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## REVERSIBLE ČECH CLOSURE SPACES

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**Abstract.** In this paper we present the notion of reversibility in Čech-closure spaces. Some characterization theorems are obtained similar to topological spaces. The relation between reversible Čech-closure spaces with the underlying topological spaces, complete homogeneity and reversibility in Čech closure spaces are also investigated.

**Keywords:** Čech closure spaces; Č-continuous functions; Č-homeomorphism; reversible Čech closure spaces; completely homogeneous Čech closure spaces.

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### 1. INTRODUCTION

The concept of Čech closure operators on a set  $\mathfrak{X}$ , was introduced by Edward Čech [2], by generalizing the notion of Kuratowski closure operators (topological closure operators). A Čech closure space  $(\mathfrak{X}, \mu)$  is a set  $\mathfrak{X}$  with a Čech closure operator  $\mu$ , which need not satisfy the idempotent law of topological closure. Many properties which hold in topological spaces hold in Čech closure spaces as well. Using the above information, in this paper we extend the notion

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of reversibility, introduced by M. Rajagopalan and A. Willansky [7] and check how far the results good in Čech closure spaces.

## 2. PRELIMINARIES

Let  $\mathfrak{X}$  be a set, and  $P(\mathfrak{X})$  denotes the power set of  $\mathfrak{X}$ , a mapping  $\mu : P(\mathfrak{X}) \longrightarrow P(\mathfrak{X})$  is called a Čech closure operator provided it satisfies the following three axioms .

- (1)  $\forall K \subset \mathfrak{X}, K \subset \mu(K)$ .
- (2)  $\mu(\phi) = \phi$
- (3)  $\mu(K \cup L) = \mu(K) \cup \mu(L), \forall K, L \subseteq \mathfrak{X}$ .

Then  $\mathfrak{X}$ , together with the a Čech closure operator  $\mu$  is called Čech closure space and is denoted by  $(\mathfrak{X}, \mu)$ . We denote a Čech closure space throughout this paper as closure space for convenience. If  $\mu(\mu(\mathcal{A})) = \mu(\mathcal{A}), \forall \mathcal{A} \subset \mathfrak{X}$ , then  $(\mathfrak{X}, \mu)$  is a topological space. Thus the concept of closure spaces, generalizes the definition of topological spaces.

The closed sets in a closure space  $(\mathfrak{X}, \mu)$  are those subset  $\mathcal{F}$  of  $\mathfrak{X}$ , such that  $\mathcal{F} = \mu(\mathcal{F})$  and a subset  $\mathfrak{D}$  of  $\mathfrak{X}$  is open provided its complement  $\mathfrak{X} - \mathfrak{D}$  is closed.

Note that the collection of all open sets in a closure space is a topology on  $\mathfrak{X}$ , denoted by  $\tau(\mu)$ , and  $\tau(\mu)$  is the topology associated with the closure space  $(\mathfrak{X}, \mu)$ .

In a closure space  $(\mathfrak{X}, \mu)$ ,  $\mu$  is finitely generated, provided for any subset  $\mathcal{A}$  of  $\mathfrak{X}$ ,  $\mu(\mathcal{A}) = \bigcup \{\mu(a); a \in \mathcal{A}\}$ , in this scenario  $(\mathfrak{X}, \mu)$  is called finitely generated closure space. Also every finitely generated closure operator is characterised by its action on singleton sets.

Let  $\mu, \nu$  are closure operators on a set  $\mathfrak{X}$  then  $\mu \cup \nu, \mu \circ \nu$  are closure operators on  $\mathfrak{X}$ , defined by  $(\mu \cup \nu)(\mathcal{A}) = \mu(\mathcal{A}) \cup \nu(\mathcal{A})$  and  $(\mu \circ \nu)(\mathcal{A}) = \mu(\nu(\mathcal{A})), \forall \mathcal{A} \subset \mathfrak{X}$ .

Let  $(\mathfrak{X}, \mu_1), (\mathfrak{Y}, \mu_2)$  are closure spaces, a function  $\theta : \mathfrak{X} \longrightarrow \mathfrak{Y}$  is said to be Č-continuous (resp., Č- homeomorphism) if  $\theta(\mu_1(\mathcal{A})) \subset \mu_2(\theta(\mathcal{A}))$  (resp.,  $\theta$  is a one to one, onto with  $\theta(\mu_1(\mathcal{A})) = \mu_2(\theta(\mathcal{A}))$ ) for every  $\mathcal{A} \subset \mathfrak{X}$ .

Clearly, if  $\theta : (\mathfrak{X}, \mu_1) \longrightarrow (\mathfrak{Y}, \mu_2)$  is continuous, then  $\theta^{-1}(\mathcal{F})$  is a closed (open) subset of  $(\mathfrak{X}, \mu_1)$  for every closed (open) subset  $\mathcal{F}$  of  $(\mathfrak{Y}, \mu_2)$ .

If  $\mu_1$  and  $\mu_2$  are topological closure operators, in this case the definition of Č-continuity is reduced to the corresponding definition of continuity. The following results are found in [6].

**Theorem 2.1.** *Let  $\theta_1, \theta_2$  are two  $\check{C}$ -continuous ( resp.,  $\check{C}$ -homeomorphism) functions on some closure space  $(\mathfrak{X}, \mu)$ , then  $\theta_1 \circ \theta_2$  is  $\check{C}$ -continuous ( resp.,  $\check{C}$ -homeomorphism) on  $(\mathfrak{X}, \mu)$ .*

**Theorem 2.2.** *If  $\theta : (\mathfrak{X}, \mu_1) \longrightarrow (\mathfrak{Y}, \mu_2)$  is  $\check{C}$ -continuous ( resp.,  $\check{C}$ -homeomorphism), then  $\theta : (\mathfrak{X}, \tau(\mu_1)) \longrightarrow (\mathfrak{Y}, \tau(\mu_2))$  is continuous ( resp., homeomorphism).*

In [9] it is proved that the converse of the above result not always true.

A closure  $\mu$  is said to be coarser (weaker) than a closure  $\nu$  on the same set  $\mathfrak{X}$ , if  $\nu(\mathcal{Y}) \subset \mu(\mathcal{Y})$  for each subset  $\mathcal{Y}$  of  $\mathfrak{X}$ , we denote it by  $\mu < \nu$ . If  $\mu$  is coarser than  $\nu$  we also say  $\nu$  is finer (stronger) than  $\mu$ .

Let  $(\mathfrak{X}, \mu)$  and  $(\mathfrak{Y}, \nu)$  be closure spaces. A map  $\theta : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{Y}, \nu)$  is said to be closed (resp. open) if  $\theta(\mathcal{F})$  is a closed ( resp. open ) subset of  $(\mathfrak{Y}, \nu)$  whenever  $\mathcal{F}$  is a closed (resp. open) subset of  $(\mathfrak{X}, \mu)$ . Every one to one , onto  $\check{C}$ -continuous closed (open) map from a closure space to any other closure space is a  $\check{C}$ -homeomorphism.

**Theorem 2.3.** *A necessary and sufficient condition for a closure space  $(\mathfrak{X}, \mu)$  to be finer than the closure space  $(\mathfrak{X}, \nu)$  is that the identity map  $i : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{X}, \nu)$  is  $\check{C}$ -continuous, [1].*

The subspace of a closure space is defined as follows. If  $(\mathfrak{X}, \mu)$  be a closure space and  $\mathfrak{W} \subset \mathfrak{X}$ , then a closure  $\mu'$  on  $\mathfrak{W}$  is defined as  $\mu'(\mathcal{Y}) = \mathfrak{W} \cap \mu(\mathcal{Y})$  for all  $\mathcal{Y} \subseteq \mathfrak{W}$ . Then the closure space  $(\mathfrak{W}, \mu')$  is called the subspace of  $(\mathfrak{X}, \mu)$ .

For more details, relevant to closure spaces we refer to [1], [2], [3],[5], [6], [10].

### 3. REVERSIBLE CLOSURE SPACES

In [7], Rajagopalan and Willansky unified the notion of minimal and maximal topologies by introducing reversible topological spaces. Also in [7], they proved that a topological space  $(\mathcal{P}, \tau)$  is reversible, if the only continuous self-bijections on  $\mathcal{P}$  are the homeomorphisms. As an analogous way we define reversible Čech -closure space as follows.

**Definition 3.1.** A closure space  $(\mathfrak{X}, \mu)$  is reversible, if it has no strictly finer closure operator  $\nu$  such that  $(\mathfrak{X}, \mu)$  and  $(\mathfrak{X}, \nu)$  are  $\check{C}$ -homeomorphic, equivalently it has no coarser closure operator  $\nu'$  such that  $(\mathfrak{X}, \mu)$  and  $(\mathfrak{X}, \nu')$  are  $\check{C}$ -homeomorphic.

**Proposition 3.2.** *A  $\check{C}$ -continuous map remains  $\check{C}$ -continuous if the co-domain closure becomes coarser or the domain closure becomes finer.*

*Proof.* Let  $(\mathfrak{X}, \mu_1), (\mathfrak{X}, \mu_2), (\mathfrak{Y}, \nu), (\mathfrak{Y}, \nu')$  are closure spaces with  $\mu_1$  and  $\nu$  finer than  $\mu_2$  and  $\nu'$ . Let  $\rho$  be a  $\check{C}$ -continuous map from  $(\mathfrak{X}, \mu_1)$  to  $(\mathfrak{Y}, \nu)$ .

Then for all  $\mathcal{Y} \subseteq \mathfrak{X}$  we have  $\rho(\mu_1(\mathcal{Y})) \subseteq \nu(\rho(\mathcal{Y}))$ , also  $\nu$  finer than  $\nu'$  we have  $\rho(\mu_1(\mathcal{Y})) \subseteq \nu(\rho(\mathcal{Y})) \subseteq \nu'(\rho(\mathcal{Y}))$ . Hence  $\rho$  is a  $\check{C}$ -continuous map from  $(\mathfrak{X}, \mu_1)$  to  $(\mathfrak{Y}, \nu')$ .

To prove the second statement, assume  $\rho$  be a  $\check{C}$ -continuous map from  $(\mathfrak{X}, \mu_2)$  to  $(\mathfrak{Y}, \nu)$ . Also given  $\mu_1$  is finer than  $\mu_2$ , we have for all  $\mathcal{Y} \subseteq \mathfrak{X}$ ,  $\rho(\mu_1(\mathcal{Y})) \subseteq \rho(\mu_2(\mathcal{Y})) \subseteq \nu(\rho(\mathcal{Y}))$ . Thus  $\rho$  is a  $\check{C}$ -continuous map from  $(\mathfrak{X}, \mu_1)$  to  $(\mathfrak{Y}, \nu)$ .  $\square$

The following theorem characterises a reversible closure space. Analogous result and proof in topological context are found in [7].

**Theorem 3.3.** *The necessary and sufficient condition for a closure space  $(\mathfrak{X}, \mu)$  to be reversible is that, each one to one, onto  $\check{C}$ -continuous map of the space onto itself is a  $\check{C}$ -homeomorphism.*

*Proof.* Assume  $(\mathfrak{X}, \mu)$  is a reversible closure space and  $\theta : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{X}, \mu)$  is a  $\check{C}$ -continuous function, which is both one to one and onto. Let  $\nu$  be a closure operator on  $\mathfrak{X}$  such that  $\mathcal{F} \subseteq \mathfrak{X}$  is closed in  $(\mathfrak{X}, \nu)$ , if  $\mu(\theta(\mathcal{F})) = \theta(\mathcal{F})$ . Then the closure  $\nu$  on  $\mathfrak{X}$  weaker than  $\mu$ , since, let  $\mathcal{K}$  be any closed set in  $(\mathfrak{X}, \nu)$  then  $\theta(\mathcal{K})$  is closed in  $(\mathfrak{X}, \mu)$  again since,  $\theta$  is continuous  $\theta^{-1}(\theta(\mathcal{K})) = \mathcal{K}$  is a closed set in  $(\mathfrak{X}, \mu)$ . Moreover the function  $\theta : (\mathfrak{X}, \nu) \longrightarrow (\mathfrak{X}, \mu)$  is a  $\check{C}$ -homeomorphism. Since  $\theta$  is closed and let  $\mathcal{K}$  be any closed set in  $(\mathfrak{X}, \mu)$ , then  $\mathcal{K} = \theta(\theta^{-1}(\mathcal{K}))$ , hence  $\theta^{-1}(\mathcal{K})$  is closed in  $(\mathfrak{X}, \nu)$  thus  $\theta$  is  $\check{C}$ -continuous. But in our assumption  $(\mathfrak{X}, \mu)$  is reversible, hence  $\mu = \nu$ . Therefore  $\theta : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{X}, \mu)$  is a  $\check{C}$ -homeomorphism.

Conversely assume each self bijective  $\check{C}$ -continuous map on  $(\mathfrak{X}, \mu)$  is a  $\check{C}$ -homeomorphism. Assume  $(\mathfrak{X}, \nu)$  is a finer closure space such that  $(\mathfrak{X}, \mu)$  and  $(\mathfrak{X}, \nu)$  are  $\check{C}$ -homeomorphic. Let  $\theta : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{X}, \nu)$  be such a  $\check{C}$ -homeomorphism. Then by Proposition 3.2,  $\theta$  is a  $\check{C}$ -continuous bijection of  $(\mathfrak{X}, \mu)$  onto itself. Also by our assumption,  $\theta$  is a  $\check{C}$ -homeomorphism of  $(\mathfrak{X}, \mu)$  onto itself. Let  $\mathcal{F}$  be any closed set in  $(\mathfrak{X}, \nu)$ , since  $\theta : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{X}, \nu)$  be a  $\check{C}$ -homeomorphism,  $\theta^{-1}(\mathcal{F})$  is a closed set in  $(\mathfrak{X}, \mu)$ . Again  $\theta$  is a  $\check{C}$ -homeomorphism of  $(\mathfrak{X}, \mu)$  onto itself, we have

$\theta(\theta^{-1}(\mathcal{F})) = \mathcal{F}$  is a closed set in  $(\mathfrak{X}, \mu)$ . Thus  $\mu = \nu$ . Thus there is no finer closure space  $(\mathfrak{X}, \nu)$  such that  $(\mathfrak{X}, \mu)$  and  $(\mathfrak{X}, \nu)$  are  $\check{C}$ -homeomorphic. Hence  $(\mathfrak{X}, \mu)$  is reversible.  $\square$

*Remark 3.4.* Using Theorem 3.3, we got the following examples of reversible closure spaces.

(1) The discrete closure space  $(\mathfrak{X}, \mathcal{D})$ , where  $\mathcal{D}$  is given by

$$\mathcal{D}(\mathcal{A}) = \mathcal{A}, \forall \mathcal{A} \subset \mathfrak{X}, \text{ is reversible.}$$

(2) The trivial closure space  $(\mathfrak{X}, \mathcal{I})$ , where  $\mathcal{I}$  is given by

$$\mathcal{I}(\mathcal{A}) = \begin{cases} \mathfrak{X}, & \text{if } \mathcal{A} \neq \phi \\ \phi, & \text{if } \mathcal{A} = \phi \end{cases}$$

is reversible.

(3) Let  $\mathfrak{X}$  be any set, which is not finite, for all  $\mathcal{A} \subset \mathfrak{X}$ , define

$$\mu(\mathcal{A}) = \begin{cases} \phi, & \text{if } \mathcal{A} = \phi \\ \mathcal{A}, & \text{if } \mathcal{A} \text{ is finite} \\ \mathfrak{X}, & \text{in all other cases.} \end{cases}$$

Then  $(\mathfrak{X}, \mu)$  is reversible.

All examples given above are topological, whereas the following example gives a reversible non topological closure space.

**EXAMPLE 3.5.** On  $\mathbb{N}$ , let  $\mu$  be the closure operator defined as  $\forall k \in \mathbb{N}, \mu(k) = \{k, k+1\}$  and for all  $\mathcal{A} \subset \mathbb{N}$ ,

$$\mu(\mathcal{A}) = \begin{cases} \cup\{\mu(a); a \in \mathcal{A}\}, & \text{if } \mathcal{A} \neq \phi \\ \phi, & \text{if } \mathcal{A} = \phi \end{cases}$$

Then  $(\mathbb{N}, \mu)$  is a reversible closure space.

To prove this, we first prove  $(\mathbb{N}, \mu)$  is a closure space. From the definition of  $\mu$ , we have  $\mu(\phi) = \phi$ , for any subset  $\mathcal{A} \subset \mathbb{N}, \mathcal{A} \subset \mu(\mathcal{A})$  and  $\mu(\mathcal{A} \cup \mathcal{B}) = \mu(\mathcal{A}) \cup \mu(\mathcal{B}), \forall \mathcal{A}, \mathcal{B} \subset \mathbb{N}$ .

Let  $\theta : \mathbb{N} \rightarrow \mathbb{N}$  be any one to one continuous function, since  $\mathfrak{X}$  is finitely generated  $\mu(\mathcal{A}) =$

$\cup\{\mu(a) : a \in \mathcal{A}, \forall \mathcal{A} \neq \phi \subset \mathbb{N}\}$ . Here the only closed subsets are  $\mathbb{N}$  and  $\phi$ , also  $\theta(\mathbb{N}) = \mathbb{N}$ ,  $\theta(\phi) = \phi$ , so  $\theta$  is closed. Hence  $\theta$  is a  $\check{C}$ -homeomorphism. Now using Theorem 3.3  $(\mathbb{N}, \mu)$  is reversible.

In this example, the associated topological space is the indiscrete space and hence the associated topological space is also reversible.

**Proposition 3.6.** *If  $\mathfrak{T}$  is finite and  $\mu$  is any closure operator on  $\mathfrak{T}$ , then  $(\mathfrak{T}, \mu)$  is a reversible closure space.*

*Proof.* Consider the set of all bijective  $\check{C}$ -continuous functions on  $\mathfrak{T}$ , we denote this set by  $\mathfrak{G}$ , and let  $\theta \in \mathfrak{G}$ . Now using Theorem 2.1, we have  $\theta^2 \in \mathfrak{G}$ , similarly  $\theta^3, \theta^4 \dots \in \mathfrak{G}$ . Since  $\mathfrak{T}$  is finite, there exist  $n > m \in \mathbb{Z}^+$  such that  $\theta^n = \theta^m$  in  $\mathfrak{G}$ . Hence  $\theta^{n-m} = id_{\mathfrak{T}}$ , it follows that  $\theta^{n-m-1} = \theta^{-1} \in \mathfrak{G}$ . Thus  $\theta$  is a  $\check{C}$ -homeomorphism. Hence by Theorem 3.3,  $\mathfrak{T}$  is reversible. □

**Proposition 3.7.** *If  $(\mathfrak{T}, \mu)$  is a reversible closure space then  $(\mathfrak{T}, \tau(\mu))$  is a reversible topological space.*

*Proof.* Result immediately follows from Theorem 2.2. □

**Proposition 3.8.** *If  $\mu$  is a topological closure operator on a set  $\mathfrak{T}$ , then  $(\mathfrak{T}, \mu)$  is reversible if and only if the corresponding topological space is reversible.*

*Proof.* Trivial. □

**Theorem 3.9.** *If  $\mu$  be any reversible closure operator on  $\mathfrak{T}$ . Let  $\mathcal{D}$  is the discrete closure operator and  $\mathcal{I}$  the indiscrete (trivial) closure operator on the same set  $\mathfrak{T}$ , then  $(\mathfrak{T}, \mu \circ \mathcal{D}), (\mathfrak{T}, \mathcal{D} \circ \mu), (\mathfrak{T}, \mu \circ \mathcal{I}), (\mathfrak{T}, \mathcal{I} \circ \mu)$  are reversible.*

*Proof.* For any  $\mathcal{A} \subset \mathfrak{T}$ , we have  $(\mu \circ \mathcal{D})(\mathcal{A}) = \mu(\mathcal{D}(\mathcal{A})) = \mu(\mathcal{A})$ . Hence  $(\mathfrak{T}, \mu \circ \mathcal{D})$  is reversible. Similar proof work for  $(\mathfrak{T}, \mathcal{D} \circ \mu), (\mathfrak{T}, \mu \circ \mathcal{I}), (\mathfrak{T}, \mathcal{I} \circ \mu)$ , thus all are reversible closure spaces. □

**Proposition 3.10.** *If  $\mu_1, \mu_2$  are reversible closure operators on  $\mathfrak{T}$  such that either  $\mu_1 \leq \mu_2$  or  $\mu_2 \leq \mu_1$  then  $(\mathfrak{T}, \mu_1 \cup \mu_2)$  is reversible.*

*Proof.* Assume  $\mu_1 \leq \mu_2$ , then for any  $\mathcal{A} \in \mathfrak{T}$ ,  $\mu_2(\mathcal{A}) \subset \mu_1(\mathcal{A})$ . Also  $(\mu_1 \cup \mu_2)(\mathcal{A}) = \mu_1(\mathcal{A}) \cup \mu_2(\mathcal{A}) = \mu_1(\mathcal{A})$ ,  $\forall \mathcal{A} \in \mathfrak{T}$ . Hence  $(\mathfrak{T}, \mu_1 \cup \mu_2)$  is reversible. Similarly, if  $\mu_2 \leq \mu_1$ , then  $(\mathfrak{T}, \mu_1 \cup \mu_2)$  is reversible. □

Given below are some examples of non-reversible closure spaces.

EXAMPLE 3.11. A non reversible topological space can be viewed as a non reversible closure space.

EXAMPLE 3.12. Define a closure operator  $\mu$  on  $\mathbb{Z}$ , as follows  $\forall k \in \mathbb{Z}, \mu(k) = k$  if  $k < 0$  and  $\mu(k) = \{k, k + 1\}$  if  $k \geq 0$ . For any  $\mathcal{N} \subset \mathbb{Z}$  we have

$$\mu(\mathcal{N}) = \begin{cases} \cup\{\mu(n); n \in \mathcal{N}\}, & \text{if } \mathcal{N} \neq \emptyset \\ \emptyset, & \text{if } \mathcal{N} = \emptyset \end{cases}$$

Then  $(\mathbb{Z}, \mu)$  is a closure space (not topological), which is not reversible.

For, consider the map  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  as  $\theta(a) = a + 1$ , then  $\theta$  is one to one, onto,  $\check{C}$ -continuous but not a  $\check{C}$ -homeomorphism. Since  $\{-1\}$  is a closed set in  $(\mathbb{Z}, \mu)$  but  $\mu(\theta(\{-1\})) = \mu(\{0\}) = \{0, 1\}$ , which is not closed in  $(\mathbb{Z}, \mu)$ . Then by Theorem 3.3,  $(\mathbb{Z}, \mu)$  is not reversible.

*Remark 3.13.* From Example 3.12, there may exist non reversible closure spaces, which is the union of two disjoint reversible subspaces.

*Remark 3.14.* From the above example we have the following proposition.

**Proposition 3.15.** *Not every finitely generated closure space is reversible.*

*Proof.* Example 3.12 gives the counter example. □

EXAMPLE 3.16. On  $\mathbb{Z}$ , define a closure operator  $\mu_{m,n}$ ,  $m \neq n$  as follows

$$\mu_{m,n}(\mathcal{A}) = \begin{cases} \emptyset, & \text{if } \mathcal{A} = \emptyset, \\ \mathbb{Z} - \{n\}, & \text{if } \mathcal{A} = \{m\} \\ \mathbb{Z}, & \text{otherwise} \end{cases}$$

Then  $(\mathbb{Z}, \mu_{m,n})$  is a closure space, which is not reversible.

For, take  $m = 1, n = 2$  and consider the function  $\theta$  on  $\mathbb{Z}$  as  $\theta(n) = n + 1$ , then  $\theta$  is one to one, onto and  $\check{C}$ -continuous but not a  $\check{C}$ -homeomorphism. Hence from Theorem 3.3,  $(\mathbb{Z}, \mu_{m,n})$  is not reversible.

*Remark 3.17.* The above example gives a non reversible non topological closure space, whose associated topological space ( in this example the topology is the indiscrete topology) is reversible. Also from the above example we have the following proposition.

**Definition 3.18.** [8] Let  $\mathfrak{X}$  be a set, for  $x, y \in \mathfrak{X} x \neq y$ , define  $\mu_{x,y}$  on  $P(\mathfrak{X})$  as follows

$$\mu_{x,y}(\mathcal{A}) = \begin{cases} \phi, & \text{if } \mathcal{A} = \phi, \\ \mathfrak{X} - \{y\}, & \text{if } \mathcal{A} = \{x\} \\ \mathfrak{X}, & \text{otherwise} \end{cases}$$

Then  $\mu_{x,y}$  is a closure operator on  $\mathfrak{X}$ , called the infra closure operator on  $\mathfrak{X}$ .

**Proposition 3.19.** *The infra closure operator on  $\mathbb{Z}$  is non reversible.*

*Proof.* Follows from Example 3.16. □

It is known that, in topological spaces, reversibility is a topological property [7]. We state an equivalent result in closure spaces context as follows.

**Theorem 3.20.** *If  $(\mathfrak{T}, \mu)$  and  $(\mathfrak{U}, \nu)$  are  $\check{C}$ -homeomorphic closure spaces. If  $(\mathfrak{T}, \mu)$  is reversible, then  $(\mathfrak{U}, \nu)$  is also reversible.*

In [4] Larson studied the concept of complete homogeneity in topological spaces and characterized all spaces which are minimum and maximum with respect to a topological property. He also determined a characterization theorem for completely homogeneous topological spaces. In an analogous way Ramachandran [8], studied complete homogeneity in closure spaces.

**Definition 3.21.** A topological space  $(V, \mathcal{T})$  is completely homogeneous if every one to one mapping of  $V$  on to itself is a homeomorphism [4].



**Definition 3.22.** A closure space  $(\mathcal{T}, \mu)$  completely homogeneous if every one to one mapping of  $\mathcal{T}$  on to itself is a  $\check{C}$ -homeomorphism [8].

The following theorem finds the relation between complete homogeneity and reversibility in closure spaces.

**Theorem 3.23.** *Let  $(\mathcal{T}, \mu)$  be a closure space. If  $(\mathcal{T}, \mu)$  is completely homogeneous then  $(\mathcal{T}, \mu)$  is reversible.*

*Proof.* Trivial. □

*Remark 3.24.* A closure space  $(\mathcal{T}, \mu)$  is reversible doesn't imply  $(\mathcal{T}, \mu)$  is completely homogeneous.

**EXAMPLE 3.25.** On  $\mathbb{N}$ , let  $\mu$  be the closure operator defined as  $\forall n \in \mathbb{N}, \mu(k) = \{k, k + 1\}$  and for all  $\mathcal{A} \subset \mathbb{N}$ ,

$$\mu(\mathcal{A}) = \begin{cases} \cup\{\mu(a); a \in \mathcal{A}\}, & \text{if } \mathcal{A} \neq \phi \\ \phi, & \text{if } \mathcal{A} = \phi. \end{cases}$$

Then  $(\mathbb{N}, \mu)$  is reversible, but  $(\mathbb{N}, \mu)$  is not completely homogeneous. For consider the bijection on  $\mathbb{N}$  defined as follows,  $\theta(1) = 2, \theta(2) = 1, \theta(k) = k, \forall k \geq 3$ , then  $\theta$  is a bijection, but not  $\check{C}$ -continuous. Hence  $(\mathbb{N}, \mu)$  is not completely homogeneous.

Some result which are not generally valid in topological spaces are the same in closure spaces even, in view of Proposition 3.8. In [7], it is showed that reversibility is not a hereditary in topological spaces, whereas complete homogeneity is hereditary in topological spaces [4]. Using these facts, we get the proposition below.

**Proposition 3.26.** *Let  $(\mathcal{T}, \mu)$  be a completely homogeneous closure space. Then  $(\mathcal{T}, \mu)$  is hereditarily reversible.*

*Proof.* Using theorem 3.23, if the closure space  $(\mathcal{T}, \mu)$  is completely homogeneous, then it is reversible, also every subspace of a completely homogeneous closure space is completely homogeneous. Which completes the proof. □

In topology, spaces that are reversible, which are maximal or minimal in relation to some defined topological property. A compact Hausdorff space, for example, is maximal compact and minimal Hausdorff. Such space is just one example of a reversible space. In this section we try an analogous study in reversible closure spaces.

**Theorem 3.27.** *For a closure space  $(\mathfrak{X}, \mu)$ , the following statements are equivalent.*

- (1)  $(\mathfrak{X}, \mu)$  is reversible.
- (2) For some closure space property  $P$ ,  $(\mathfrak{X}, \mu)$  is minimum.
- (3) For some closure space property  $P$ ,  $(\mathfrak{X}, \mu)$  is maximum.

*Proof.* To prove (1)  $\implies$  (2), assume  $(\mathfrak{X}, \mu)$  is reversible. We define a property  $P$  as follows, a closure space  $(\mathfrak{U}, \nu)$  has a property  $P$ , if there exist a one to one, onto,  $\check{C}$ -continuous map from  $(\mathfrak{U}, \nu)$  to  $(\mathfrak{X}, \mu)$ . Clearly  $(\mathfrak{X}, \mu)$  has a property  $P$ . Assume  $\mu^*$  is a closure operator on  $\mathfrak{X}$  such that  $(\mathfrak{X}, \mu^*)$  has the property  $P$  and  $\mu^* \leq \mu$ . Then there exist a one to one, onto,  $\check{C}$ -continuous map  $\eta : (\mathfrak{X}, \mu^*) \longrightarrow (\mathfrak{X}, \mu)$ . Let us take a closure operator  $\mu'$  on  $\mathfrak{X}$  in such a way that the closed sets in  $(\mathfrak{X}, \mu')$  are those sets whose image under  $\eta$  are closed in  $(\mathfrak{X}, \mu)$ . Then clearly  $\mu'$  is coarser than  $\mu^*$ . Define  $\theta : (\mathfrak{X}, \mu') \longrightarrow (\mathfrak{X}, \mu)$  such that  $\theta(x) = \eta(x)$ , then by using similar steps in Theorem 3.3, we can prove  $\theta$  is a  $\check{C}$ -homeomorphism. Hence  $\mu' = \mu^*$ . Thus  $\eta : (\mathfrak{X}, \mu^*) \longrightarrow (\mathfrak{X}, \mu)$  is a  $\check{C}$ -homeomorphism, which contradicts the reversibility of  $(\mathfrak{X}, \mu)$ . Hence  $\mu$  must be minimum for some  $P$ .

To prove (2)  $\implies$  (1) assume  $(\mathfrak{X}, \mu)$  is minimum for some closure space property  $P$ . Let  $\eta$  be a  $\check{C}$ -continuous, self-bijective map on  $\mathfrak{X}$ . Let us define a closure operator  $\mu^*$  on  $\mathfrak{X}$  such that, a  $\mathcal{A} \subset \mathfrak{X}$  is closed in  $(\mathfrak{X}, \mu^*)$ , if  $\eta(\mathcal{A})$  is closed in  $(\mathfrak{X}, \mu)$ . Then  $(\mathfrak{X}, \mu)$  and  $(\mathfrak{X}, \mu^*)$  are  $\check{C}$ -homeomorphic through  $\eta$ . Hence  $(\mathfrak{X}, \mu^*)$  has the property  $P$ . Thus  $\mu^*$  is minimum with respect  $P$  and hence  $\mu^* \leq \mu$ . Also by assumption  $\mu$  is minimum, hence  $\mu \leq \mu^*$ . Thus  $\mu = \mu^*$  and  $\eta : (\mathfrak{X}, \mu) \longrightarrow (\mathfrak{X}, \mu = \mu^*)$  is a  $\check{C}$ -homeomorphism. Therefore  $(\mathfrak{X}, \mu)$  is reversible.

To prove (1) and (3) are equivalent a similar proof will work, if we redefine the property  $P$  as, a closure space  $(\mathfrak{U}, \nu)$  has a property  $P$ , if there exist a one to one, onto,  $\check{C}$ -continuous map from  $(\mathfrak{X}, \mu)$  to  $(\mathfrak{U}, \nu)$ . □

Analogous theorem and proof in complete homogeneity context(topological) are found in [4].

#### 4. CONCLUDING REMARKS

In this paper we define reversible closure space analogous to reversible topological spaces. We obtain some results analogous to topological spaces. The relation between reversible topological spaces and the underlying topological spaces is obtained. We prove not every finitely generated closure space is reversible also find a relation between complete homogeneity and reversibility in closure space context. We also find some non topological non reversible closure spaces whose associated topological space is reversible. These are some questions for further investigation.

- (1) Identify the lattice of reversible closure operator and investigate its properties.
- (2) Does there exist a reversible non topological closure space which is the union of two disjoint non reversible closure spaces ? The answer to this question in topological context is yes.
- (3) Identify and study reversible non topological closure operators in detail.
- (4) Extend this study in fuzzy context is also recommended.

#### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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