ON CO-PRIME ORDER GRAPHS OF FINITE ABELIAN $p$-GROUPS

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Abstract. For a finite group $G$, the co-prime order graph $\Theta(G)$ of $G$ is defined as the graph with vertex set $G$, the group itself, and two distinct vertices $u, v$ in $\Theta(G)$ are adjacent if and only if $\gcd(o(u), o(v)) = 1$ or a prime number. In this paper, some properties and some topological indices such as Wiener, Hyper-Wiener, first and second Zagreb, Schultz, Gutman and eccentric connectivity indices of the co-prime order graph of finite abelian $p$-group are studied. We also figure out the metric dimension and resolving polynomial of the co-prime order graph of finite abelian $p$-group.

Keywords: resolving polynomial of a graph; co-prime order graph; finite abelian $p$-group; Wiener index; Zagreb indices; Schultz index.

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1. INTRODUCTION

An obvious phenomenon is to generate graphs from groups. The notion of ”co-prime order graph of a finite group” has been introduced by Subarsha Banerjee in 2019 [8]. They defined it as a simple graph with vertex set as the elements of a group, and there is adjacency between two

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vertices $u$ and $v$ if and only if $\gcd(o(u), o(v)) = 1$ or a prime number. For more studies about co-prime order graphs, we refer the reader to see [1, 10].

Suppose that $\Gamma$ is a simple graph, which is undirected and contains no multiple edges or loops. Here, the set of vertices of $\Gamma$ is denoted by $V(\Gamma)$ and the corresponding set of edges is denoted by $E(\Gamma)$. We write $uv \in E(\Gamma)$ if $u$ and $v$ form an edge in $\Gamma$. The size of the vertex-set is denoted by $|V(\Gamma)|$ for the set $\Gamma$ and the number of its edges is denoted by $|E(\Gamma)|$. The degree of a vertex is defined as number of vertices adjacent to $u$ and is represented as $\deg(u)$. The distance between any pair of vertices $u$ and $v$ denoted by $d(u, v)$, is the shortest $u - v$ path in graph $\Gamma$ and the eccentricity of any vertex $u$ is given as $\text{ecc}(u)$ and it is the largest distance between $u$ and any other vertex in $\Gamma$. The diameter of the graph $\Gamma$, denoted by $\text{diam}(\Gamma)$, is given by $\text{diam}(\Gamma) = \max\{\text{ecc}(u) : u \in V(\Gamma)\}$. A graph $\Gamma$ is called complete if every pair of vertices of $\Gamma$ are adjacent. If $D \subseteq V(\Gamma)$ and no vertices of $D$ are adjacent, then $D$ is called an independent set. The cardinality of the largest independent set is called an independent number of the graph $\Gamma$. A graph $\Gamma$ is called bipartite one if $V(\Gamma)$ can be partitioned in such a way into two disjoint independent sets that each edge in $\Gamma$ has its ends in different independent sets. A graph $\Gamma$ is called split if its vertex set can be splitted up into two different sets $U$ and $K$ such that $U$ is an independent set and the induced subgraph by $K$ is a complete graph.

Let $W = \{v_1, v_2, v_3, ..., v_k\} \subseteq V(\Gamma)$ and let $v$ be any vertex of $\Gamma$. The representation of $v$ with respect to $W$ is the $k$-vector $r(v|W) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. If different vertices have different representations with respect to $W$, then $W$ is called a resolving set of $\Gamma$. A basis of $\Gamma$ is a minimum resolving set for $\Gamma$ and the cardinality of a basis of $\Gamma$ is called metric dimension of $\Gamma$ and it is denoted by $\beta(\Gamma)$ [3]. Suppose $r_i$ is the number of resolving sets for $\Gamma$ of cardinality $i$. Then the resolving polynomial of a graph $\Gamma$ of order $n$, denoted by $\beta(\Gamma, x)$, is defined as $\beta(\Gamma, x) = \sum_{i=\beta(\Gamma)}^n r_i x^i$. The sequence $(r_{\beta(\Gamma)}, r_{\beta(\Gamma)+1}, ..., r_n)$ formed from the coefficients of $\beta(\Gamma, x)$ is called the resolving sequence.

For a graph $\Gamma$, the Wiener index and Hyper–Wiener index are defined by $W(\Gamma) = \sum_{\{u,v\} \subseteq V(\Gamma)} d(u,v)$ [5] and $WW(\Gamma) = \frac{1}{2}W(\Gamma) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(\Gamma)} d(u,v)^2$ [2]. The Zagreb indices mainly first and second are defined by $M_1(\Gamma) = \sum_{v \in V(\Gamma)} (\deg(v))^2$ and $M_2(\Gamma) = \sum_{uv \in E(\Gamma)} [\deg(u) \times \deg(v)]$ [6]. The Schultz index of $\Gamma$ is defined by $MTI(\Gamma) = \sum_{v \in V(\Gamma)} \deg(v)^3$. The hyper-Wiener index of $\Gamma$ is defined by $\text{HWW}(\Gamma) = \sum_{\{u,v\} \subseteq V(\Gamma)} d(u,v)^3$ [2].
\[ \sum_{(u,v) \in V(\Gamma)} d(u,v)[\deg(u) + \deg(v)] \] [4]. The Gutman index of \( \Gamma \) is defined by 
\[ \text{Gut}(\Gamma) = \sum_{(u,v) \in V(\Gamma)} d(u,v)[\deg(u) \times \deg(v)] \] [7]. The eccentric connectivity index of \( \Gamma \) is defined by 
\[ \xi^c(\Gamma) = \sum_{v \in V(\Gamma)} \deg(v) \text{ecc}(v) \] [9].

In [10], Xuanlong Ma and Zhonghua Wang studied \( \Theta(G) \) of all finite groups which are complete and classify all finite groups which are planar. In this paper, the focus will be on the co-prime order graph of a finite abelian \( p \)-group which is defined as 
\[ G_p = \{ \prod_{i=1}^{r} x_i | o(x_i) = p^{\alpha_i}, x_i \neq x_j \ and \ x_i \neq x_j \ where \ i, j = 1, 2, \ldots, r \} \cong Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}} \times \ldots \times Z_{p^{\alpha_r}} \] where \( p \) is a prime number.

When \( \alpha_i = 0 \) for all \( i \), then co-prime order graph is null graph, which is not of our interest.

Throughout sections 2,3 and 4, we use notations and assumptions given below 
\( p \) is a prime number, \( r \geq 1 \), \( \alpha_i \geq 1 \) for all \( i = 1, 2, \ldots, r \), \( G_p = Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}} \times \ldots \times Z_{p^{\alpha_r}} \), \( \Omega_1 = \{ x \in G_p | o(x) = 1 or p \} \) and \( \Omega_2 = G - \Omega_1 \).

This paper is organized as follows. In Section 2, some basic properties of the graph \( \Theta(G_p) \) are investigated. We see that the graph \( \Theta(G_p) \) is split. In Section 3, we find some topological indices of the graph \( \Theta(G_p) \) such as the Wiener, Hyper-Wiener and Zagreb indices. In Section 4, we find the metric dimension and the resolving polynomial of the graph \( \Theta(G_p) \).

2. SOME PROPERTIES OF THE GRAPH \( \Theta(G_p) \)

**Lemma 1.** Let \( G_p \) be a finite abelian \( p \)-group. Then \( |\Omega_1| = p^r \) and \( |\Omega_2| = p^{\sum_{i=1}^{r} \alpha_i} - p^r \).

**Proof.** Firstly, we count elements of order 1 and \( p \) in the group \( G_p \).

Let \( x \) be arbitrary element of order 1 or \( p \). Then \( x = x_1 x_2 \cdots x_r \) such that \( x_i \in Z_{p^{\alpha_i}} \) and \( o(x_i) = 1 \) or \( p \) where \( i = 1, 2, \ldots, r \).

Possibilities for each \( x_i \) are \( p \), hence possibilities for \( x \) are \( p^r \). Remaining \( p^{\sum_{i=1}^{r} \alpha_i} - p^r \) has neither order 1 nor \( p \). Hence, we get desired result.

**Lemma 2.** A sub-graph induced by \( \Omega_1 \) from the graph \( \Theta(G_p) \) is \( K_{p^r} \) and a sub-graph induced by \( \Omega_2 \) from the graph \( \Theta(G_p) \) has no edges. Furthermore, each vertex of the set \( \Omega_1 \) must be adjacent with every vertex in \( \Omega_2 \).

**Proof.** Using the definition of co-prime order graph, every pair of vertices in \( \Omega_1 \) must be adjacent because of the order of every vertex is either 1 or \( p \). Also, each vertex of the set \( \Omega_1 \) must be
adjacent with every vertex in $\Omega_2$. If $\Omega_2 \neq \phi$, then no pair of vertices in $\Omega_2$ are adjacent because of the order of each vertex is a power of $p$ other than 1 or $p$.

Using above lemmas, we can state the following results

**Theorem 1.** Let $\Theta(G_p)$ be the co-prime order graph on $G$. Then $\Theta(G_p) = K_{p^r} \lor (p^{\sum_{i=1}^r a_i - p^r})K_1$.

**Theorem 2.** In the graph $\Theta(G_p)$, $\deg(u) = \begin{cases} |\Omega_1| + |\Omega_2| - 1, & \text{if } u \in \Omega_1 \\ |\Omega_1|, & \text{if } u \in \Omega_2 \end{cases}$

**Corollary 1.** In the graph $\Gamma = \Theta(G_p)$, $|E(\Gamma)| = \binom{|\Omega_1|}{2} + |\Omega_1||\Omega_2|$

*Proof.* Every pair of vertices $u, v \in \Omega_1$ are adjacent, and the total number of edges in $\Omega_1$ is $\binom{|\Omega_1|}{2}$. Also, there is an edge between any vertex $u \in \Omega_1$ and every vertex $v \in \Omega_2$, and the total number of such edges is $|\Omega_1||\Omega_2|$. No pairs of vertices $u, v \in \Omega_2$ are adjacent. In total, we get $|E(\Gamma)| = \binom{|\Omega_1|}{2} + |\Omega_1||\Omega_2|$. □

**Theorem 3.** In the graph $\Theta(G_p)$, $\text{ecc}(u) = \begin{cases} 1, & \text{if } u \in \Omega_1, \\ 2, & \text{if } u \in \Omega_2 \end{cases}$

*Proof.* If $u \in \Omega_1$, then $u$ is directly connected with every vertex of the graph because of the order of $u$ is either 1 or $p$. So, the maximum distance between $u$ and any vertex of the graph is 1. If $u \in \Omega_2$, then the order of $u$ is $p^k$, where $k \geq 2$. So, there exists at least $\phi(p^k)$ vertices of order the same as $u$. By the use of the concept $k \geq 2$, we know that $\phi(p^k) \geq 2$. Take $v$ be another element different from $u$ whose order is neither 1 nor $p$, then the maximum distance between $u$ and $v$ is 2. □

**Corollary 2.** The diameter of the graph $\Gamma = \Theta(G_p)$ is 2, that is $\text{diam}(\Gamma) = 2$.

**Corollary 3.** In the graph $\Gamma = \Theta(G_p)$, $d(u, v) = \begin{cases} 2, & \text{if } u, v \in \Omega_2 \\ 1, & \text{otherwise} \end{cases}$

**Corollary 4.** In the graph $\Gamma = \Theta(G_p)$, the independent set $S$ of $\Gamma$ is either a singleton set or a set with the property that no vertex has order 1 or $p$.
Proof. If \( S \) is a singleton set, then it is an independent set.

If \( S \) contains more than one element with at least one element of order 1 or \( p \), say \( x \), then using the definition of the co-prime order graph, \( x \) must be adjacent with every vertex of \( S \) except \( x \). So, \( S \) is not an independent set.

We conclude that if \( S \) is an independent set with more than one element, then \( S \) does not contains any element of order 1 or \( p \). \( \square \)

**Corollary 5.** In the graph \( \Gamma = \Theta(G_p) \), the largest independent set is \( \Omega_2 \) or \( \{e\} \) according as group \( G_p \) has an element of order \( p^2 \) or not, respectively.

**Proof.** If every element of the group \( G_p \) has order 1 or \( p \), then by the above corollary, the set \( \{e\} \) is largest independent set.

If the group \( G_p \) has at least one element of order \( p^2 \), then by the above corollary, the set \( S = \{x \in G_p|o(x) \neq 1, o(x) \neq p\} = \Omega_2 \) is an independent set. The set \( S \) is the largest independent set because the remaining elements are either of order 1 or \( p \), which cannot belong to the independent set. \( \square \)

**Corollary 6.** If \( H = \{x \in G_p|o(x) = 1 \text{ or } p\} \cup L \), where \( L = \emptyset \) or \( L = \{y\} \) such that \( o(y) \neq 1 \text{ or } p \) then \( \Theta(H) \) is the maximal clique of the graph \( \Theta(G_p) \).

**Proof.** By Corollary 2, \( d(u,v) = 1 \) if one of \( u \) or \( v \) has order 1 or \( p \), then \( \Theta(H) \) is a clique. Now we show that \( \Theta(H) \) is a maximal clique.

Using the definition of the co-prime order graph, two vertices with orders neither 1 nor \( p \) can be adjacent. Hence, they cannot a part of a clique. So, clique can have at most one vertex whose order is neither 1 nor \( p \).

Take \( L \) contains singleton element if at-least one of \( \alpha_i \geq 2 \) otherwise \( \emptyset \) and set \( \{x \in G_p|o(x) = 1 \text{ or } p\} \) contains all elements of order 1 or \( p \). Therefore, \( H \) is maximal clique. \( \square \)

**Corollary 7.** The graph \( \Gamma = \Theta(G_p) \) is a complete split graph.

**Proof.** There are two cases to consider.

Case 1:- If \( \alpha_i \geq 2 \) for some \( i \).

The vertex set of the graph \( \Gamma \) can be partitioned as \( K = \{x \in G_p|o(x) = 1 \text{ or } p\} \) and \( L = \)
$G_p - K$. Here $K$ is non-empty because $K$ must contains the identity element of $G_p$ and the largest independent set which is determined in Corollary 4. Hence $K$ must be an independent set.

It is given that at least one of the $\alpha_i \geq 2$, so the group $G_p$ contains at least two elements of order not equal to 1 or $p$. So $L$ is non-empty. Also, $L$ contains a maximal clique of the co-prime order graph of the group $G_p$ which is determined in Corollary 6, hence the subgraph induced by $L$ must be a clique. Also, each vertex of $K$ must be adjacent to every vertex of $L$, so the graph $\Gamma$ is a complete split graph.

Case 2:- If $\alpha_i = 1$ for all $i$.

In this case the group $G_p$ is a complete graph with more than one vertex. Take $K = \{e\}$ and $L = G_p - K$. Then $K$ is independent because $K$ is a singleton set which is not adjacent with any vertex of $K$, and $L$ is a clique because it induces a subgraph of a complete graph. □

3. Some Topological Indices of the Graph $\Theta(G_p)$

**Theorem 4.** Let $\Gamma = \Theta(G_p)$ be a co-prime order graph of the group $G_p$. Then $W(\Gamma) = \left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|+2\left(\frac{|\Omega_2|}{2}\right)$

*Proof.* Let $u, v \in \Gamma$. It follows from the Corollary 3 that the number of possibilities of $d(u,v) = 1$ is $\left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|$ and number of possibilities of $d(u,v) = 2$ is $\left(\frac{|\Omega_2|}{2}\right)$. Thus, $W(\Gamma) = \left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|+2\left(\frac{|\Omega_2|}{2}\right)$ □

**Theorem 5.** Let $\Gamma = \Theta(G_p)$ be a co-prime order graph of the group $G_p$. Then $WW(\Gamma) = \left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|+3\left(\frac{|\Omega_2|}{2}\right)$

*Proof.* From Theorem 4 and Corollary 3, we can see that $WW(\Gamma) = \frac{1}{2}\left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|+2\left(\frac{|\Omega_2|}{2}\right) + \frac{1}{2}\left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|+4\left(\frac{|\Omega_2|}{2}\right) = \left(\frac{|\Omega_1|}{2}\right) + |\Omega_1||\Omega_2|+3\left(\frac{|\Omega_2|}{2}\right)$ □

In the next two theorems, the first and second Zagreb indices for the co-prime order graph of the group $G_p$ are presented.

**Theorem 6.** Let $\Theta(G_p)$ be a co-prime order graph of the group $G_p$. Then $M_1(\Theta(G_p)) = |\Omega_1|(|\Omega_1|+|\Omega_2|-1)^2 + (|\Omega_1|)^2|\Omega_2|$
Proof. It follows from Theorem 2 that $M_1(\Theta(G_p)) = |\Omega_1|(|\Omega_1| + |\Omega_2|) + (|\Omega_1|)2|\Omega_2|$.

\textbf{Theorem 7.} Let $\Theta(G_p)$ be a co-prime order graph of the group $G_p$. Then $M_2(\Theta(G)) = (\Omega_2||\Omega_1| + |\Omega_2| + 1)^2 + (|\Omega_1|)2|\Omega_2|(|\Omega_1| + |\Omega_2| - 1)$.

Proof. From Theorem 2 and Corollary 1, we have $(\Omega_2||\Omega_1| + |\Omega_2| - 1)$ edges with end vertices of degree $|\Omega_1| + |\Omega_2| - 1$, and $|\Omega_1||\Omega_2|$ edges with one end vertex of degree $|\Omega_1|$ and the other end vertex of degree $|\Omega_1| + |\Omega_2| - 1$. Hence we get desired result.

\textbf{Theorem 8.} Let $\Theta(G_p)$ be a co-prime order graph of the group $G_p$. Then $MTI(\Theta(G_p)) = 2(\Omega_2||\Omega_1| + |\Omega_2| - 1) + |\Omega_1||\Omega_2|(2|\Omega_1| + |\Omega_2| - 1) + 4|\Omega_1|(\Omega_2||\Omega_1|)$.

Proof. There are three possibilities for $u, v \in G_p$.

Case 1:- $u, v \in \Omega_1$.

It follows from Theorem 2 and Corollary 1 that $d(u, v) = 1$ and $deg(u) + deg(v) = 2(|\Omega_1| + |\Omega_2| - 1)$. There are $(\Omega_2||\Omega_1|)$ possibilities for this case.

Case 2:- $u \in \Omega_1$ and $v \in \Omega_2$.

It follows from Theorem 2 and Corollary 1 that $d(u, v) = 1$ and $deg(u) + deg(v) = (2|\Omega_1| + |\Omega_2| - 1)$. There are $|\Omega_1||\Omega_2|$ possibilities for this case.

Case 3:- $u, v \in \Omega_2$.

It follows from Theorem 2 and Corollary 1 that $d(u, v) = 2$ and $deg(u) + deg(v) = 2|\Omega_1|$. There are $(\Omega_2||\Omega_1|)$ possibilities for this case.

Hence, we get $MTI(\Theta(G_p)) = 2(\Omega_2||\Omega_1| + |\Omega_2| - 1) + |\Omega_1||\Omega_2|(2|\Omega_1| + |\Omega_2| - 1) + 4|\Omega_1|(\Omega_2||\Omega_1|)$.

\textbf{Theorem 9.} Let $\Theta(G_p)$ be a co-prime order graph of the group $G_p$. Then $Gut(\Theta(G_p)) = (\Omega_2||\Omega_1| + |\Omega_2| - 1)^2 + |\Omega_1||\Omega_2|(2|\Omega_1| + |\Omega_2| - 1) + 2|\Omega_1|^2(\Omega_2||\Omega_1|)$.

Proof. There are three possibilities for $u, v \in G_p$.

Case 1:- $u, v \in \Omega_1$.

It follows from Theorem 2 and Corollary 1 that $d(u, v) = 1$ and $deg(u)deg(v) = (|\Omega_1| + |\Omega_2| - 1)^2$. There are $(\Omega_2||\Omega_1|)$ possibilities for this case.

Case 2:- $u \in \Omega_1$ and $v \in \Omega_2$. 
It follows from Theorem 2 and Corollary 1 that \( d(u, v) = 1 \) and \( \deg(u)\deg(v) = |\Omega_1|(|\Omega_1| + |\Omega_2| - 1) \). There are \(|\Omega_1||\Omega_2|\) possibilities for this case.

Case 3: \( u, v \in \Omega_2 \).

It follows from Theorem 2 and Corollary 1 that \( d(u, v) = 2 \) and \( \deg(u)\deg(v) = |\Omega_1|^2 \). There are \( \binom{|\Omega_2|}{2} \) possibilities for this case.

Hence, we get \( \text{Gut}(\Theta(G_\rho)) = \binom{|\Omega_1|}{2}(|\Omega_1| + |\Omega_2| - 1)^2 + |\Omega_1|^2|\Omega_2|(|\Omega_1| + |\Omega_2| - 1) + 2|\Omega_1|^2\binom{|\Omega_2|}{2} \). \( \square \)

**Theorem 10.** Let \( \Theta(G_\rho) \) be a co-prime order graph of the group \( G_\rho \). Then \( \xi^c(\Theta(G_\rho)) = (|\Omega_1|^2 + 3|\Omega_1||\Omega_2| - |\Omega_1|) \).

**Proof.** There are two possibilities for \( u \in G_\rho \).

Case 1: \( u \in \Omega_1 \).

It follows from Theorem 2 and Theorem 3 that \( \deg(u) = (|\Omega_1| + |\Omega_2| - 1) \) and \( \text{ecc}(u) = 1 \). There are \(|\Omega_1|\) possibilities for this case.

Case 2: \( u \in \Omega_2 \).

It follows from Theorem 2 and Theorem 3 that \( \deg(u) = |\Omega_1| \) and \( \text{ecc}(u) = 2 \). There are \(|\Omega_2|\) possibilities for this case.

Hence, we get \( \xi^c(\Theta(G_\rho)) = (|\Omega_1|^2 + 3|\Omega_1||\Omega_2| - |\Omega_1|) \). \( \square \)

**4. Metric Dimension and Resolving Polynomial of the Graph \( \Theta(G_\rho) \)**

For a graph \( \Gamma \), we define the open neighborhood of a vertex \( u \in \Gamma \), \( N(u) \), by \( N(u) = \{v \in V(\Gamma) : uv \in E(\Gamma)\} \) and the closed neighborhood of \( u \), \( N[u] \), by \( N[u] = N(u) \cup \{u\} \). Two distinct vertices \( u \) and \( v \) of \( \Gamma \) are said to be twins if \( N(u) = N(v) \) or \( N[u] = N[v] \). A subset \( S \) of \( V(\Gamma) \) is said to be a twin set in \( \Gamma \) if every pair of vertices in \( S \) are twins.

**Lemma 3.** If \( \Gamma \) is a connected graph of order \( n \) and \( S \subseteq V(\Gamma) \) is a twin in \( \Gamma \) set of size \( l \geq 2 \), then every resolving set of \( \Gamma \) contains at least \( l - 1 \) vertices of \( S \).

**Corollary 8.** If \( S \) is a resolving set for a connected graph \( \Gamma \) and \( u \) and \( v \) are twin vertices in \( \Gamma \), then \( u \in S \) or \( v \in S \). Furthermore, if \( u \in S \) and \( v \notin S \), then \( (S \setminus \{u\}) \cup \{v\} \) is also a resolving set for \( \Gamma \).
Theorem 11. Let \( \Gamma = \Theta(G_p) \) be a co-prime order graph of group \( G_p \). Then \( \beta(\Gamma) = p^{\sum_{i=1}^{\alpha_i}} - 2 \).

Proof. We see that the sets \( \Omega_1 \) and \( \Omega_2 \) are twin sets in \( \Gamma \) of cardinality \( p^r \) and \( p^{\sum_{i=1}^{\alpha_i}} - 2 \), respectively. From Lemma 3, we have \( \beta(\Gamma) \geq p^{\sum_{i=1}^{\alpha_i}} - 2 \). In addition, we see that the set \( W := \Omega_1 \cup \Omega_2 - \{e, x\} \), where \( x \in G_p \) with \( o(x) \) is neither 1 nor \( p \), is a resolving set for \( \Gamma \) of cardinality \( p^{\sum_{i=1}^{\alpha_i}} - 2 \), this implies that \( \beta(\Gamma) \leq p^{\sum_{i=1}^{\alpha_i}} - 2 \). \( \square \)

Lemma 4. For a connected graph \( \Gamma \) of order \( n \), \( r_n = 1 \) and \( r_{n-1} = n \).

Theorem 12. Let \( \Gamma = \Theta(G_p) \) be a co-prime order graph of group \( G_p \). Then \( \beta(\Gamma, x) = x^{p^{\sum_{i=1}^{\alpha_i}} - 2} \left( |\Omega_1| + |\Omega_2| \right) x + x^2 \).

Proof. From Theorem 11, we have that \( \beta(\Gamma) = |\Omega_1| + |\Omega_2| - 2 \). We need to find the resolving sequence \( (r_{|\Omega_1| + |\Omega_2| - 2}, r_{|\Omega_1| + |\Omega_2| - 1}, r_{|\Omega_1| + |\Omega_2|}) \) of length three.

For \( r_{|\Omega_1| + |\Omega_2| - 2} \): By the principal of multiplication and Corollary 8, we see that

\[
r_{|\Omega_1| + |\Omega_2| - 2} = \binom{|\Omega_1|}{|\Omega_1| - 1} \binom{|\Omega_2|}{|\Omega_2| - 1} = |\Omega_1| + |\Omega_2|.
\]

By Lemma 4, we have \( r_{|\Omega_1| + |\Omega_2| - 1} = |\Omega_1| + |\Omega_2| \) and \( r_{|\Omega_1| + |\Omega_2|} = 1 \). \( \square \)

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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