Riemann-Liouville fractional differential equation with

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integral boundary conditions is proved in [11, 12]. Monotone method for Riemann-Liouville fractional differential equations with initial conditions is established by McRae [7]. Vasundhara Devi considered [2] the general monotone method for periodic boundary value problem of Caputo fractional differential equation. Recently, initial value problems involving Riemann-Liouville fractional derivative was studied by authors [5, 13]. Yaker et.al. proved existence and uniqueness of solutions of fractional differential equations with initial time difference for locally Holder continuous functions [14]. Authors have generalized these results for the class of continuous functions [10] and extended for system. In this paper, we consider the finite system of Riemann-Liouville fractional differential equations with initial time difference when the function on the right hand side is mixed quasi-monotone and construct two monotone convergent sequences to obtain existence and uniqueness of solution for the finite system.

The paper is organized as follows: In section 2, basic definitions and results are given. Section 3 is devoted to obtain main results.

2. BASIC RESULTS

Some basic definitions and results used for the development of monotone technique for the problem are given in this section.

The Riemann-Liouville fractional derivative of order \( q \) \((0 < q < 1) \) [?] is defined as

\[
D^a_{q}u(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \int_{a}^{t} (t-\tau)^{n-q-1} u(\tau) d\tau, \quad \text{for } a \leq t \leq b.
\]

**Lemma 2.1.** [2] Let \( m \in C_p(J, \mathbb{R}) \) and for any \( t_1 \in (t_0, T] \) we have \( m(t_1) = 0 \) and \( m(t) < 0 \) for \( t_0 \leq t \leq t_1 \). Then it follows that \( D^q m(t_1) \geq 0 \).

**Theorem 2.1.** [11] Let \( v, w \in C_p([t_0, T], \mathbb{R}), f \in C([t_0, T] \times \mathbb{R}, \mathbb{R}) \) and

(i) \( D^q v(t) \leq f(t, v(t)) \)

and

(ii) \( D^q w(t) \geq f(t, w(t)) \),

\( t_0 < t \leq T \). Assume \( f(t, u) \) satisfy one sided Lipschitz condition

\[
f(t, u) - f(t, v) \leq L(u - v), \quad u \geq v, L > 0.
\]
Then \( v_0 < w_0 \), where \( v_0 = v(t)(t-t_0)^{1-q}|_{t=t_0} \) and \( w_0 = w(t)(t-t_0)^{1-q}|_{t=t_0} \), implies \( v(t) \leq w(t), t \in [t_0, T] \).

**Corollary 2.1.** The function \( f(t,u) = \sigma(t)u \), where \( \sigma(t) \leq L \), is admissible in Theorem 2.1 to yield \( u(t) \leq 0 \) on \( t_0 \leq t \leq T \).

The results proved by Yakar et.al. for the following problem

\[
D^q u(t) = f(t,u), \quad u(t)(t-t_0)^{1-q}|_{t=t_0} = u_0
\]

where \( 0 < q < 1, f \in C[R^+ \times \mathbb{R}, \mathbb{R}] \), are generalized by authors [10] for class of continuous functions \( u(t) \).

The corresponding Volterra fractional integral is given by

\[
u(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,u(s)) ds
\]

where

\[
u^0(t) = \frac{u(t)(t-t_0)^{1-q}}{\Gamma(q)}\]

and that every solution of (2.2) is a solution of (2.1).

In this paper, we develop monotone technique coupled with lower and upper solutions for the class of continuous functions for the following finite system of Riemann-Liouville fractional differential equations with initial time difference and obtain existence and uniqueness of solution for the problem.

\[
D^q u_i(t) = f_i(t,u_1(t),u_2(t),\ldots,u_N(t)), \quad u_i(t)(t-t_0)^{1-q}|_{t=t_0} = u_i^0
\]

where \( i = 1,2,\ldots,N, t \in J = [t_0, T] \) \( f_i \) in \( C(J \times \mathbb{R}^n, \mathbb{R}) \), \( 0 < q < 1 \).

**Definition 2.1.** A pair of functions \( v = (v_1,v_2,\ldots,v_N) \) and \( w = (w_1,w_2,\ldots,w_N) \) in \( C_p(J, \mathbb{R}), p = 1-q \) are said to be ordered lower and upper solutions \( (v_1,v_2,\ldots,v_N) \leq (w_1,w_2,\ldots,w_N) \) of the problem (2.3) if

\[
D^q v_i(t) \leq f_i(t,v_1(t),v_2(t),\ldots,v_N), \quad v_i(t)(t-t_0)^{1-q}|_{t=t_0} = v_i^0
\]
and

\[ D^q w_i(t) \geq f_i(t, w_1(t), w_2(t), \ldots, w_N(t)), \quad w_i(t)(t - t_0)^{1-q}|_{t=t_0} = w_i^0. \]

**Definition 2.2.** A function \( f_i \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N) \) is said to satisfy mixed quasimonotone property if for each \( i \), \( f_i(t, u, [u]_{r_i}, [u]_{s_i}) \) is monotone nondecreasing in \( [u]_{r_i} \) and monotone nonincreasing in \( [u]_{s_i} \).

When either \( r_i \) or \( s_i \) is equal to zero a special case of the mixed quasimonotone property is defined as follows:

**Definition 2.3.** A function \( f_i \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N) \) is said to be quasimonotone nondecreasing (nonincreasing) if for each \( i \), \( u_i \leq v_i \) and \( u_j = v_j, i \neq j \), then \( f_i(t, u_1, u_2, \ldots, u_N) \leq f_i(t, v_1, v_2, \ldots, v_N) \left( f_i(t, u_1, \ldots, u_N) \geq f_i(t, v_1, \ldots, v_N) \right) \).

### 3. Qualitative Results

This section is devoted to develop monotone method for system of Riemann-Liouville fractional differential equations with initial time difference and obtain existence and uniqueness of solution of the problem (2.3).

**Theorem 3.1.** Assume that

\[ (E_1) \quad v = (v_1, v_2, \ldots, v_N) \in C_p[0, t_0 + T], \mathbb{R}], t_0, T > 0, w = (w_1, w_2, \ldots, w_N) \in C_p, [\tau_0, \tau_0 + T], \mathbb{R}[ \] is continuous and \( p = 1 - q \) where

\[ C_p(J, R) = \{ u(t) \in C(J, R) \text{ and } u(t)(t - t_0)^p \in C(J, R) \}, J = [0, t_0 + T], \]

\[ C_p^*(J, R) = \{ u(t) \in C(J^*, R) \text{ and } u(t)(t - t_0)^p \in C(J^*, R) \}, J^* = [\tau_0, \tau_0 + T], \]

\( f \in C[[0, \tau_0 + T] \times \mathbb{R}, \mathbb{R}] \) and

\[ D^q v_i(t) = f_i(t, v_1(t), v_2(t), \ldots, v_N(t)), \quad t_0 \leq t \leq t_0 + T, \]

\[ D^q w_i(t) = f_i(t, w_1(t), w_2(t), \ldots, w_N(t)), \quad \tau_0 \leq t \leq \tau_0 + T, \]

\[ v_i^0 \leq u_i^0 \leq w_i^0, \quad \text{where} \quad v_i^0 = v_i(t)(t - t_0)^{1-q}|_{t=t_0}, w_i^0 = w_i(t)(t - \tau_0)^{1-q}|_{t=\tau_0} \]

**(E2)** \( f_i(t, u_1, u_2, \ldots, u_N) \) is mixed quasimonotone in \( t \) for each \( u_i \) and \( v_i(t) \leq w_i(t + \eta), t_0 \leq t \leq t_0 + T, \) where \( \eta = \tau_0 - t_0 \)
\((E_3)\) \(f_i\) satisfies one-sided Lipschitz condition,

\[
f_i(t, u_1, u_2, \ldots, u_N) - f_i(t, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N) \geq -M_i[u_i - \bar{u}_i], \text{for } \bar{u}_i \leq u_i, M_i \geq 0.
\]

Then there exists monotone sequences \(\{v^n(t)\}\) and \(\{w^n(t)\}\) such that

\[
\{v^n(t)\} \to v(t) = (v_1, v_2, \ldots, v_n) \quad \text{and} \quad \{w^n(t)\} \to w(t) = (w_1, w_2, \ldots, w_n) \text{ as } n \to \infty
\]

where \(v(t)\) and \(w(t)\) are minimal and maximal solutions of the problem (2.3) respectively.

**Proof.** Let \(w_{i0}(t) = w_i(t + \eta)\) and \(v_{i0}(t) = v_i(t)\) \(i = 1, 2\) for \(t_0 \leq t \leq t_0 + T\), where \(\eta = \tau_0 - t_0\). Since \(f(t, u_1, u_2, \ldots, u_N)\) is quasimonotone nondecreasing in \(t\) for each \(u_i\) we have

\[
D^q w_{0}(t) = D^q w_i(t + \eta) \geq f(t + \eta, w_1(t + \eta), w_2(t + \eta), \ldots, u_N(t + \eta)) \geq f(t, w_1(t), w_2(t), \ldots, w_N(t))
\]

and

\[
w_{0}(t) = w_{i0}(t)(t - t_0)^{1 - q} \big|_{t = t_0} = w + i(t + \eta)(t - t_0)^{1 - q} \big|_{t = t_0} = w_i(t)(t - t_0)^{1 - q} \big|_{t = t_0} = w_i^0
\]

Also,

\[
D^q v_{i0}(t) = D^q v_i(t) \leq f(t, v_{10}(t), v_{20}(t), \ldots, v_{N0}(t))
\]

and

\[
v_{i0}(t) = v_{i0}(t)(t - t_0)^{1 - q} \big|_{t = t_0} = v_i(t)(t - t_0)^{1 - q} \big|_{t = t_0} = v_i^0, v_i^0 \leq u_i^0 \leq w_i^0
\]

which proves that \(v_{i0}\) and \(w_{i0}\) are lower and upper solutions of IVP (2.3) respectively.

For any \(\theta(t) = (\theta_1, \theta_2, \ldots, \theta_N)\) in \(C_p(J, \mathbb{R})\) such that for \(\alpha_{i0} \leq \theta_i \leq \beta_{i0}, \alpha_{i0} \leq \theta_i \leq \beta_{i0}\) on \(J\), consider the following linear system of fractional differential equations

\[
D^q u_i(t) = f_i(t, \theta_1(t), \ldots, \theta_N(t)) - M_i[u_i(t) - \theta_i(t)],
\]

\[
u_i(0) = u_i(t)(t - t_0)^{1 - q} \big|_{t = t_0}
\]

Since the right hand side of IVP (3.1) satisfies Lipschitz condition, unique solution of IVP (3.1) exists on \(J\).

For each \(\eta(t)\) and \(\mu(t)\) in \(C_p(J, \mathbb{R})\) such that \(v_i^0(0) \leq \eta_i(t), w_i^0(0) \leq \mu_i(t)\), define a mapping \(A\) by \(A[\eta, \mu] = u(t)\) where \(u(t)\) is the unique solution of the problem (3.1). Firstly, we prove that

\[
(A_1) \quad v_i^0 \leq A[v_i^0, w_i^0], \quad w_i^0 \geq A[w_i^0, v_i^0]
\]
(A2) $A$ possesses the monotone property on the segment $[v_0, w_0] = \{ (u_1, u_2, \ldots, u_N) \in C(J, \mathbb{R}) : v_i^0 \leq u_i \leq w_i^0 \}$. 

Set $A[v_0, w_0] = v^1(t)$, where $v^1(t) = (v_1^1, v_2^1, \ldots, v_n^1)$ is the unique solution of system (3.1) with $\eta_i = v_i^0(0)$.

Setting $p_i(t) = v_i^0(t) - v_i^1(t)$ we see that

\[ D^\alpha p_i(t) \leq f_i(t, v_i^0(t), v_i^0(t), \ldots, v_N^0(t)) - f_i(t, v_i^1(t), v_i^1(t), \ldots, v_N^0(t)) \]
\[ \leq -M_i p_i(t) \]

and $p_i(t) \leq 0$.

Applying Corollary 2.1, we get $p_i(t) \leq 0$ on $0 \leq t \leq T$ and hence $v_i^0(t) - v_i^1(t) \leq 0$ which implies $v_i^0 \leq A[v_0, w_0]$. Set $A[v_0, w_0] = v^1(t)$, where $w^1(t) = (w_1^1, w_2^1)$ is the unique solution of the problem (3.1) with $\mu_i = w_i^0(t)$. Setting $p_i(t) = w_i^0(t) - w_i^1(t)$, similarly by Corollary 2.1, we have $w_i^0 \geq w_i^1$. Hence $w^0 \geq A[w_0, v^0]$. Let $\eta, \beta, \mu \in [v_0, w_0]$ with $\eta \leq \beta$. Suppose that $A[\eta, \mu] = u(t), A[\beta, \mu] = v(t)$. Then setting $p_i(t) = u_i(t) - v_i(t)$ we find that

\[ D^\alpha p_i(t) = f_i(t, \eta_1, \ldots, \eta_N) - f_i(t, \beta_1, \ldots, \beta_N) - M_i [u_i(t) - \eta_i(t)] \]
\[ + M_i [v_i(t) - \beta_i(t)] \]
\[ \leq -M_i p_i(t) \]

and $p_i(t) \leq 0$.

As before in (A1), we have $A[\eta, \mu] \leq A[\beta, \mu]$.

Similarly, if $\eta(t), \gamma(t), \mu(t) \in [v_0, w_0]$ be such that $\gamma(t) \leq \mu(t)$. Suppose that $A[\eta, \gamma] = u(t), A[\eta, \mu] = v(t)$ we can prove that $A[\eta, \gamma] \geq A[\eta, \mu]$. Thus it follows that the mapping $A$ possesses monotone property on the segment $[v_0, w_0]$.

Define the sequences

\[ v_i^n(t) = A[v_i^{n-1}, w_i^{n-1}], \quad w_i^n(t) = A[w_i^{n-1}, v_i^{n-1}] \]
Using corresponding Volterra fractional integral equations

\[ D^q v^n_i(t) = f_i(t, v^{n-1}_1, \ldots, v^{n-1}_N) - M_i[v^n_i - v^{n-1}_i], \quad v^n_i(t)(t-t_0)^{1-q}|_{t=t_0} = v^{n_0}_i \]

\[ D^q w^n_i(t) = f_i(t, w^{n-1}_1, \ldots, w^{n-1}_N) - M_i[w^n_i - w^{n-1}_i], \quad w^n_i(t)(t-t_0)^{1-q}|_{t=t_0} = w^{n_0}_i \]

From (A1), we have \( v^0_i \leq v^1_i, \quad w^0_i \leq w^1_i \). Assume that \( v^{k-1}_i \leq v^k_i, \quad w^{k-1}_i \leq w^k_i \). To prove

\[ v^k_i \leq v^{k+1}_i, \quad w^k_i \geq w^{k+1}_i \]

and

\[ p(t) \leq 0. \]

It follows from Corollary 2.1 that \( p_i(t) \leq 0 \), which gives \( v^k_i(t) \leq v^{k+1}_i(t) \). Similarly we can prove \( w^k_i(t) \geq w^{k+1}_i(t) \) and \( v^k_i(t) \geq w^k_i(t) \). By induction, it follows that

\[ v^0_i(t) \leq v^1_i(t) \leq \ldots \leq v^q_i(t) \leq \ldots \leq v^n_i(t) \leq w^n_i(t) \leq w^{n-1}_i(t) \leq \ldots \leq w^1_i(t) \leq w^0_i(t) \]

Thus the sequences \( \{v^n(t)\} \) and \( \{w^n(t)\} \) are bounded from below and bounded from above respectively and monotonically nondecreasing and monotonically non-increasing on \( J \). Hence point-wise limit exist and are given by

\[ \lim_{n \to \infty} v^n_i(t) = v_i(t), \quad \lim_{n \to \infty} w^n_i(t) = w_i(t) \text{ on } J \]

Using corresponding Volterra fractional integral equations

\[ v^n_i(t) = v^0_i + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \left\{ f_i(s, v^n_1(s), \ldots, v^n_N(s)) - M_i[v^n_i - v^{n-1}_i] \right\} ds \]

\[ w^n_i(t) = w^0_i + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \left\{ f_i(s, w^n_1(s), \ldots, w^n_N(s)) - M_i[w^n_i - w^{n-1}_i] \right\} ds, \]
as $n \to \infty$, we get

\begin{align*}
v_i(t) &= \frac{v_i^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,v_1^n(s),\ldots,v_N^n(s)) ds \\
w_i(t) &= \frac{w_i^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,w_1^n(s),\ldots,w_N^n(s)) ds
\end{align*}

where $v_i^0 = v_i(t)(t-t_0)^{1-q}|_{t=t_0}$. It follows that $v(t)$ and $w(t)$ are solutions of system (2.3). Lastly, we prove that $v(t)$ and $w(t)$ are the minimal and maximal solution of the problem (2.3).

Let $u(t) = (u_1,\ldots,u_N)$ be any solution of (2.3) other than $v(t)$ and $w(t)$, so that there exists $k$ such that $v_i^k(t) \leq u_i(t) \leq w_i^k(t)$ on $[t_0,T]$ and set $p_i(t) = v_i^{k+1}(t) - u_i(t)$ so that

\begin{align*}
D^q p_i(t) &= f_i(t,v_i^k,\ldots,v_N^k) - M_i[v_i^{k+1} - v_i^k] - f_i(t,u_1,\ldots,u_N) \\
&\geq -M_i p_i(t) \\
\text{and } p_i(t) &\geq 0.
\end{align*}

Thus $v_i^{k+1} \leq u_i(t)$ on $J$. Since $v_i^0 \leq u_i(t)$ on $J$, by induction it follows that $v_i^k(t) \leq u_i(t)$ for all $k$. Similarly, we can prove $u_i \leq w_i^k$ for all $k$ on $J$. Hence $v_i^k(t) \leq u_i(t) \leq w_i^k(t)$ on $J$. Taking limit as $n \to \infty$, it follows that $v_i(t) \leq u_i(t) \leq w_i(t)$ on $J$. Now, we obtain the uniqueness of solution of the problem (2.3) in the following

**Theorem 3.2.** Assume that

(U$_1$) Assumptions (E$_1$) and (E$_3$) of Theorem 3.1 holds.

(U$_2$) $f_i = f_i(t,u_1,u_2,\ldots,u_N)$ satisfies Lipschitz condition,

$$|f_i(t,u_1,u_2,u_N) - f_i(t,\overline{u}_1,\overline{u}_2,\overline{u}_N)| \geq -M_i|u_i - \overline{u}_i|, M_i \geq 0$$

then there exists unique solution of the problem (2.3).
**Proof.** We know that $v(t) \leq w(t)$. It is sufficient to prove $v(t) \geq w(t)$. For this, if $p_i(t) = w_i(t) - v_i(t)$ then we have

$$D^q p_i(t) \leq D^q w_i(t) - D^q v_i(t)$$

$$\leq M_i(w_i(t) - v_i(t))$$

$$\leq -M_i p_i(t)$$

and $p_i(t) = 0$.

Applying Corollary 2.1, we obtain $p_i(t) \leq 0$ implies $w_i(t) \leq v_i(t)$. Thus $v(t) = u(t) = w(t)$ is the unique solution of (2.3) on $[t_0, t_0 + T]$.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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