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APPROXIMATIONS TO THE SOLUTION OF QUADRATIC FRACTIONAL INTEGRAL EQUATION (QFIE) WITH GENERALIZED MITTAG-LEFFLER Q FUNCTION

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Abstract. In this paper, we will find the solution to the quadratic fractional integral equation involving the Q function which is the generalization of Mittag-Leffler function with the help of forming the sequence of solutions converging to the solution of the fractional integral equation involving the Q function. We will study in this paper the existence and convergence of a nonlinear quadratic fractional integral equation with the new Q function which is the generalization of Mittag-Leffler function, on a closed and bounded interval of the real line with the help of some conditions.

Keywords: quadratic fractional integral equation; fractional derivatives and integrals; approximate solution.

2010 AMS Subject Classification: 45G10, 47H09, 47H10.

1. INTRODUCTION

Linear and nonlinear integral equations form an essential class of problems in mathematics. The theory of integral operators and integral equations is an imperative part of nonlinear analysis. It is initiated by the fact that this theory is often applicable in other branches of mathematics

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and some equations define mathematical models in physics, engineering or biology as well in describing problems linked with real world. Many authors have demonstrated applications of fractional calculus in the nonlinear oscillation of earthquakes [13], fluid-dynamic traffic model [14], to model frequency dependent damping behavior of many viscoelastic materials [15, 16], continuum and statistical mechanics [17], colored noise [18], solid mechanics [19], economics [20], bioengineering [21, 22, 23], anomalous transport [24], and dynamics of interfaces between nanoparticles and substrates[25]. There are also such equations whose interest lies in other branch of pure mathematics. Integral equations of fractional order create an interesting and important branch of the theory of integral equations. The theory of such integral equations is developed intensively in recent years together with the theory of differential equations of fractional order ([1, 2, 3, 4, 5, 6, 7]). On the other hand the theory of quadratic integral equations is also intensively studied and finds numerous applications in describing real world problems ([8, 9, 10, 11]). Let us mention that this theory was initiated by considering a quadratic integral equation of Chandrasekhar type ([2, 11, 12]).In this paper we prove the existence as well as approximations of the solutions of a certain generalized quadratic integral equation via an algorithm based on successive approximations under weak partial Lipschitz and compactness type conditions.

Given a closed and bounded interval $J = [0, T]$ of the real line \mathbb{R} for some $T > 0$, we consider the quadratic fractional integral equation (in short QFIE)

$$(1.1) \quad x(t) = x(t^{q-1})Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) f(s, x(s)) ds$$

where $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ and $q : J \rightarrow \mathbb{R}$ are continuous functions, $1 \leq q < 2$ and Γ is the Euler gamma function, and $Q_{\alpha,\beta,\delta}^{\gamma,q,r}(x)$ is generalized mittag leffler function.

By a *solution* of the QFIE (1.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation (1.1) on J , where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J .

2. AUXILIARY RESULTS

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation \preceq and the norm $\| \cdot \|$. It is known that E is regular if $\{x_n\}_{n \in \mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \rightarrow x^*$

as $n \rightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [39] and the references therein.

In this section, we present some basic definitions and preliminaries which are useful in further discussion.

Definition 2.1. (Mittag-Leffler Function) [28] *The Mittag - Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as,*

$$(2.1) \quad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k$$

where $z, \alpha \in C, \operatorname{Re}(\alpha) > 0$.

If we put $\alpha = 1$, then the above equation becomes

$$(2.2) \quad E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Definition 2.2. (Mittag-Leffler Function for two parameters) *The generalization of $E_\alpha(z)$ was studied by Wiman (1905) [33], Agarwal [26] and Humbert and Agarwal [29] defined the function as,*

$$(2.3) \quad E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k$$

where $z, \alpha, \beta \in C, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0$,

In 1971, The more generalized function is introduced by Prabhakar [38] as

$$(2.4) \quad E_{\alpha, \beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)}.$$

where $z, \alpha, \beta, \gamma \in C, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\gamma) > 0$,

where $\gamma \neq 0, (\gamma)_k = \gamma(\gamma+1)(\gamma+2)\dots(\gamma+k-1)$ is the Pochhammer symbol [31], and

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$$

In 2007, Shulka and Prajapati [31] introduced the function which is defined as,

$$(2.5) \quad E_{\alpha, \beta}^{\gamma, q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{k! \Gamma(\alpha k + \beta)}$$

where $z, \alpha, \beta, \gamma \in C, \min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0, 1) \cup N$

In 2012, further generalization of Mittag - Leffler function was defined by Salim [32] and Chauhan [27] as,

$$(2.6) \quad E_{\alpha, \beta}^{\gamma, \delta, q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_{(qk)} \Gamma(\alpha k + \beta)}$$

where $z, \alpha, \beta, \gamma \in C, \min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0, 1) \cup N$

$$(\gamma)_{qk} = \frac{\Gamma(\gamma + qk)}{\Gamma(\gamma)} \text{ and } (\delta)_{qk} = \frac{\Gamma(\delta + qk)}{\Gamma(\delta)}$$

denote the generalized Pochhammer symbol [31],

Definition 2.3. [30] *The generalization of Mittag - Leffler function denoted by $Q_{\alpha, \beta, \delta}^{\gamma, q, r}(x)$ and defined by*

$$(2.7) \quad \begin{aligned} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(x) &= Q_{\alpha, \beta, \delta}^{\gamma, q, r}(a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r, x) \\ &= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, s) (\gamma)_{qs}}{\prod_{n=1}^r \beta(a_n, s) (\delta)_{qs} \Gamma(\alpha s + \beta)} x^s, \end{aligned}$$

where $x, \alpha, \beta, \gamma, \delta, a_i, b_i \in C$,

$\min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0, 1) \cup N$,

$$(\gamma)_{qk} = \frac{\Gamma(\gamma + qk)}{\Gamma(\gamma)} \text{ and } (\delta)_{qk} = \frac{\Gamma(\delta + qk)}{\Gamma(\delta)}$$

Definition 2.4. A mapping $\mathcal{T} : E \rightarrow E$ is called isotone or nondecreasing if it preserves the order relation \preceq , that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.

Definition 2.5 ([36]). A mapping $\mathcal{T} : E \rightarrow E$ is called partially continuous at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. \mathcal{T} called partially continuous on E if it is partially continuous at every point of it. It is clear that if \mathcal{T} is partially continuous on E , then it is continuous on every chain C contained in E .

Definition 2.6. A mapping $\mathcal{T} : E \rightarrow E$ is called *partially bounded* if $\mathcal{T}(C)$ is bounded for every chain C in E . \mathcal{T} is called *uniformly partially bounded* if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. \mathcal{T} is called *bounded* if $\mathcal{T}(E)$ is a bounded subset of E .

Definition 2.7. A mapping $\mathcal{T} : E \rightarrow E$ is called *partially compact* if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E . \mathcal{T} is called *uniformly partially compact* if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E . \mathcal{T} is called *partially totally bounded* if for any totally ordered and bounded subset C of E , $\mathcal{T}(C)$ is a relatively compact subset of E . If \mathcal{T} is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E .

Definition 2.8 ([36]). The order relation \preceq and the metric d on a non-empty set E are said to be **compatible** if $\{x_n\}_{n \in \mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x^* . Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation \preceq and the norm $\|\cdot\|$ are said to be compatible if \preceq and the metric d defined through the norm $\|\cdot\|$ are compatible.

Definition 2.9 ([34]). A upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a \mathcal{D} -function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \rightarrow E$ is called *partially nonlinear \mathcal{D} -Lipschitz* if there exists a \mathcal{D} -function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(2.8) \quad \|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|)$$

for all comparable elements $x, y \in E$. If $\psi(r) = kr$, $k > 0$, then \mathcal{T} is called a *partially Lipschitz* with a Lipschitz constant k .

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$(2.9) \quad \mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}.$$

The elements of \mathcal{K} are called the positive vectors of the normed linear algebra E . The following lemma follows immediately from the definition of the set \mathcal{K} and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.10 ([35]). *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1 u_2 \preceq v_1 v_2$.*

Definition 2.11. *An operator $\mathcal{T} : E \rightarrow E$ is said to be positive if the range $R(\mathcal{T})$ of \mathcal{T} is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

Theorem 2.12 ([37]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation \preceq and the norm $\|\cdot\|$ in E are compatible in every compact chain of E . Let $\mathcal{A}, \mathcal{B} : E \rightarrow \mathcal{K}$ be two nondecreasing operators such that*

- (a) \mathcal{A} is partially bounded and partially nonlinear \mathcal{D} -Lipschitz with \mathcal{D} -functions $\Psi_{\mathcal{A}}$,
- (b) \mathcal{B} is partially continuous and uniformly partially compact, and
- (c) $M\Psi_{\mathcal{A}}(r) < r, r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C \text{ is a chain in } E\}$, and
- (d) there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the operator equation

$$(2.10) \quad \mathcal{A}x + \mathcal{B}x = x$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n, n = 0, 1, \dots$, converges monotonically to x^* .

3. MAIN RESULT

The QFIE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J . We define a norm $\|\cdot\|$ and the order relation \leq in $C(J, \mathbb{R})$ by

$$(3.1) \quad \|x\| = \sup_{t \in J} |x(t)|$$

and

$$(3.2) \quad x \leq y \iff x(t) \leq y(t)$$

for all $t \in J$ respectively. Clearly, $C(J, \mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation \leq . The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $(C(J, \mathbb{R}), \leq, \|\cdot\|)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation \leq defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and \leq are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

Definition 3.2. *A function $v \in C(J, \mathbb{R})$ is said to be a lower solution of the QFIE (1.1) if it satisfies*

$$v(t) \leq v(t^{q-1})Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) f(s, v(s)) ds$$

for all $t \in J$. Similarly, a function $u \in C(J, \mathbb{R})$ is said to be an upper solution of the QFIE (1.1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

- (A₁) The functions $f : J \times \mathbb{R} \rightarrow \mathbb{R}_+$, $q : J \rightarrow \mathbb{R}_+$ where q is continuous function.
- (A₂) There exists constant $M_f, M > 0$ such that $0 \leq f(t, x) \leq M_f$ and $x(t)Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) < M$ for all $t \in J$ and $x \in \mathbb{R}$.
- (A₃) There exists a \mathcal{D} -function ψ_f such that

$$0 \leq f(t, x) - f(t, y) \leq \psi_f(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}, x \leq y$.

- (A₄) $f(t, x)$ is nondecreasing in x for all $t \in J$.
- (A₅) The QFIE (1.1) has a lower solution $v \in C(J, \mathbb{R})$.

Theorem 3.3. *Assume that hypotheses (A₁)-(A₅) holds*

then the QFIE (1.1) has a solution x^ defined on J and the sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ of successive approximations defined by*

(3.3)

$$x_{n+1}(t) = x_n(t^{q-1})Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) f(s, x_n(s)) ds$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^* .

Proof. Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\| \cdot \|$ and the order relation \leq in E .

Define two operators \mathcal{A} and \mathcal{B} on E by

$$(3.4) \quad \mathcal{A}x(t) = x(t^{q-1})\mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q), t \in J,$$

$$(3.5) \quad \mathcal{B}x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) f(s,x(s)) ds, t \in J,$$

From the continuity of the integral and the hypotheses (A₁)-(A₅), it follows that \mathcal{A} and \mathcal{B} define the maps $\mathcal{A}, \mathcal{B} : E \rightarrow \mathcal{K}$. Now by definitions of the operators \mathcal{A} and \mathcal{B} , the QFIE (1.1) is equivalent to the operator equation

$$(3.6) \quad \mathcal{A}x(t) + \mathcal{B}x(t) = x(t), t \in J.$$

We shall show that the operators \mathcal{A} and \mathcal{B} satisfy all the conditions of Theorem 2.12. This is achieved in the series of following steps.

Step I: \mathcal{A} and \mathcal{B} are nondecreasing on E .

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (A₃) and (A₄), we obtain

$$\mathcal{A}x(t) = x(t^{q-1})\mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) \leq y(t^{q-1})\mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) = \mathcal{A}y(t),$$

and

$$\begin{aligned} \mathcal{B}x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) f(s,x(s)) ds, t \in J, \\ &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) f(s,y(s)) ds, t \in J, \\ &= \mathcal{B}y(t) \end{aligned}$$

for all $t \in J$. This shows that \mathcal{A} and \mathcal{B} are nondecreasing operators on E into E . Thus, \mathcal{A} and \mathcal{B} are nondecreasing positive operators on E into itself.

Step II: \mathcal{A} is partially bounded and partially \mathcal{D} -Lipschitz on E .

Let $x \in E$ be arbitrary. Then by (A₂),

$$|\mathcal{A}x(t)| \leq |x_n(t^{q-1})Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q)| \leq M,$$

for all $t \in J$. Taking supremum over t , we obtain $\|\mathcal{A}x\| \leq M$ and so, \mathcal{A} is bounded. This further implies that \mathcal{A} is partially bounded on E .

Now, let $x, y \in E$ be such that $x \leq y$. Then, by hypothesis,

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |x(t^{q-1})Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) - y(t^{q-1})Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q)| \\ &\leq Q_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q)|x(t^{q-1}) - y(t^{q-1})| \\ &\leq M(|x - y|), \end{aligned}$$

for all $t \in J$. Taking supremum over t , we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq M(\|x - y\|)$$

for all $x, y \in E$ with $x \leq y$. Hence \mathcal{A} is partially nonlinear \mathcal{D} -Lipschitz operators on E which further implies that it is also a partially continuous on E into itself.

Step III: \mathcal{B} is a partially continuous operator on E .

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a chain C of E such that $x_n \rightarrow x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{B}x_n(t) &= \lim_{n \rightarrow \infty} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) f(s, x_n(s)) ds, \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) \left[\lim_{n \rightarrow \infty} f(s, x_n(s)) \right] ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) f(s, x(s)) ds \\ &= \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J .

Next, we will show that $\{\mathcal{B}x_n\}_{n \in \mathbb{N}}$ is an equicontinuous sequence of functions in E . Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\left| \mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1) \right|$$

$$\begin{aligned}
 &\leq \left| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_2 - s)^q) f(s, x_n(s)) ds \right. \\
 &\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_2 - s)^q) f(s, x_n(s)) ds \right. \\
 &\quad \left. - \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right| \\
 &+ \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right| \\
 &+ \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right. \\
 &\quad \left. - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \left| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_2 - s)^q) - \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) \right| |f(s, x_n(s))| ds \\
 &+ \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) f(s, x_n(s)) ds \right| \\
 &+ \frac{1}{\Gamma(q)} \int_0^{t_1} \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) |f(s, x_n(s))| ds \\
 &\leq \frac{1}{\Gamma(q)} \int_0^T (t_2 - s)^{q-1} \left| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_2 - s)^q) - \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) \right| M_f ds \\
 &+ \frac{1}{\Gamma(q)} \int_0^T \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) M_f ds \\
 &+ \frac{1}{\Gamma(q)} \int_0^T \left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) M_f ds \\
 &\leq \frac{M_f}{\Gamma(q)} \left(\int_0^T |(t_2 - s)^{q-1}|^2 ds \right)^{1/2} \left(\int_0^T \left| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_2 - s)^q) - \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) \right|^2 ds \right)^{1/2} \\
 &+ 2 \left(\int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|^2 ds \right)^{1/2} \left(\int_0^T \left| \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t_1 - s)^q) \right|^2 ds \right)^{1/2} \frac{M_f}{\Gamma(q)}
 \end{aligned}$$

(3.7)

Since the functions $Q_{\alpha,\beta,\delta}^{\gamma,q,r}$, q are continuous on compact interval J and interval is continuous on compact set $J \times J$, they are uniformly continuous there. Therefore, from the above inequality (3.7) it follows that

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \rightarrow \mathcal{B}x$ is uniform and hence \mathcal{B} is partially continuous on E .

Step IV: \mathcal{B} is uniformly partially compact operator on E .

Let C be an arbitrary chain in E . We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E . First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathcal{B}x$. Now, by hypothesis (A₁),

$$\begin{aligned} |y(t)| &\leq \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t-s)^q) f(s,x(s)) ds \right| \\ &\leq r \end{aligned}$$

for all $t \in J$. Taking the supremum over t , we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E . Moreover, $\|\mathcal{B}(C)\| \leq r$ for all chains C in E . Hence, \mathcal{B} is a uniformly partially bounded operator on E .

Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E . Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$\begin{aligned} &\left| \mathcal{B}x(t_2) - \mathcal{B}x(t_1) \right| \\ &\leq \left| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t_2-s)^q) f(s,x(s)) ds \right. \\ &\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1-s)^q) f(s,x(s)) ds \right| \\ &\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t_2-s)^q) f(s,x(s)) ds \right. \\ &\quad \left. - \int_0^{t_2} (t_2-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1-s)^q) f(s,x(s)) ds \right| \\ &\quad + \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2-s)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1-s)^q) f(s,x(s)) ds \right. \end{aligned}$$

$$\begin{aligned}
 & - \int_0^{t_1} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) f(s, x(s)) ds \Big| \\
 & + \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) f(s, x(s)) ds \right. \\
 & \quad \left. - \int_0^{t_1} (t_1 - s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) f(s, x(s)) ds \right| \\
 & \leq \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \left| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_2 - s)^q) - \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) \right| |f(s, x(s))| ds \\
 & + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) f(s, x(s)) ds \right| \\
 & + \frac{1}{\Gamma(q)} \int_0^{t_1} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) |f(s, x(s))| ds \\
 & \leq \frac{1}{\Gamma(q)} \int_0^T (t_2 - s)^{q-1} \left| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_2 - s)^q) - \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) \right| M_f ds \\
 & + \frac{1}{\Gamma(q)} \int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) M_f ds \\
 & + \frac{1}{\Gamma(q)} \int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) M_f ds \\
 & \leq \frac{M_f}{\Gamma(q)} \left(\int_0^T |(t_2 - s)^{q-1}|^2 ds \right)^{1/2} \left(\int_0^T \left| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_2 - s)^q) - \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) \right|^2 ds \right)^{1/2} \\
 & + 2 \left(\int_0^T |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|^2 ds \right)^{1/2} \left(\int_0^T \left| \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}((t_1 - s)^q) \right|^2 ds \right)^{1/2} \frac{M_f}{\Gamma(q)} \\
 & \rightarrow 0 \quad \text{as } t_1 \rightarrow t_2,
 \end{aligned}$$

uniformly for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E . Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E , so it is compact. Consequently, \mathcal{B} is a uniformly partially compact operator on E into itself.

Step V: v satisfies the operator inequality $v \leq \mathcal{A}v + \mathcal{B}v$.

By hypothesis (A₅), the QFIE (1.1) has a lower solution v defined on J . Then, we have (3.8)

$$v(t) \leq v(t^{q-1}) \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(-a(t-s)^q) f(s, v(s)) ds$$

for all $t \in J$. From the definitions of the operators \mathcal{A} , \mathcal{B} and \mathcal{C} it follows that $v(t) \leq \mathcal{A}v(t) + \mathcal{B}v(t)$ for all $t \in J$. Hence $v \leq \mathcal{A}v + \mathcal{B}v$.

Step VI: The \mathcal{D} -functions $\psi_{\mathcal{A}}$ satisfy the growth condition $M\psi_{\mathcal{A}}(r) < r$, for $r > 0$.

Finally, the \mathcal{D} -function $\psi_{\mathcal{A}}$ of the operator \mathcal{A} satisfy the inequality given in hypothesis (d) of Theorem 2.12, viz.,

$$M\psi_{\mathcal{A}}(r) < r$$

for all $r > 0$.

Thus \mathcal{A} and \mathcal{B} satisfy all the conditions of Theorem 2.12 and we conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution. Consequently the QFIE (1.1) has a solution x^* defined on J . Furthermore, the sequence $\{x_n\}_{n \in \mathbb{N}}$ of successive approximations defined by (3.3) converges monotonically to x^* . This completes the proof. \square

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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