CONTINUOUS MONOTONIC DECOMPOSITION OF JUMP GRAPH OF PATHS AND COMPLETE GRAPHS

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Abstract. The Jump graph $J(G)$ of a graph $G$ is the graph whose vertices are edges of $G$ and two vertices of $J(G)$ are adjacent if and only if they are not adjacent in $G$. In this article, we have given characterization for the Jump graph of paths into Continuous monotonic star decomposition. Also we have given characterization for the Jump graph of complete graphs into Continuous monotonic tree decomposition.

Keywords: decomposition; jump graph; path; complete graph.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph without loops or multiple edges. A path on $n$ vertices is denoted by $P_n$, cycle on $n$ vertices is denoted by $C_n$ and complete graph on $n$ vertices is denoted by $K_n$. The neighbourhood of a vertex $v$ in $G$ is the set $N(v)$ consisting of all vertices that are adjacent to $v$. $|N(v)|$ is called the degree of $v$ and is denoted by $d(v)$. A complete bipartite graph with partite sets $V_1$ and $V_2$, where $|V_1| = r$ and $|V_2| = s$, is denoted by $K_{r,s}$. The graph
$K_{1,r}$ is called a star and is denoted by $S_r$. Claw is a star with three edges. For any set $S$ of points of $G$, induced subgraph $<S>$ is the maximal subgraph of $G$ with point set $S$. An edge induced subgraph $<E'>$ of $G$ is the subgraph of $G$ whose vertex set is the set of ends of edges in $E'$ and whose edge set is $E'$. The terms not defined here are used in the sense of [3].

A decomposition of a graph $G$ is a family of edge-disjoint subgraphs $\{G_1, G_2, \ldots, G_k\}$ such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. If each $G_i$ is isomorphic to $H$ for some subgraph $H$ of $G$, then the decomposition is called a $H$-decomposition of $G$. A decomposition, $\{G_1, G_2, \ldots, G_k\}$ for all $k \in \mathbb{N}$ is said to be a Continuous Monotonic Decomposition (CMD) if each $G_i$ is connected and $|E(G_i)| = i$ for all $i \in \mathbb{N}$. The concept of CMD was introduced by Paulraj Joseph and Gnanadhas [4].

The Jump graph $J(G)$ of a graph $G$ is the graph whose vertices are edges of $G$ and two vertices of $J(G)$ are adjacent if and only if they are not adjacent in $G$. Equivalently complement of line graph $L(G)$ is the Jump graph $J(G)$ of $G$. This concept was introduced by Chartrand in [1]. Coconut tree $CT(m,n)$ is a graph obtained from the path $P_n$ by appending $m$ new pendant edges at an end vertex of $P_n$. Double coconut tree $D(n,r,m)$ is a graph obtained by attaching $n > 1$ pendant vertices to one end of the path $P_r$ and $m > 1$ pendant vertices to the other end of the path $P_r$.

2. **Continuous Monotonic Star Decomposition of Jump Graph of Paths**

Let $J(P_n)$ denote the Jump graph of paths. Then $J(P_n)$ is a connected graph if and only if $n \geq 5$. Let us consider the connected jump graph of paths. Let the edges of path $P_n$ be labelled as $x_1, x_2, \ldots, x_{n-1}$. Since the number of edges of path $P_n$ is $(n-1)$, the number of vertices of $J(P_n)$ is $(n-1)$. The number of edges of Jump graph of paths $J(P_n)$ is $\binom{n-2}{2}$.

**Definition 2.1.** [4] If $G$ admits a CMD $\{G_1, G_2, \ldots, G_k\}$ for all $k \in \mathbb{N}$, where each $G_i$ is a star, then we say that $G$ admits a Continuous Monotonic Star Decomposition (CMSD).

**Theorem 2.2.** [4] Let $G$ be a connected simple graph of order $p$ and size $q$. Then $G$ admits a CMD $\{H_1, H_2, \ldots, H_n\}$ if and only if $q = \binom{n+1}{2}$.

**Lemma 2.3.** Let $m \geq 2$. The set $\{1, 2, \ldots, m\}$ can be partitioned into two sets $B_1$ and $B_2$ such that $\sum_{x \in B_1} x = \frac{(n-3)(n-4)}{2}$ and $\sum_{y \in B_2} y = n - 3$ where $\frac{m(m+1)}{2} = \binom{n-2}{2}$. 

Proof. Let \( m \geq 2 \) and \( n = m + 3 \). Let us prove this lemma by induction on \( m \).

When \( m = 2 \), \( n = 5 \). If \( B_1 = 1 \) and \( B_2 = 2 \) then \( \sum_{x \in B_1} x = 1 \) and \( \sum_{y \in B_2} y = 2 \). Hence the result is true for \( m = 2 \).

Assume that the result is true for \( m - 1 \). Hence the set \( \{1, 2, \ldots, m - 1\} \) can be partitioned into two sets \( B_1 \) and \( B_2 \) such that \( \sum_{x \in B_1} x = \frac{(m-1)(m-2)}{2} \) and \( \sum_{y \in B_2} y = m - 1 \). Then the set \( \{1, 2, \ldots, m\} \) can be partitioned into two sets \( B'_1 \) and \( B'_2 \) where \( B'_1 = B_1 \cup \{m - 1\} \) and \( B'_2 = m \).

Clearly
\[
\sum_{x \in B_1'} x = \sum_{x \in B_1} x + \{m - 1\} = \frac{(m-1)(m)}{2} = \frac{(n-3)(n-4)}{2}
\]
and
\[
\sum_{y \in B_2'} y = m = n - 3.
\]
Hence the induction and lemma holds.

\[\square\]

**Theorem 2.4.** Jump graph of path \( J(P_n) \) admits Continuous monotonic star decomposition \( \{S_1, S_2, \ldots, S_m\} \) if and only if there exists an integer \( m \) such that (i) \( m = n - 3 \) and (ii) \( \frac{m(m+1)}{2} = \frac{(n-2)}{2} \).

Proof. We have \(|E[J(P_n)]| = \binom{n-2}{2} \).

Assume that \( J(P_n) \) admits Continuous monotonic star decomposition \( \{S_1, S_2, \ldots, S_m\} \). By theorem 2.2, \(|E[J(P_n)]| = \frac{m(m+1)}{2} \). Hence, \( \frac{m(m+1)}{2} = \binom{n-2}{2} \). Hence (ii).

Since \( J(P_n) \) admits Continuous monotonic star decomposition \( \{S_1, S_2, \ldots, S_m\} \), \( \binom{n-2}{2} = 1 + 2 + \ldots + m = \frac{m(m+1)}{2} \). This implies \( \frac{m(m+1)}{2} = \binom{(n-2)(n-3)}{2} \).

Thus \( m = n - 3 \). Hence (i).

Conversely, assume that \( m = n - 3 \) and \( \frac{m(m+1)}{2} = \binom{n-2}{2} \).

Define \( T_1 = \{x_i x_j/3 \leq i \leq n-2 \}; i > j \}; 1 \leq j \leq n-3; j \neq i-1 \} \) and
\( T_2 = \{x_{n-1} x_j/1 \leq j \leq n-3 \} \).

Now, \(|T_1| = \frac{(n-3)(n-4)}{2} \) and \(|T_2| = n - 3 \). Thus \(|T_1| + |T_2| = \binom{(n-2)}{2} = \frac{m(m+1)}{2} = 1 + 2 + \ldots + m \).

By lemma 2.3, \( \{1, 2, \ldots, m\} = B_1 \cup B_2 \) where \( \sum_{x \in B_1} x = \frac{(n-3)(n-4)}{2} \) and \( \sum_{y \in B_2} y = n - 3 \).

Decompose \( T_1 \) and \( T_2 \) into stars \( S_i \) as follows:
\( T_1 = \bigcup S_i \) where \( i \in B_1 \) and \( S_i = \{x_{i+2}; x_1, x_2, \ldots, x_i\} \). Here \( x_{i+2} \) forms center of the star \( S_i \).
\( T_2 = S_m \) where \( m \in B_2 \) and \( S_m = \{x_{n-1}; x_1, x_2, \ldots, x_{n-3}\} \). Here \( x_{n-1} \) forms center of the star \( S_m \). Also \(|E(S_i)| = i; 1 \leq i \leq m \). Thus \( J(P_n) \) admits Continuous monotonic star decomposition \( \{S_1, S_2, \ldots, S_m\} \). \[\square\]
Illustration 2.5. As an illustration let us decompose $J(P_{12})$.

Let $E(P_{12}) = \{1, 2, \ldots, 11\}$. Therefore $V[J(P_{12})] = \{1, 2, \ldots, 11\}$.

$P_{12}$ and $J(P_{12})$ are given in Figure 2.1 and Figure 2.2 respectively.

Here $|E[J(P_{12})]| = 45$.

Define $T_1 = \{31, 41, 42, 51, 52, 53, 61, 62, 63, 64, 71, 72, 73, 74, 75, 81, 82, 83, 84, 85, 86, 91, 92, 93, 94, 95, 96, 97, (10)1, (10)2, (10)3, (10)4, (10)5, (10)6, (10)7, (10)8\}$ and


$|T_1| + |T_2| = 45 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = \binom{10}{2}$.

Here $B_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B_2 = \{9\}$. 
$T_1$ is decomposed as $S_1 \cup S_2 \cup \ldots \cup S_8$ where $S_1 = \{31\}, S_2 = \{41, 42\}$,
$S_3 = \{51, 52, 53\}, S_4 = \{61, 62, 63, 64\}, S_5 = \{71, 72, 73, 74, 75\}$,
$S_6 = \{81, 82, 83, 84, 85, 86\}, S_7 = \{91, 92, 93, 94, 95, 96, 97\}$,
$S_8 = \{(10)1, (10)2, (10)3, (10)4, (10)5, (10)6, (10)7, (10)8\}$.
$T_2$ is decomposed as $S_9$ where $S_9 = \{\{11\}1, (11)2, (11)3, (11)4, (11)5, (11)6, (11)7, (11)8, (11)9\}$. Clearly $\{S_1, S_2, \ldots, S_9\}$ forms a CMSD of $J(P_{12})$.

3. Continuous Monotonic Tree Decomposition of Jump Graph of Complete Graphs

Let $J(K_n)$ denote the Jump graph of complete graphs. Then $J(K_n)$ is a connected graph if and only if $n \geq 5$. Let us consider the connected jump graph of complete graphs. Let $V(K_n) = \{1, 2, \ldots, n\}$ and $E(K_n) = \{12, 13, \ldots, 1n, 23, 24, \ldots, 2n, \ldots, (n-2)(n-1), (n-2)n, (n-1)n\}$. Since the number of edges of complete graphs $K_n$ is $\frac{n(n-1)}{2}$, the number of vertices of $J(K_n)$ is $\frac{n(n-1)}{2}$. The number of edges of Jump graph of complete graphs $J(K_n)$ is $\frac{n(n-1)(n-2)(n-3)}{8}$.

**Theorem 3.1.** Let $n \geq 5$. Then $J(K_n)$ is decomposed into $\{T_2S_1, T_3S_2, \ldots, T_{n-2}S_{n-3}\}$ where $T_l = \frac{l(l+1)}{2}; 2 \leq l \leq n-2$.

**Proof.** Let $n \geq 5$. Let $V(K_n) = \{1, 2, \ldots, n\}$ and $E(K_n) = \{12, 13, \ldots, 1n, 23, 24, \ldots, 2n, \ldots, (n-2)(n-1), (n-2)n, (n-1)n\}$. Since edges of $K_n$ are taken as vertices of $J(K_n)$ we have, $V[J(K_n)] = \{12, 13, \ldots, 1n, 23, 24, \ldots, 2n, \ldots, (n-2)(n-1), (n-2)n, (n-1)n\}$. Two vertices $uv, xy$ in $J(K_n)$ are adjacent if $u, v, x, y$ all are distinct elements.

Define $U_1 = \{(n-3)(n-2), (n-3)(n-1), (n-3)n\}$ and $V_1 = \{(n-2)(n-1), (n-2)n, (n-1)n\}$. Thus $|U_1| = 3$ and $|V_1| = 3$.

Now $\langle\{(n-3)(n-2), (n-3)(n-1), (n-3)n, (n-2)(n-1), (n-2)n, (n-1)n\}\rangle \cong 3(S_1)$.

Thus we get $T_2$ copies of $S_1$.

Define $U_2 = \{(n-4)(n-3), (n-4)(n-2), (n-4)(n-1), (n-4)n\}$ and $V_2 = \{U_1, V_1\}$. Thus $|U_2| = 4$ and $|V_2| = 6$. Now $\langle\{U_1, U_2\}\rangle \cong 3(S_2)$. Also, $\langle\{V_1, U_2\}\rangle \cong 3(S_2)$. Therefore we get $T_3$ copies of $S_2$.

Define $U_3 = \{(n-5)(n-4), (n-5)(n-3), (n-5)(n-2), (n-5)(n-1), (n-5)n\}$ and
\[ V_3 = \{U_2, V_2\}. \text{ Thus } |V_3| = 5 \text{ and } |V_3| = 10. \]

Now, \( \langle \{U_2, U_3\} \rangle \cong |U_2|(S_3) \). Also, \( \langle \{U_1, U_3\} \rangle \cong |U_1|(S_3) \) and \( \langle \{V_1, U_3\} \rangle \cong |V_1|(S_3) \). \[ |U_1| + |U_2| + |V_1| = 3 + 3 + 4 = \frac{(4)(5)}{2} = T_4. \] Thus we get \( T_4 \) copies of \( S_3 \). Proceed like this. Finally we define \( U_{(n-3)} = \{12, 13, \ldots, 1n\} \) and \( V_{(n-3)} = \{U_{(n-4)}, V_{(n-4)}\} \).

Thus \( |U_{(n-3)}| = n - 1 \) and \( |V_{(n-3)}| = |U_{(n-4)}| + |V_{(n-4)}| \).

Now, \( \langle \{U_{(n-4)}, U_{(n-3)}\} \rangle \cong |U_{(n-4)}|(S_{n-3}) \)

Also, \( \langle \{U_{(n-5)}, U_{(n-3)}\} \rangle \cong |U_{(n-5)}|(S_{n-3}), \)

\[ \vdots \]

\( \langle \{U_1, U_{(n-3)}\} \rangle \cong |U_1|(S_{n-3}), \]

\( \langle \{V_1, U_{(n-3)}\} \rangle \cong |V_1|(S_{n-3}). \)

Now \( |V_1| + |U_1| + |U_2| + \ldots |U_{(n-5)}| + |U_{(n-4)}| = 3 + 3 + 4 + 5 + \ldots + (n - 3) + (n - 2) = \frac{(n-2)(n-1)}{2} = T_{n-2}. \) Thus we get \( T_{(n-2)} \) copies of \( S_{(n-3)} \). Hence we have

\[ E[J(K_n)] = \frac{S_1 \cup S_1 \cup S_2 \cup \ldots \cup S_{n-3} \cup \ldots \cup S_{(n-3)} \cup \ldots \cup S_{(n-3)}}{T_2 \text{ times} \quad T_3 \text{ times} \quad T_4 \text{ times} \quad T_{n-2} \text{ times}} = \]

is decomposed into \( \{T_2S_1, T_3S_2, \ldots, T_{n-2}S_{n-3}\} \) where \( T_l = \frac{l(l+1)}{2}, \) \( 2 \leq l \leq n - 2. \) \( \square \)

**Lemma 3.2.** Let \( m = \frac{n(n-3)}{2} \) where \( n \geq 5. \) The set \( \{1, 2, \ldots, m\} \) can be partitioned into two sets \( B_1 \) and \( B_2 \) such that \( \sum_{x \in B_1} x = \frac{n(n-3)(n^2-3n-2)}{8} \) and \( \sum_{y \in B_2} y = \frac{n(n-3)}{2} \) where \( \frac{m(m+1)}{2} = \frac{n(n-1)(n-2)(n-3)}{8}. \)

**Proof.** Let \( n \geq 5. \) Let us prove this lemma by induction on \( n. \) When \( n = 5, \) \( m = 5. \) If \( B_1 = \{1, 2, 3, 4\} \) and \( B_2 = 5 \) then \( \sum_{x \in B_1} x = 10 \) and \( \sum_{y \in B_2} y = 5. \) Hence the result is true for \( n = 5. \)

Assume that the result is true for \( n - 1. \) Hence the set \( \{1, 2, \ldots, \frac{(n-1)(n-4)}{2}\} \) can be partitioned into two sets \( B_1 \) and \( B_2 \) such that \( \sum_{x \in B_1} x = \frac{(n-1)(n-4)(n^2-3n-2)}{8} \) and \( \sum_{y \in B_2} y = \frac{(n-1)(n-4)}{2}. \)

Then the set \( \{1, 2, \ldots, m\} \) can be partitioned into two sets \( B_1' \) and \( B_2' \) where \( B_1' = B_1 \cup B_2 \cup \left\{ \frac{(n-1)(n-4)}{2} + 1, \frac{(n-1)(n-4)}{2} + 2, \ldots, \frac{(n-1)(n-4)}{2} + (n-3) \right\} \) and \( B_2' = \frac{(n-1)(n-4)}{2} + (n-2). \) Clearly \( \sum_{x \in B_1'} x = \sum_{x \in B_1} x + \sum_{y \in B_2} y + \frac{(n-1)(n-4)}{2} + 1 + \frac{(n-1)(n-4)}{2} + 2 + \ldots + \frac{(n-1)(n-4)}{2} + (n-3) = \frac{n(n^3-6n^2+7n+6)}{8} = \frac{n(n-3)(n^2-3n-2)}{8}. \) Also \( \sum_{y \in B_2'} y = \frac{(n-1)(n-4)}{2} + (n-2) = \frac{n(n-3)}{2}. \)

Hence the induction and lemma holds. \( \square \)
Theorem 3.3. Let \( n > 5 \). Then the jump graph of complete graph \( J(K_n) \) is \( \{3\text{DCT}(2,2,2), S_{T_l}, CT(T_l - 1,t), DCT(T_l - 1,r,s)\} \) decomposable where \( T_l = \frac{l(l+1)}{2}; l = 3,4,\ldots,n-3 \).

Proof. Let \( V(K_n) = \{1,2,\ldots,n\} \) and

\[
E(K_n) = \{12, 13, \ldots, 1n, 23, 24, \ldots, 2n, \ldots, (n-2)(n-1), (n-2)n, (n-1)n\}. \quad \text{Then}
\]

\[
V[J(k_n)] = \{12, \ldots, 1n, 23, \ldots, 2n, \ldots, (n-2)(n-1), (n-2)n, (n-1)n\}.
\]

Take \( D_1 = \{(n-4)(n-3), (n-4)(n-2), (n-4)(n-1), (n-4)n\} \),

\( D_2 = \{(n-3)(n-2), (n-1)n\} \),

\( D_3 = \{(n-3)(n-1), (n-2)n\} \) and

\( D_4 = \{(n-3)n, (n-2)(n-1)\} \).

Now, \( \langle\{D_1, D_2\} \rangle \cong \text{DCT}(2,2,2) \),

\( \langle\{D_1, D_3\} \rangle \cong \text{DCT}(2,2,2) \),

\( \langle\{D_1, D_4\} \rangle \cong \text{DCT}(2,2,2) \).

Now \( \{(n-5)(n-4); (n-3)(n-2), (n-3)(n-1), (n-3)n, (n-2)(n-1), (n-2)n, (n-1)n\} \cong S_6 \), \( \{(n-5)(n-3); (n-4)(n-2), (n-4)(n-1), (n-2)n, (n-1)n\} \cong S_5 \) and

\( \langle\{(n-5)(n-3), (n-4)n, (n-5)(n-2)\}\rangle \cong P_3 \). Thus \( \langle E(S_5) \cup E(P_3) \rangle \cong CT(5,3) \).

Now \( \{(n-5)(n-1); (n-4)(n-2), (n-3)(n-2), (n-3)n, (n-4)n, (n-2)n\} \cong S_5 \) and

\( \langle\{(n-5)(n-1), (n-4)(n-3), (n-5)(n-2), (n-4)(n-1)\}\rangle \cong P_4 \).

Thus \( \langle E(S_5) \cup E(P_4) \rangle \cong CT(5,4) \).

Now \( \{(n-5)n; (n-4)(n-3), (n-4)(n-2), (n-4)(n-1), (n-3)(n-2), (n-2)(n-1)\} \cong S_5 \),

\( \langle\{(n-5)n, (n-3)(n-1), (n-5)(n-2)\}\rangle \cong P_3 \) and \( \{(n-5)(n-2); (n-3)n, (n-1)n\} \cong S_2 \).

Thus \( \langle E(S_5) \cup E(P_3) \cup E(S_2) \rangle \cong \text{DCT}(5,3,2) \).

Now \( \{(n-6)(n-5); (n-4)(n-3), (n-4)(n-2), (n-4)(n-1), (n-4)n, (n-3)(n-2), (n-3)(n-1), (n-3)n, (n-2)(n-1), (n-2)n, (n-1)n\} \cong S_{10} \),

\( \{(n-6)(n-4); (n-5)(n-3), (n-5)(n-2), (n-5)n, (n-3)(n-2), (n-3)(n-1), (n-3)n, (n-2)(n-1), (n-2)n, (n-1)n\} \cong S_9 \) and

\( \langle\{(n-6)(n-4), (n-5)(n-1), (n-6)(n-3)\}\rangle \cong P_3 \).

Thus \( \langle E(S_9) \cup E(P_3) \rangle \cong CT(9,3) \).

Now \( \{(n-6)(n-2); (n-5)(n-3), (n-5)(n-1), (n-5)n, (n-4)(n-3), (n-4)(n-1), (n-4)n, (n-2)(n-1), (n-3)n, (n-1)n\} \cong S_9 \) and
\[ \langle (n - 6)(n - 2), (n - 5)(n - 4), (n - 6)(n - 3), (n - 5)(n - 2) \rangle \cong P_4. \]

Thus \( E(S_9) \cup E(P_3) \cong CT(9, 4) \).

Now \((n - 6)(n - 1); (n - 5)(n - 4), (n - 5)(n - 3), (n - 5)(n - 2), (n - 4)(n - 3), (n - 4)(n - 2), (n - 4)n, (n - 3)(n - 2), (n - 3)n, (n - 2)n \cong S_9, \)

\[ \langle (n - 6)(n - 1), (n - 5)n, (n - 6)(n - 3) \rangle \cong P_3 \text{ and } \]

\[ \langle (n - 6)(n - 3); (n - 2)(n - 1), (n - 4)(n - 1) \rangle \cong S_2. \]

Thus \( E(S_9) \cup E(P_3) \cup E(S_2) \cong DCT(9, 3, 2). \)

Now \((n - 6)n; (n - 5)(n - 4), (n - 5)(n - 3), (n - 5)(n - 2), (n - 5)(n - 1), (n - 4)(n - 3), (n - 4)(n - 2), (n - 3)(n - 2), (n - 3)(n - 1), (n - 2)(n - 1) \cong S_9, \)

\[ \langle (n - 6)n, (n - 4)(n - 2), (n - 6)(n - 3) \rangle \cong P_3 \text{ and } \]

\[ \langle (n - 6)(n - 3); (n - 4)n, (n - 2)n, (n - 1)n \rangle \cong S_2. \]

Thus \( E(S_9) \cup E(P_3) \cup E(S_2) \cong DCT(9, 3, 3). \)

Proceed like this,

Now \((12; 34, 35, \ldots, 3n, \ldots, (n - 2)(n - 1), (n - 2)n, (n - 1)n \cong S_{n-3}^{n-2}. \)

Also, \((13; 24, 25, \ldots, 2n, \ldots, (n - 2)(n - 1), (n - 2)n, (n - 1)n \cong S_{n-3}^{n-2} - 1, \text{ and } \)

\[ \langle (13), (26), (14) \rangle \cong P_3. \]

Thus \( E(P_3) \cup E\left(S_{n-3}^{n-2} - 1\right) \cong CT\left(\frac{(n-3)(n-2)}{2} - 1, 3\right). \)

\[ \vdots \]

\[ (1n; 23, 24, \ldots, 2(n-1), \ldots, (n - 2)(n - 1)) \cong S_{n-3}^{n-2} - 1, \]

\[ \langle (1n), (35), (14) \rangle \cong P_3 \text{ and } (14; 3n, 5n, \ldots, (n - 1)n) \cong S_{n-4}. \]

Thus we get

\[ \langle E(S_{n-3}^{n-2} - 1) \cup E(P_3) \cup E(S_{n-4}) \rangle \cong DCT\left(\frac{(n-3)(n-2)}{2} - 1, 3, n - 4\right). \]

Thus \( E[J(K_n)] = E[3DCT(2, 2, 2)] \cup E[S_{T_l}] \cup E[CT(T_l - 1, 3)] \cup E[CT(T_l - 1, 4)] \cup E[DCT(T_l - 1, 3, 2)] \cup E[DCT(T_l - 1, 3, 3)] \cup \ldots \cup E[DCT(T_l - 1, 3, n - 4)] \) where \( T_l = \frac{l(l+1)}{2}; \)

\( l = 3, 4, \ldots, n - 3. \)

\[ \square \]

**Theorem 3.4.** Jump graph of complete graph \( J(K_n) \) admits Continuous Monotonic tree decomposition \( \{H_1, H_2, \ldots, H_m\} \) if and only if there exists an integer \( m \) such that (i) \( m = \frac{n(n-3)}{2} \) and (ii) \( \frac{m(m+1)}{2} = \frac{n(n-1)(n-2)(n-3)}{8} \).
Proof. Let $|E[J(K_n)]| = \frac{n(n-1)(n-2)(n-3)}{8}$. Assume $J(K_n)$ admits Continuous Monotonic decomposition $\{H_1, H_2, \ldots, H_m\}$. By theorem 2.2, $|E[J(K_n)]| = \frac{m(m+1)}{2}$. Hence, $\frac{m(m+1)}{2} = \frac{n(n-1)(n-2)(n-3)}{8}$. Since $J(K_n)$ admits Continuous Monotonic decomposition $\{H_1, H_2, \ldots, H_m\}$, $\frac{n(n-1)(n-2)(n-3)}{8} = 1 + 2 + \ldots + m = \frac{m(m+1)}{2}$. This implies $\frac{m(m+1)}{2} = \frac{n(n-1)(n-2)(n-3)}{8}$. Hence (ii).

(ii) implies $m(m+1) = \frac{n(n-3)}{2} [\frac{n-3}{2} + 1]$. Thus $m = \frac{n(n-3)}{2}$. Hence (i).

Conversely, assume that $m = \frac{n(n-3)}{2}$ and $\frac{m(m+1)}{2} = \frac{n(n-1)(n-2)(n-3)}{8}$. By theorem 3.3, $J(K_n)$ is decomposed into $\{3DCT(2,2,2), S_T, CT(T_l - 1, t), DCT(T_l - 1, r, s)\}$ where $T_l = \frac{l(l+1)}{2}; l = 3, 4, \ldots, n - 3$.

Define $T_1 = \{E[3DCT(2,2,2)] \cup E[S_T] \cup E[CT(T_l - 1, 3)] \cup E[CT(T_l - 1, 4)] \cup E[DCT(T_l - 1, 3, 2)] \cup E[DCT(T_l - 1, 3, 3)] \cup \ldots \cup E[DCT(T_l - 1, 3, n - 5)]\}$ where $l = 3, 4, \ldots, n - 3$ and $T_2 = E[DCT(T_l - 1, 3, n - 4)]$ where $l = n - 3$.

Here $|T_1| = \frac{n(n-3)(n^2-3n-2)}{8}$ and $|T_2| = \frac{n(n-3)}{2}$. Also $|T_1| + |T_2| = \frac{n(n-3)(n^2-3n-2)}{8} + \frac{n(n-3)}{2} = \frac{m(m+1)}{2} = 1 + 2 + \ldots + m$.

By lemma 3.2, $\{1, 2, \ldots, m\} = B_1 \cup B_2$ where $\Sigma_{x \in B_1} x = \frac{n(n-3)(n^2-3n-2)}{8}$ and $\Sigma_{y \in B_2} y = \frac{n(n-3)}{2}$.

Decompose $T_1$ and $T_2$ into trees $H_i$. For $i \in B_1$ and $B_2$, we choose $H_i$ in such a way that $|E(H_i)| = i$.

$T_1 = \cup H_i; i \in B_1$ where $H_i \in \{3DCT(2,2,2), S_T, CT(T_l - 1, 3), CT(T_l - 1, 4), DCT(T_l - 1, 3, 2), DCT(T_l - 1, 3, 3), \ldots, DCT(T_l - 1, 3, n - 5)\}$.

$T_2 = \cup H_i; i \in B_2$ where $H_i = \{DCT(T_l - 1, 3, n - 4)\}$ and $|E(H_i)| = i$ for all $1 \leq i \leq m$.

Clearly $J(K_n)$ admits Continuous Monotonic tree decomposition $\{H_1, H_2, \ldots, H_m\}$. Hence the theorem.\qed

Illustration 3.5. As an illustration let us decompose $J(K_6)$.

Let $E(K_6) = \{12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56\}$.

Thus $V[J(K_6)] = \{12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56\}$.

$K_6$ and $J(K_6)$ are given in Figure 3.1 and Figure 3.2 respectively.
Figure 3.1. $K_6$

Figure 3.2. $J(K_6)$
Here $|E[J(K_6)]| = 45$.


Now $|T_1| + |T_2| = 45 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = \frac{n(n-1)(n-2)(n-3)}{8}$.

Here $B_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B_2 = \{9\}$.

$T_1$ is decomposed as $H_1 \cup H_2 \cup \ldots \cup H_8$ where $H_1 = \{(34)(25)\}$,

$H_2 = \{(35)(24), (35)(26)\}$, $H_3 = \{(35)(46), (46)(23), (46)(25)\}$,

$H_4 = \{(34)(26), (34)(56), (56)(23), (56)(24)\}$,

$H_5 = \{(36)(24), (36)(25), (36)(45), (45)(23), (45)(26)\}$,

$H_6 = \{(12)(34), (12)(35), (12)(36), (12)(45), (12)(46), (12)(56)\}$,

$H_7 = \{(13)(24), (13)(25), (13)(45), (13)(46), (13)(56), (13)(26), (14)(26)\}$,


Also $T_2$ is decomposed as


Clearly $\{H_1, H_2, \ldots, H_9\}$ forms a CMTD of $J(K_6)$.

**CONFLICT OF INTERESTS**

The authors declare that there is no conflict of interests.

**REFERENCES**


