D-HOMOTHETIC DEFORMATION OF $(\kappa, \mu)$ MANIFOLD

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Abstract. In this paper, we study the invariance of certain curvature conditions in $(\kappa, \mu)$-contact metric manifold under $D$-homothetic deformation. Finally we give an example to verify the results.

Keywords: $(\kappa, \mu)$-manifold; $D$-homothetic deformation; extended Ricci-recurrent; $\eta$-parallel.

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1. INTRODUCTION

The class of $(\kappa, \mu)$-contact metric manifolds encaoses both Sasakian and non-Sasakian structures. This class of manifolds are invariant under $D$-homothetic transformation. It is noted that the class of spaces acquired through $D$-homothetic deformation [13] is a contact metric manifold whose curvature satisfies $R(X, Y)\xi = 0$. In [13], [14], the authors used $D$-homothetic deformation on Sasakian and $K$-contact structures to get results on the first Betti number, second Betti number and harmonic forms. A plane section in the tangent space $T_p(M)$ is called a $\phi$-section if there exist a unit vector $X$ in $T_p(M)$ orthogonal to $\xi$ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature $K(X, \phi X) = g(R(X, \phi X)X, \phi X)$ is called a $\phi$-sectional curvature. A contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be of constant $\phi$-sectional curvature if at any point $p \in M$, the sectional curvature $K(X, \phi X)$ is independent of

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choice of non-zero \( X \in \mathcal{D}_p \), where \( \mathcal{D} \) denotes the contact distribution of the contact metric manifold defined by \( \eta = 0 \).

The Riemannian curvature tensor \( R \) of Sasakian manifold of constant \( \phi \)-sectional curvature is determined by Ogiue [9]. The geometry of contact Riemannian manifolds of constant \( \phi \)-sectional curvature is obtained by Tanno [15]. If the \( \phi \)-sectional curvature \( H \) is constant on \( K \)-contact Riemannian manifold \( M(\phi, \xi, \eta, g) \) then \( H \) can be deformed by a \( D \)-homothetic deformation of the structure tensors [16]. An extensive research about \( D \)-homothetic deformation on contact geometry is carried out in recent years. The \( D \)-homothetic deformation is related to the following tensor structures. In other words, it means that the changing of the tensor form

\[
\eta' = a \eta, \quad \xi' = \left( \frac{1}{a} \right) \xi, \quad \phi' = \phi, \quad g' = ag + (a - 1) \eta \otimes \eta,
\]

where \( a \) is a positive constant. In particular, some authors (Carraio et al [3]), (De et al [4]) studied \( D \)-homothetic deformations of certain structures . An almost contact metric manafold is said to be \( \eta \)-Einstein if its Ricci tensor \( S \) is of the form

\[
S = \alpha g + \beta \eta \otimes \eta,
\]

where \( \alpha \) and \( \beta \) are smooth functions on the manifold.

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to different structures. As a weaker version of local symmetry Takahashi [12] introduced the notion of a local \( \phi \)-symmetry on a Sasakian manifold. Generalizing the notion of a local \( \phi \)-symmetry of Takahashi [12]. De et al. [6] introduced the idea of \( \phi \)-recurrent Sasakian manifolds. The notion of a generalized recurrent manifold has been introduced by Dubey [7] and studied by others. Again, the notion of a generalized Ricci recurrent manifold has been introduced and studied by De et. al. [5]. The properties of the extended generalized \( \phi \)-recurrent \( \beta \)-Kenmotsu, Sasakian and \((LCS)_{2n+1}\)-manifolds have been studied in [11], [10] and [18] respectively. Motivated by the above studies, in this paper we characterize the \((\kappa, \mu)\)-contact metric manifolds under \( D \)-homothetic deformation. We study the invariance properties of extended generalized \( \phi \)-recurrent, locally \( \phi \)-Ricci symmetric \((\kappa, \mu)\) manifolds under \( D \)-homothetic deformation. Also \( \eta \)-parallel Ricci tensor is considered in \((\kappa, \mu)\)-contact metric manifolds. Finally, we give an example of such manifold.
2. Preliminaries

Let $M$ be $(2n + 1)$-dimensional almost contact metric manifold. Then it carries two fields $\phi$ and $\xi$ and a 1-form $\eta$. The field $\phi$ represents the endomorphism of the tangent spaces, the field $\xi$ is called characteristic vector field and $\eta$ is a 1-form satisfying

\begin{equation}
\phi^2 = -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X),
\end{equation}

\begin{equation}
\eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0,
\end{equation}

\begin{equation}
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\end{equation}

\begin{equation}
g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \phi Y) = d\eta(X, Y),
\end{equation}

for any vector fields $X, Y \in \chi(M)$. In a contact metric manifold, we characterize a $(1, 1)$ tensor field $h$ by $h = \frac{1}{2} L_{\xi} \phi$, where $L$ denotes the Lie differentiation. At this point $h$ is symmetric and satisfies $h\phi = -\phi h$. Also we have $Tr h = Tr \phi h = 0$ and $h\xi = 0$. The $(\kappa, \mu)$-nullity distribution of a Riemannian manifold $(M, g)$ is a distribution

\begin{equation}
N(\kappa, \mu) : p \mapsto N_p(\kappa, \mu) = \{Z \in \chi_p(M) : R(X, Y)Z = \kappa[g(Y, Z)X - g(X, Z)Y]
+ \mu[g(Y, Z)hX - g(X, Z)hY]\}
\end{equation}

for any $X, Y, Z \in \chi_p(M)$ and $\kappa$ and $\mu$ being constants, where $R$ denotes the Riemannian curvature tensor and $\chi_p(M)$ denotes the tangent vector space of $M$ at any point $p \in M$. If the characteristic vector field of a contact metric manifold belongs to the $(\kappa, \mu)$ nullity distribution, then the relation

\begin{equation}
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)
\end{equation}

holds. A contact metric manifold with $\xi \in N(\kappa, \mu)$ is called a $(\kappa, \mu)$-contact metric manifold [1]. In a $(\kappa, \mu)$-contact metric manifold $M$ the following relations hold [1], [2]:

\begin{equation}
h^2 = (\kappa - 1)\phi^2,
\end{equation}

\begin{equation}
\nabla_X \xi = -\phi X - \phi hX,
\end{equation}
(2.9) \( (\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \)

(2.10) \( (\nabla_X \eta)Y = g(X + hX, \phi Y), \)

(2.11) \( R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX), \)

(2.12) \( S(X, Y) = [2(n - 1) - n\mu]g(X, Y) + [2(n - 1) + \mu]g(hX, Y) + [2(1 - n) + n(2\kappa + \mu)]\eta(X)\eta(Y), n \geq 1 \)

(2.13) \( S(X, \xi) = 2n\kappa\eta(X), \)

(2.14) \( r = 2n[2n - 2 + \kappa - n\mu], \)

(2.15) \( (\nabla_X h)(Y) = [(1 - \kappa)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)[(1 - \kappa)\phi X + \phi hX] - \mu \eta(X)\phi hY, \)

where \( S \) and \( r \) are the Ricci tensor and scalar curvature respectively and \( Q \) is the Ricci operator, i.e., \( g(QX, Y) = S(X, Y). \)

3. THE \( D \)-HOMOTHETIC DEFORMATION IN \((\kappa, \mu)\) CONTACT METRIC MANIFOLD

Let \((M, \phi, \xi, \eta, g)\) be \((2n + 1)\) dimensional \((\kappa, \mu)\)-contact metric manifold and \((\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) be obtained from \((M, \phi, \xi, \eta, g)\) by homothetic deformation (1.1). Throught the paper the quantity with bar denote quantities in \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) and the quantity without bar are for \((M, \phi, \xi, \eta, g)\). The relation between \( \bar{R} \) and \( R \) of \((M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) follows: [8].

\[
\bar{R}(X, Y)Z = R(X, Y)Z + (1 - a)[g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y + 2\eta(X)\eta(Z)hY
- 2\eta(Y)\eta(Z)hX + 2g(\phi Y, X)\phi Z + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi]
+ \frac{(1 - a)}{a} [2\eta(Y)g(hX, Z)\xi - 2\eta(X)g(hY, Z)\xi + (1 - \kappa)\eta(Y)g(X, Z)\xi
- \eta(X)g(Y, Z)\xi + g(\phi hX, Z)\phi Y - g(\phi hY, Z)\phi hX] + (a^2 - 1)[\eta(Y)\eta(Z)X
- \eta(X)\eta(Z)Y],
\]
for any vector fields $X, Y, Z$ on $M$.

Using (3.1), we derive

$$
\bar{S}(Y, Z) = aS(Y, Z) + (a - 1)[(a^2 - 2a - \kappa + 1)g(Y, Z) + (2na^2 + 2na + 2a - a^2 + \kappa - 1)
\eta(Y)\eta(Z) + a(2 + \mu)g(hY, Z)].
$$

(3.2)

**Theorem 3.1.** Under a D-homothetic deformation the expression $Q\phi - \phi Q$ of a $(2n + 1)$-dimensional $(\kappa, \mu)$-contact metric manifold is invariant, provided $\mu = -2$.

**Proof:** From (3.1) we have

(3.3)

$$
\bar{Q}X = QX + \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)X + (2na^2 + 2na + 2a - a^2 + \kappa - 1)\eta(X)\xi + a(2 + \mu)hX].
$$

Operating $\bar{\phi} = \phi$ on both sides of above equation from the left, we have,

(3.4) $$
\bar{\phi}\bar{Q}X = \phi QX + \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)\phi X + a(2 + \mu)\phi hX].
$$

Again, putting $\bar{\phi}X = \phi X$ in (3.2) we have

(3.5) $$
\bar{Q}\bar{\phi}X = Q\phi X + \frac{a-1}{a}[(a^2 - 2a - \kappa + 1)\phi X + a(2 + \mu)\phi hX].
$$

From (3.3) and (3.5) we get

(3.6) $$
(\bar{\phi}\bar{Q} - \bar{Q}\bar{\phi})X = (\phi Q - Q\phi)X + 2a(a - 1)\{2 + \mu\}\phi hX.
$$

Hence the proof.

**Lemma 3.1.** In a $(2n + 1)$-dimensional $\eta$-Einstein $(\kappa, \mu)$ manifold $M(\phi, \xi, \eta, g)$, the Ricci tensor is expressed as

(3.7) $$
S(X, Y) = (\frac{r}{2n} - \kappa)g(X, Y) - (\frac{r}{2n} - 2n\kappa - \kappa)\eta(X)\eta(Y).
$$

**Proof:** On contracting (1.2) we have

(3.8) $$
r = (2n + 1)\alpha + \beta,
$$

where $r$ is the scalar curvature of the manifold. Again putting $X = \xi$ in (2.13) we obtain,

(3.9) $$
\alpha + \beta = 2n\kappa.
$$
Solving (3.8) and (3.9) we obtain values for \( \alpha = \frac{r}{2n} - \kappa \) and \( \beta = -\frac{r}{2n} + (2n + 1)\kappa \). Putting the values of \( \alpha \) and \( \beta \) in (1.2), we get (3.7).

**Theorem 3.2.** Under D-homothetic deformation, a \((2n + 1)\)-dimensional \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold transforms to \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold provided \(\mu = -2\).

**Proof:** Let \(M(\phi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold which becomes \(M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) under a D-homothetic deformation. Then from (3.1) it follows by virtue of (3.7) that

\[
\bar{S}(X, Y) = \bar{A}\bar{g}(X, Y) = \bar{B}\bar{\eta}(X)\bar{\eta}(Y) + \left(\frac{2 + \mu}{a}\right)\bar{g}(hX, Y),
\]

where \(\bar{A}\) and \(\bar{B}\) are smooth functions given by

\[
\bar{A} = \frac{1}{a}\left(\frac{r}{2n} - \kappa + \left(\frac{a - 1}{a}\right)(a^2 - 2a - \kappa - 1)\right)
\]

and

\[
\bar{B} = -\left(\frac{a - 1}{a}\right)\left(\frac{r}{2n} - \kappa + \left(\frac{a - 1}{a}\right)(a^2 - 2a - \kappa + 1)\right) - \frac{1}{a^2}\left(\frac{r}{2n} - 2n\kappa - \kappa - \left(\frac{a - 1}{a}\right)\right) [2na^2 + 2na + 2a - a^2 + \kappa - 1])
\]

The Proof follows by (3.10).

**Theorem 3.3.** Under D-homothetic deformation, the \(\phi\)-sectional curvature of a \((2n + 1)\)-dimensional \((\kappa, \mu)\)-contact metric manifold is invariant, provided \(\kappa = (1 - 3a)\).

**Proof:** Here we consider the \(\phi\)-sectional curvature on a \((2n + 1)\)-dimensional \((\kappa, \mu)\)-contact metric manifold. From (3.1) it can be easily seen that

\[
\bar{K}(X, \phi X) - K(X, \phi X) = -(1 - a)(3a + \kappa - 1).
\]

Hence we have the proof of the theorem.
4. Extended Generalized $\phi$-Reccurrent, Locally $\phi$-Ricci Symmetry and $\eta$-Parallel $(\kappa, \mu)$-Manifold

Firstly, we study the properties of the extended generalized $\phi$-reccurrent $(\kappa, \mu)$-manifolds under $D$-homothetic deformation.

**Definition 4.1.** A $(\kappa, \mu)$-manifold $M(\phi, \xi, \eta, g)$, is said to be an extended generalized $\phi$-reccurrent manifold under $D$-homothetic deformation if its curvature tensor $\bar{\bar{R}}$ satisfies

$$\phi^2((\nabla_W \bar{\bar{R}})(X,Y)Z = A(W)\phi^2(\bar{\bar{R}}(X,Y)Z) + B(W)\phi^2(G(X,Y)Z),$$

for $X, Y, Z, W \in \chi(M)$, where $A$ and $B$ are non-vanishing 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and $G$ is a tensor field of type $(1, 3)$ defined as

$$G(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

The 1-forms $A$ and $B$ are called the associated 1-forms of the manifold.

**Definition 4.2.** A $(\kappa, \mu)$ manifold $M(\phi, \xi, \eta, g)$ is said to be generalized Ricci-recurrent manifold under $D$-homothetic deformation if its non-vanishing Ricci tensor $\bar{\bar{S}}$ satisfies the relation

$$\nabla_W \bar{\bar{S}}(Y,Z) = A(W)\bar{\bar{S}}(Y,Z) + B(W)g(Y,Z),$$

for all vector fields $W, X, Y \in \chi(M)$.

**Theorem 4.1.** If an extended generalized $\phi$-reccurrent $(\kappa, \mu)$-manifold $M$ under $D$-homothetic deformation is a generalized Ricci-recurrent manifold, then the 1-forms $A$ and $B$ are related as

$$2n(1 - a^2 - \kappa a)A(W) + (4n^2 - 2n - 1)B(W) = 0.$$

**Proof:** Let us suppose that the manifold $M(\phi, \xi, \eta, g)$, is an extended generalized $\phi$-reccurrent $(\kappa, \mu)$-manifold under $D$-homothetic deformation. Then from (2.1), (2.2), (2.3) and (4.1), we have

$$-(\nabla_W \bar{\bar{R}})(X,Y)Z + \eta((\nabla_W \bar{\bar{R}})(X,Y)Z)\xi = A(W)[-\bar{\bar{R}}(X,Y)Z + \eta(\bar{\bar{R}}(X,Y)Z)\xi] + B(W)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi],$$

(4.3)
from which it follows that
\[-g((\nabla_W \tilde{R})(X,Y)Z,U) + \eta((\nabla_W \tilde{R})(X,Y)Z) \eta(U) = A(W)[-g(\tilde{R}(X,Y)Z,U)\]
(4.4) 
\[+ \eta(\tilde{R}(X,Y)Z) \eta(U)] + B(W)[-g(G(X,Y)Z,U)\]
\[+ \eta(G(X,Y)Z) \eta(U)].\]

Let \(\{e_i, i = 1, 2, \ldots, 2n + 1\}\) be an orthonormal basis of the tangent space at any point of the manifold. Replacing \(X = U = e_i\) in (4.4) and taking summation over \(i, 1 \leq i \leq 2n + 1\), we have
\[(\nabla_W \tilde{S})(Y,Z) - g((\nabla_W \tilde{R})(\tilde{\xi},Y)Z, \tilde{\xi}) = A(W)[\tilde{S}(Y,Z) - \eta(\tilde{R}(\tilde{\xi},Y)Z)]\]
(4.5) 
\[+ B(W)[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)].\]

In consequence of (2.1), (2.2), (3.1) we have
\[\eta(\tilde{R}(\tilde{\xi},Y)Z) = (a^2-1+\kappa)g(Y,Z) - \eta(Y)\eta(Z) + (\mu - \frac{2(1-a)}{a})g(hY,Z).\]
(4.6)

The covariant derivative of the above equation along the vector field \(W\) gives
\[g((\nabla_W \tilde{R})(\tilde{\xi},Y)Z, \tilde{\xi}) = (a^2-1+\kappa)g(\phi W,Y)\eta(Z)\]
\[- \mu (1-\kappa) + \frac{2(1-\kappa)(1-a)}{a}g(\phi W,Y)\eta(Z)\]
(4.7) 
\[+ \frac{a^2 - 1 + \kappa}{a} - \mu + \frac{2(1-a)}{a}g(\phi hW,Y)\eta(Z) - \mu (1-\kappa)\]
\[g(\phi W,Z)\eta(Y) - \mu g(\phi hW,Z)\eta(Y) - \mu g(\phi hW,Z)\eta(Y)\]
\[- \mu (\mu - \frac{2(1-a)}{a})g(\phi hY,Z)\eta(W).\]

In view of (4.6), (4.7), (4.5) becomes
\[(\nabla_W \tilde{S})(Y,Z) = (a^2-1+\kappa)g(Y,Z) - \eta(Y)\eta(Z) + (\mu - \frac{2(1-a)}{a})g(hY,Z)\]
(4.8) 
\[+ B(W)[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)].\]
From (4.8) and the definition (4.2), it follows that an extended generalized \( \phi \)-recurrent \((\kappa, \mu)\)-manifold under \(D\)-homothetic deformation is a generalized Ricci-recurrent manifold if and only if

\[
\left[ \frac{a^2 - 1 + \kappa}{a} - \mu (1 - \kappa) + \frac{2(1 - \kappa)(1 - a)}{a} \right] g(\phi W, Y) \eta(Z) + \left[ \frac{a^2 - 1 + \kappa}{a} - \mu \right]

+ \frac{2(1 - a)}{a} g(\phi hW, Y) \eta(Z) - \mu (1 - \kappa) g(\phi W, Z) \eta(Y) - \mu g(\phi hW, Z) \eta(Y)

= 0.
\]

Let \( \{ e_i : i = 1, 2, \ldots, 2n + 1 \} \) be an orthonormal basis of the tangent space at any point of the manifold. Setting \( Y = Z = e_i \) in (4.9) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we have

\[
2n(1 - a^2 - \kappa a) A(W) + (4n^2 - 2n - 1) B(W) = 0.
\]

Next, we deal with the study of locally \( \phi \)-Ricci symmetric \((\kappa, \mu)\)-manifolds under \(D\)-homothetic deformation.

**Theorem 4.2.** The property of locally \( \phi \)-Ricci symmetry on an \((\kappa, \mu)\)-manifold is invariant under the \(D\)-homothetic deformation provided \( \mu = -2 \).

**Proof:** Differentiating (3.2) covariantly with respect to \( W \) we have

\[
(\nabla_W \bar{Q}) X = (\nabla_W Q) X + \left( \frac{a - 1}{a} \right) (2na^2 + 2n + 2a - a^2 + \kappa - 1) ((\nabla_W \eta)(X)) \xi

+ \eta(X)(-\phi W - \phi hW) + (2 + \mu)(\nabla_W h) X.
\]

Simplifying by using (2.10) and (2.15) and operating \( \phi^2 \) on both sides and suppose that \( X \) is orthogonal to \( \xi \), we find that

\[
\bar{\phi}^2(\nabla_W \bar{Q})(X) = \phi^2(\nabla_W Q)(X) + (2 + \mu) \mu \eta(W) \phi hX.
\]

Hence the proof.

Now, we deal with the study of \( \eta \)-parallel \((\kappa, \mu)\)-manifolds under \(D\)-homothetic deformation.
**Theorem 4.3.** Under D-homothetic deformation, an $\eta$-Parallel Ricci tensor in a $(\kappa, \mu)$-manifold remains $\eta$-parallel, provided $\mu = -2$.

**Proof:** Differentiating (3.1) covariantly with respect to $W$ and then using (2.10) and (2.15) we have

\[(4.13)\]

\[
(\nabla_W S)(X, Y) = (\nabla_W S)(X, Y) + \left(\frac{a-1}{a}\right)(2na + 2a - a^2 + \kappa - 1)(\eta(Y)(\nabla_W \eta)(X) + \eta(X)(\nabla_W \eta)(Y)) + (a-1)(2+\mu)[(1-\kappa)g(W, \phi X)\eta(Y) - g(W, \phi h X)\eta(Y)] - (1-\kappa)\eta(X)g(\phi W, Y) - \eta(X)g(\phi h W, Y) - \mu \eta(W)g(\phi h X, Y).
\]

Replacing the vector fields $X$ by $\phi X$ and $Y$ by $\phi Y$ in (4.13) and then by using (2.1) and (2.2) we obtain

\[(4.14)\]

\[
(\nabla_W S)(X, Y) = (\nabla_W S)(X, Y) - (a-1)(2+\mu)\mu \eta(W)g(X, \phi Y).
\]

Hence the Proof.

**5. Example**

We consider 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on $M$ given by $E_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y}$ and $E_3 = 2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. $[E_1, E_2] = 0$, $[E_2, E_3] = 2E_1$, $[E_1, E_3] = 2E_2$. Let $g$ be a metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$. Let $\eta$ be the 1-form defined by $\eta(V) = g(V, E_1)$ for any $V \in \chi(M)$. Let $\phi$ be the $(1, 1)$-tensor field defined by $\phi E_1 = 0$, $\phi E_2 = E_3$, $\phi E_3 = -E_2$ and $hE_1 = 0$, $hE_2 = E_2$ and $hE_3 = -E_3$. Using the linearity of $\phi$ and $g$, we have $\eta(E_1) = 1$, $\phi^2 V = -V + \eta(V)\xi$ and $g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W)$, for any $V, W \in \chi(M)$.

The Riemannian connection $\nabla$ of the metric tensor $g$ is given by

\[2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).\]

Using Koszul’s formula, we get the following,

\[
\nabla_{E_1} E_3 = 2E_2, \nabla_{E_2} E_2 = -2E_3, \nabla_{E_1} E_1 = 0, \nabla_{E_2} E_3 = 0, \nabla_{E_2} E_2 = 0, \nabla_{E_2} E_1 = -2E_3,
\]

\[(5.1)\]

\[
\nabla_{E_3} E_3 = 0, \nabla_{E_3} E_2 = 0, \nabla_{E_3} E_1 = 0.
\]
Similarly we find
\[ S(5.3) \]
In view of the expression of the curvature tensor we find the Ricci tensor as follows:
\[ R(E_1, E_2)E_3 = 0, R(E_2, E_3)E_3 = -4E_2, R(E_1, E_3)E_3 = 0, \]
\[ R(E_1, E_2)E_2 = 0, R(E_2, E_3)E_2 = 4E_3, R(E_1, E_3)E_2 = 0, \]
\[ R(E_1, E_2)E_1 = -4E_2, R(E_2, E_3)E_1 = 0, R(E_1, E_3)E_1 = 4E_3. \]
In view of the expression of the curvature tensor we find the Ricci tensor as follows:
\[ S(E_1, E_1) = g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) = 0. \]
Similarly we find \( S(E_2, E_2) = -4 = S(E_3, E_3) \). Hence \( r = -8 \).

It is well known that in a 3-dimensional manifold, the curvature tensor \( R \) satisfies the relation
\[
R(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y].
\]
From (2.12) we have
\[ S(X, Y) = -\mu g(X, Y) + \mu [hX, Y] + (2\kappa + \mu) \eta(X) \eta(Y). \]
From (5.5) we can find that
\[ R(X, Y)Z = 2\mu [g(X, Z)Y - g(Y, Z)X] + \mu [hX, Z]X - g(hX, Z)Y + g(Y, Z)hX - g(X, Z)hY \]
\[ + (2\kappa + \mu) [\eta(Y)X - \eta(X)Y] \eta(Z) + (2\kappa + \mu) [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] \xi \]
\[ - \frac{r}{2} [g(Y, Z)X - g(X, Z)Y]. \]
which is equivalent to
\[
^tR(X, Y, Z, W) = \mu [g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] + \mu [hY, Z]g(X, W) \]
\[ - g(hX, Z)g(Y, W) + g(Y, Z)g(hX, W) - g(X, Z)g(hY, W)] \]
\[ + (2\kappa + \mu) [\eta(Y)g(X, W) - \eta(X)g(Y, W)] \eta(Z) \]
\[ + (2\kappa + \mu) [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] \eta(W) \]
\[ - \frac{r}{2} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \]
In view of above relation we get
\[ K(E_1, \phi E_1) = 0, \]
\[ K(E_2, \phi E_2) = g(R(E_2, \phi E_2)E_2, \phi E_2) = g(R(E_1, E_3)E_2, E_3) = 2\mu + \frac{r}{2}. \]

Similarly we have \( K(E_3, \phi E_3) = 2\mu + \frac{r}{2} \). Again from (3.1) it can be easily shown that
\[ \bar{K}(E_2, \phi E_2) - K(E_2, \phi E_2) = -(1 - a)(3a - 1). \]
Similarly we have \( \bar{K}(E_3, \phi E_3) - K(E_3, \phi E_3) = -(1 - a)(3a - 1) \), therefore \((\kappa, \mu)\)-manifold satisfies the relation (3.13) and hence Theorem (3.3) is verified.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**


