THE FIRST ENTIRE ZAGREB INDEX OF VARIOUS CORONA PRODUCTS AND THEIR BOUNDS

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Abstract. The First Entire Zagreb Index (FEZI) of a (molecular) graph was introduced by Alwardi et al. [2] as the sum of the squares of degree of all the vertices and edges of the given graph. In this paper, the exact expressions for the FEZI of two graphs of several types of Corona products are established. Finally, the obtained results are applied to compute the bounds for the FEZI of two graphs.

Keywords: degree (of vertex or edge); degree-based topological indices; first entire Zagreb index; corona product; graph operation.

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1. Introduction

In this paper, we use only molecular graphs. Molecular graphs [16] are simple, connected graphs and in which nodes and edges are assumed to be atoms and chemical bonds compounds, respectively. We consider the notations $V(G)$ and $E(G)$ as the node and edge sets of a graph $G$, respectively. The degree of a vertex $u \in V(G)$, $d(u/G)$, is the cardinality of the set of edges which are incident to $u$. In chemical graph theory, a topological index of a graph could be represented by a single numerical number that characterizes the some properties of the corresponding

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molecular graph. Relationships like quantitative structure-property relationship (QSPR), quantitative structure-activity relationship (QSAR) of molecules or the biological activity with their structure are to be predicted by using the topological indices. The two most popular and extensively studied vertex-degree-based topological indices are the first and second Zagreb indices introduced by Gutman et al. [6] in 1972, denoted as $M_1(G)$ and $M_2(G)$, respectively and are defined as follows

$$M_1(G) = \sum_{u \in V(G)} d^2(u/G) = \sum_{uv \in E(G)} [d(u/G) + d(v/G)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d(u/G)d(v/G).$$

In the paper [7], another vertex-degree-based topological index was introduced. It was denoted as $F(G)$ and was defined by

$$F(G) = \sum_{u \in V(G)} d^3(u/G) = \sum_{uv \in E(G)} [d^2(u/G) + d^2(v/G)].$$

This index was not further studied for a long time but it was studied by Furtula et al. in 2015 [4] in which this index was named as forgotten topological index or F-index.

In 2004, Milicevic et al. [15] introduced the first reformulated Zagreb index in terms of edge degrees instead of vertex degrees. The first reformulated Zagreb index of a graph $G$ is defined by

$$EM_1(G) = \sum_{e \in E(G)} d^2(e/G) = \sum_{e=uv \in E(G)} \left(d(u/G) + d(v/G) - 2\right)^2,$$

where the degree of the edge $e = uv$ is defined as $d(e/G) = d(u/G) + d(v/G) - 2$.

We refer our interested readers to [3, 8, 12, 13, 17] for some more study of the topological indices of graph operations. The readers interested in more information on bounds for various topological indices can be referred to [1, 5].

2. Preliminaries

The chemical relations (forces) of the inter-molecular forces of molecules subsist between the atoms as well as the atoms and bonds in the corresponding molecular graphs. Using this
chemical reason, Alwardi et al. [2] introduced a new graph invariant, namely, the First Entire Zagreb Index (FEZI).

**Definition 1.** [2] The FEZI of a graph $G = (V, E)$ is defined by $M^E_1(G) = \sum_{u \in V(G) \cup E(G)} d^2(u/G)$, where $d(u/G)$ is the degree of a vertex or an edge $u$ in $G$.

The following Proposition is very important for computing the expressions of the FEZI.

**Proposition 1.** [2, 5] For the graph $G$, the following formulas are

(i) $M^E_1(G) = M_1(G) + EM_1(G)$

(ii) $M^E_1(G) = 4|E(G)| - 3M_1(G) + 2M_2(G) + F(G)$.

**Figure 1.** A graph $G$ with $M^E_1(G) = 66$

**Figure 2.** An example of various types of Corona products based on $R$-graphs of two graphs $P_4$ and $P_3$ such as (i) RVCP, (ii) RECP, (iii) RVNCP and (iv) RENCP
3. **Main Results and Discussions**

Throughout this section, we present some explicit expressions of the FEZI of two graphs for some graph operations in which one is based on $R$-graphs (triangle parallel graphs) such as $R$-Vertex Corona product (RVCP), $R$-Edge Corona product (RECP), $R$-Vertex Neighborhood Corona product (RVNCP) and $R$-Edge Neighborhood Corona product (RENCP) and another one is based on $S$-graphs (subdivision graphs) such as $S$-Vertex Corona product (SVCP), $S$-Edge Corona product (SECP), $S$-Vertex Neighborhood Corona product (SVNCP) and $S$-Edge Neighborhood Corona product (SENCP). We obtain the expression of FEZI for the Vertex-Edge Corona product (VECP) of two graphs. Also we compute some bounds for the FEZI of the nine different types of Corona product of graphs. It is to be noted that the subdivision graph $S(G)$ of a graph is the graph obtained from $G$ by inserting a new vertex into every edge of $G$ and the triangle parallel graph $R(G)$ is constructed from $G$ by adding a new vertex $v_e$ on each edge of $G$ and then joining every newly inserted vertex to the end vertices of the corresponding $e \in E(G)$, respectively. There are two simple, connected graphs $G$ and $H$ having $n_1, n_2$ vertices and $m_1, m_2$ edges, respectively. To illustrate, we assume the familiar notations $P_n$ and $C_n$ as a path and cycle graph with $n$ number of vertices, respectively. The maximum and minimum vertex degree of $G$ and $H$ are denoted by $\Delta_1, \delta_1$ and $\Delta_2, \delta_2$, respectively. For each $u \in V(G)$ and $v \in V(H)$, we have

\[
\begin{align*}
\Delta_1 &\geq d(u/G), \quad \delta_1 \leq d(u/G), \\
\Delta_2 &\geq d(v/H) \text{ and } \delta_2 \leq d(v/H). 
\end{align*}
\]

The equality holds if and only if $G$ and $H$ are regular graphs.

3.1. **The $R$-Vertex Corona product (RVCP).** The RVCP of two graphs is a new graph operation based on $R$-graphs and it was introduced by Lan et al. [9] in 2015.

**Definition 2.** [9] The RVCP of $G$ and $H$, denoted by $G \ast H$, is the new graph obtained from vertex disjoint $R(G)$ and $|V(G)|$ copies of $H$ by joining the $i$th vertex of $V(G)$ to every vertex in the $i$th copy of $H$.

It has the number of vertices $\left( |V(G)| + |E(G)| + |V(G)||V(H)| \right)$ and the number of edges $\left( 3|E(G)| + |V(G)||E(H)| + |V(G)||V(H)| \right)$, (see Fig. 2). Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, $I(G) = \{v_1, v_2, \ldots, v_{n_2}\}$, $E(G) = \{(u_i, v_j) : 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$, $E(H) = \{(v_j, u_i) : 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$, where $\ast$ denotes the RVCP of two graphs. The RVCP of two graphs is constructed by joining each vertex of the $i$th copy of $H$, $1 \leq i \leq |V(G)|$, with each vertex of $G_i$, the $i$th copy of $G$.
Applying the Definition 1, we have

\[ V(R(G)) \setminus V(G) = \{u_{e_1}, u_{e_2}, \ldots, u_{e_m}\} \quad \text{and} \quad V(H) = \{v_1, v_2, \ldots, v_{n_2}\}, \quad \text{so that} \quad V(R(G)) = V(G) \cup I(G). \]

Let \( V(H^i) = \{v_{i1}, v_{i2}, \ldots, v_{in}\} \) be the vertex set of the \( i \)th copy of \( H \) for \( i = 1, 2, \ldots, n_1 \). Thus the vertex set and edge set of \( R(G) \ast H \) are given by \( V(G \ast H) = V(G) \cup I(G) \cup \left( \bigcup_{i=1}^{n_1} V(H^i) \right) \) and \( E(G \ast H) = E_1 \cup E_2 \cup E_3 \cup E_4 \ast, \) where \( E_1 = \{v_i, v_k \in E(G \ast H) | v_i \in V(G), v_k \in I(G); i \neq k \} \) and \( l = 1, 2, \ldots, n_1 \), \( E_2 = \{u_iu_{e_l} \in E(G \ast H) | u_i \in V(G), u_{e_l} \in I(G) \} \) for \( i = 1, 2, \ldots, n_1 \) and \( l = 1, 2, \ldots, m_1 \), \( E_3 = \{u_iu_{e_l} \in E(G \ast H) | u_i \in V(G), u_{e_l} \in I(G) \} \) for \( i = 1, 2, \ldots, n_1 \) and \( j = 1, 2, \ldots, n_2 \) and \( E_4 = \{v_{p}v_{q} \in E(G \ast H) | v_{p}v_{q} \in V(H) \} \) for \( i = 1, 2, \ldots, n_1 \) and \( p, q = 1, 2, \ldots, n_2 \). The degrees of the vertices of \( G \ast H \) are given by

\[
\begin{align*}
\frac{d(u_i/G \ast H)}{d(u_{e_l}/G \ast H)} &= 2d(u_i/G) + n_2 \quad \text{for} \quad i = 1, 2, \ldots, n_1, \\
\frac{d(u_{e_l}/G \ast H)}{d(v_j/H)} &= 2 \quad \text{for} \quad l = 1, 2, \ldots, m_1, \\
\frac{d(v_j/H)}{d(u_{e_l}/G \ast H)} &= d(v_{j}/G) + 1 \quad \text{for} \quad i = 1, 2, \ldots, n_1 \quad \text{and} \quad j = 1, 2, \ldots, n_2.
\end{align*}
\]

We establish a formula of the FEZI for RVCP of two graphs.

**Theorem 1.** The FEZI of \( G \ast H \) is given by

\[
M_1^2(G \ast H) = 8F(G) + n_1F(H) + 4(4n_2 - 1)M_1(G) + 2n_1M_1(H) + 8M_2(G) + 2n_1M_2(H) + 10m_1n_2 + 4m_2n_1n_2 + n_1n_2(n_2 + 1) + (4m_1 + n_1n_2)(n_2 + 1)^2 + 4m_1(4m_2 + 1).
\]

**Proof.** Applying the Definition 1, we have

\[
M_1^2(G \ast H) = \sum_{u \in V(G \ast H) \cup E(G \ast H)} \frac{d^2(u/G \ast H)}{d(u/G \ast H)}
\]

\[
= \sum_{u \in V(G \ast H)} \left( \frac{d(u/G \ast H)}{d(u/G \ast H)} + \frac{d(v/G \ast H)}{d(v/G \ast H)} \right) + \sum_{u \in V(G \ast H)} \left( \frac{d(u/G \ast H)}{d(u/G \ast H)} + \frac{d(v/G \ast H)}{d(v/G \ast H)} - 2 \right)^2
\]

\[
= A_1 + A_2 \quad \text{(say), where} \quad A_1 \quad \text{and} \quad A_2 \quad \text{denote the sums of the above terms in order.}
\]

Now \( A_1 = \sum_{u \in V(G \ast H)} \left( \frac{d(u_i/G \ast H)}{d(u_i/G \ast H)} + \frac{d(u_k/G \ast H)}{d(u_k/G \ast H)} \right) + \sum_{u \in V(G \ast H)} \left( \frac{d(u_i/G \ast H)}{d(u_i/G \ast H)} + \frac{d(u_{e_l}/G \ast H)}{d(u_{e_l}/G \ast H)} \right) + \sum_{u \in V(G \ast H)} \left( \frac{d(u_i/G \ast H)}{d(u_i/G \ast H)} + \frac{d(v_j/G \ast H)}{d(v_j/G \ast H)} \right) + \sum_{u \in V(G \ast H)} \left( \frac{d(v_j/G \ast H)}{d(v_j/G \ast H)} \right)\right)
\]

\[
= \sum_{u \in V(G \ast H)} \left\{ 2 \left( \frac{d(u_i/G)}{d(u_i/G)} + \frac{d(u_k/G)}{d(u_k/G)} \right) + 2n_2 \right\} d(u_i/G) + \sum_{u \in V(G \ast H)} \left\{ 2d(u_i/G) + (n_2 + 2) \right\} d(u_i/G) + \sum_{u \in V(G \ast H)} \left( d(v_j/G) + d(v_j/H) + (n_2 + 1) \right) + \sum_{i=1}^{n_1} \sum_{v \in V(H)} \left( d(v_p/H) + d(v_q/H) \right) + 2
\]

\[
= 2M_1(G) + 2n_2m_1 + 2M_1(G) + 2(n_2 + 2)m_1 + 4m_2n_2 + 2n_1m_2 + n_1n_2(n_2 + 1) + n_1 \left( M_1(H) + 2m_2 \right).
\]
From the Proposition 1 and using the Equation 2, we have

\[ A_2 = \sum_{u_i, u_k \in E_1} \left[ (d(u_i/G \ast H) + d(u_k/G \ast H) - 2)^2 + \sum_{u_i, u_j \in E_2} (d(u_i/G \ast H) + d(u_j/G \ast H) - 2)^2 + \sum_{v_j, v_q \in E_3} (d(v_j/G \ast H) + d(v_q/G \ast H) - 2)^2 + \sum_{v_j, v_q \in E_4} (d(v_j/G \ast H) + d(v_q/G \ast H) - 2)^2 \right] \]

\[ = \sum_{u_i, u_j \in E(G)} \left\{ 2(d(u_i/G) + d(u_j/G)) + 2(n_2 - 1) \right\}^2 + \sum_{u_i \in V(G)} \left( 2d(u_i/G) + n_2 \right)^2 d(u_i/G) \]

\[ + \sum_{u_i \in V(G)} \sum_{v_j \in V(H)} \left( 2d(u_i/G) + d(v_j/H) + n_2 \right)^2 + \sum_{i = 1}^{n_1} \sum_{v_{p, q} \in E(H)} \left( d(v_{p}/H) + d(v_{q}/H) \right)^2 \]

\[ = \sum_{u_i, u_j \in E(G)} \left\{ (d(u_i/G) + d(u_j/G))^2 + (n_2 - 1)^2 + 2(n_2 - 1)(d(u_i/G) + d(u_j/G)) \right\} + \sum_{u_i \in V(G)} \left( 4d^2(u_i/G) + 4n_2d^2(u_i/G) + n_2^2d(u_i/G) \right) \]

\[ + \sum_{u_i \in V(G)} \sum_{v_j \in V(H)} \left\{ 4d^2(u_i/G) + d^2(v_j/H) + (n_2 - 1)^2 + 4(n_2 - 1)d(u_i/G) + 2(n_2 - 1)d(v_j/H) + 4d(u_i/G)d(v_j/H) \right\} \]

\[ + \sum_{i = 1}^{n_1} \sum_{v_{p, q} \in E(H)} \left( d^2(u_{p}/H) + d^2(v_{q}/H) + 2d(u_{p}/H)d(v_{q}/H) \right) \]

\[ = 4 \left( F(G) + 2M_2(G) + m_1(n_2 - 1)^2 + 2(n_2 - 1)m_1(G) \right) + 4F(G) + 4n_2M_1(G) + 2m_1n_2 + 4n_2M_1(G) + n_1M_1(H) + n_1n_2(n_2 - 1)^2 + 8m_1(n_2 - 1)n_2 + 4n_1(n_2 - 1)n_2 + 16m_1n_2 + n_1 \left( F(H) + 2M_2(H) \right). \]

By summing \( A_1 \) and \( A_2 \), we have the required result. □

Applying the Theorem 1, we illustrate some examples below.

**Example 1.**

(i) \( M^f_1(P_n \ast P_m) = (m^3n + 17m^2n + 94mn - 14m^2 - 104m + 30n - 144). \)

(ii) \( M^f_1(C_n \ast C_m) = n(m^3 + 17m^2 + 98m + 88). \)

(iii) \( M^f_1(C_n \ast P_m) = n(m^3 + 17m^2 + 94m + 46). \)

We compute the bounds of the FEZI for RVCP of two graphs.

**Theorem 2.** The bounds for the FEZI of \( G \ast H \) are given by

\[ A \geq M^f_1(G \ast H) \geq B, \] where \( A = 2m_1(2\Delta_1 + n_2)(6\Delta_1 + 3n_2 - 2) + 2m_2n_1(2\Delta_2^2 + \Delta_2 + 1) + n_1n_2(2\Delta_1 + \Delta_2 + n_2)(2\Delta_1 + \Delta_2 + n_2 - 1) + 2n_1n_2 + 8m_1 \] and \( B = 2m_1(2\delta_1 + n_2)(6\delta_1 + 3n_2 - 2) + 2m_2n_1(2\delta_2^2 + \delta_2 + 1) + n_1n_2(2\delta_1 + \delta_2 + n_2)(2\delta_1 + \delta_2 + n_2 - 1) + 2n_1n_2 + 8m_1. \) The equality holds if and only if \( G \) and \( H \) are regular graphs.

**Proof.** From the Proposition 1 and using the Equation 2, we have

\[ M^f_1(G \ast H) = M_1(G \ast H) + EM_1(G \ast H) \]

\[ = \sum_{u_i, u_k \in E(G)} \left\{ 2(d(u_i/G) + d(u_k/G)) + 2n_2 \right\} + \sum_{u_i \in V(G)} \left\{ 2d(u_i/G) + (n_2 + 2) \right\} d(u_i/G) \]
\[
+ \sum_{u_i \in V(G)} \sum_{v_j \in V(H)} \left( 2d(u_i/G) + d(v_j/H) + (n_2 + 1) \right) + \sum_{i=1}^{n_1} \sum_{v_p \in E(H)} \left( (d(v_p/H) + d(v_q/H)) + 2 \right) + \sum_{u_i,u_j \in E(G)} \left\{ 2 \left( d(u_i/G) + d(u_k/G) \right) + 2(n_2 - 1) \right\}^2 + \sum_{u_i \in V(G)} \left( d(u_i/G) + n_2 \right)^2 d(u_i/G) \\
+ \sum_{u_i \in V(G)} \sum_{v_j \in V(H)} \left( 2d(u_i/G) + d(v_j/H) + n_2 - 1 \right)^2 + \sum_{i=1}^{n_1} \sum_{v_p \in E(H)} \left( d(v_p/H) + d(v_q/H) \right)^2.
\]

Also, from the Equation 1, we can write
\[
\leq 2m_1 \left( 2\Delta_1 + n_2 \right) + 2m_1 \left( 2\Delta_1 + n_2 + 2 \right) + n_1n_2 \left( 2\Delta_1 + \Delta_2 + n_2 + 1 \right) + 2n_1m_2 \left( \Delta_2 + 1 \right) + 4m_1 \left( 2\Delta_1 + n_2 - 1 \right)^2 + 2m_1 \left( 2\Delta_1 + n_2 \right)^2 + n_1n_2 \left( 2\Delta_1 + \Delta_2 + n_2 - 1 \right)^2 + 4n_1m_2 \Delta_2^2 \\
= 2m_1 \left( 2\Delta_1 + n_2 \right) + 2m_1 \left( 2\Delta_1 + n_2 \right) + 4m_1 + n_1n_2 \left( 2\Delta_1 + \Delta_2 + n_2 \right) + n_1n_2 + 2n_1m_2 \Delta_2 + 1 + 4m_1 \left( 2\Delta_1 + n_2 + 2 \right) + n_1n_2 \left( 2\Delta_1 + \Delta_2 + n_2 \right) + 4m_1 + 2m_1 \left( 2\Delta_1 + n_2 \right)^2 + n_1n_2 \left( 2\Delta_1 + n_2 + 2 \right)^2 - 2n_1n_2 \left( 2\Delta_1 + \Delta_2 + n_2 \right) + n_1n_2 + 4n_1m_2 \Delta_2^2 \\
= 2m_1 \left( 2\Delta_1 + n_2 \right) \left( 6\Delta_1 + 3n_2 - 2 \right) + 2m_2n_1 \left( 2\Delta_2^2 + \Delta_2 + 1 \right) + n_1n_2 \left( 2\Delta_1 + \Delta_2 + n_2 \right) \left( 2\Delta_1 + \Delta_2 + n_2 - 1 \right) + 2n_1n_2 + 8m_1.
\]

Similarly, for the reverse bound we can do that \( M'_1(G \ast H) \geq 2m_1 \left( 2\delta_1 + n_2 \right) + 2m_1 \left( 2\delta_1 + n_2 + 2 \right) + n_1n_2 \left( 2\delta_1 + \delta_2 + n_2 + 1 \right) + 2n_1m_2 \left( \delta_2 + 1 \right) + 4m_1 \left( 2\delta_1 + n_2 - 1 \right)^2 + 2m_1 \left( 2\delta_1 + n_2 \right)^2 + n_1n_2 \left( 2\delta_1 + \delta_2 + n_2 - 1 \right)^2 + 4n_1m_2 \Delta_2^2. \) After simplification we get the desired result. \( \square \)

**Corollary 1.** If \( G \) and \( H \) are \( r_1 \) and \( r_2 \)-regular graphs (i.e. \( \Delta_1 = \delta_1 = d(u/G) = r_1 \) and \( \Delta_2 = \delta_2 = d(v/H) = r_2 \)) with the orders \( n_1, n_2 \) and the sizes \( m_1, m_2 \) respectively, then \( M'_1(G \ast H) = 2m_1 \left( 2r_1 + n_2 \right) \left( 6r_1 + 3n_2 - 2 \right) + 2m_2n_1 \left( 2r_2^2 + r_2 + 1 \right) + n_1n_2 \left( 2r_1 + r_2 + n_2 + 1 \right) + 2n_1n_2 + 8m_1. \)

### 3.2. The R-Edge Corona product (RECP).

The RECP of two graphs is a one kind of graph operation based on \( R \)-graphs. It was introduced by Lan et al. [9] in 2015.

**Definition 3.** [9] The RECP of two vertex-disjoint graphs \( G \) and \( H \), denoted by \( G \ast H \), is a new graph obtained from one copy of the semi-total point graph \( R(G) \) and also connected \( |I(G)| \) copies of graph \( H \) by joining the \( l \)th vertex of \( I(G) \) to every vertex in the \( l \)th copy of \( H \).

The graph \( G \ast H \) has \( (|V(G)| + |E(G)| + |E(G)||V(H)|) \) vertices and \( (|E(G)||E(H)| + |E(G)||V(H)|) \) edges (see Fig. 2). Let \( V(G) = \{u_1, u_2, \ldots, u_{n_1}\}, I(G) = \{u_{e_1}, u_{e_2}, \ldots, u_{e_{m_1}}\} \) and \( V(H) = \{v_1, v_2, \ldots, v_{n_2}\} \). Also, for \( l = 1, 2, \ldots, m_1 \) let \( V(H^l) = \)
\{v_1^l, v_2^l, \ldots, v_{n^l}\} \text{ be the vertex set of the } l^{th} \text{ copy of } H. \text{ So, } V(R(G)) = V(G) \cup I(G) \text{ and } V(G*H) = V(G) \cup I(G) \cup \left( V(H^1) \cup V(H^2) \cup \ldots \cup V(H^{m_1}) \right) \text{ and } E(G*H) = E_1^* \cup E_2^* \cup E_3^* \cup E_4^*.

where \( E_1^* = \{ u_iu_k \in E(G*H) | u_i, u_k \in V(G); i \neq k \text{ and } i, k = 1, 2, \ldots, n_1 \} \), \( E_2^* = \{ u_1u_l \in E(G*H) | u_i \in V(G), u_{el} \in I(G) \text{ for } i = 1, 2, \ldots, n_1 \text{ and } l = 1, 2, \ldots, m_1 \} \), \( E_3^* = \{ u_1v_p \in E(G*H) | u_{ei} \in I(G), v_p \in V(H^l) \text{ for } l = 1, 2, \ldots, m_1 \} \) and \( E_4^* = \{ v_p, v_q \in V(H^l) \text{ for } l = 1, 2, \ldots, m_1 \} \) and \( p, q = 1, \ldots, n_2 \). \text{ The degrees of the vertices of } G*H \text{ are given by}

\[
\begin{align*}
&d(u_i/G*H) = 2d(u_i/G) \text{ for } i = 1, 2, \ldots, n_1, \\
d(u_{el}/G*H) = 2 + n_2 \text{ for } l = 1, 2, \ldots, m_1, \\
d(v_j^l/G*H) = d(v_j^l/H) + 1 \text{ for } l = 1, 2, \ldots, m_1 \text{ and } j = 1, 2, \ldots, n_2.
\end{align*}
\]

In the following Theorem, we determine the FEZI for RECP of two graphs.

**Theorem 3.** The FEZI of \( G*H \) is given by

\[
M_1^*(G*H) = 4EM_1(G) + 4F(G) + m_1F(H) + 4(n_2 + 3)M_1(G) + 2m_1M_1(H) + 2m_1M_2(H) + m_1n_2(n_2 + 1)^2 + 3m_1n_2^2 + 4m_1m_2(n_2 + 1) + 5m_1n_2 + 4m_1m_2 - 8m_1.
\]

**Proof.** To calculate the FEZI of \( G*H \), we follow the Definition 1 and the Equations 1 and 3.

Let us consider \( T_1 = \text{The contribution of } M_1(G*H) \text{ and } EM_1(G*H) \text{ in } E_1^* \)

\[
= \sum_{u_iu_k \in E_1^*} \left( d(u_i/G*H) + d(u_k/G*H) \right) + \sum_{u_iu_k \in E_1^*} \left( d(u_i/G*H) + d(u_k/G*H) - 2 \right)^2
\]

\[
= \sum_{u_iu_k \in E(G)} \left( 2d(u_i/G) + 2d(u_k/G) \right) + \sum_{u_iu_k \in E(G)} \left\{ 2 \left( d(u_i/G) + d(u_k/G) - 2 \right) + 2 \right\}^2
\]

\[
= 2M_1(G) + 4EM_1(G) + 8M_1(G) - 12m_1.
\]

Next, let \( T_2 = \text{The contribution of } M_1(G*H) \text{ and } EM_1(G*H) \text{ in } E_2^* \)

\[
= \sum_{u_1u_{el} \in E_2^*} \left( d(u_i/G*H) + d(u_{el}/G*H) \right) + \sum_{u_1u_{el} \in E_2^*} \left( d(u_i/G*H) + d(u_{el}/G*H) - 2 \right)^2
\]

\[
= \sum_{u_i \in V(G), u_{el} \in I(G)} \left( 2d(u_i/G) + n_2 + 2 \right) + \sum_{u_i \in V(G), u_{el} \in I(G)} \left( 2d(u_i/G) + n_2 \right)^2
\]

\[
= 2M_1(G) + 2m_1(n_2 + 2) + 4F(G) + 4n_2M_1(G) + 2m_1n_2.
\]

Similarly, let us consider \( T_3 = \text{The contribution of } M_1(G*H) \text{ and } EM_1(G*H) \text{ in } E_3^* \)

\[
= \sum_{u_{ei}v_p \in E_3^*} \left( d(u_{ei}/G*H) + d(v_p/G*H) \right) + \sum_{u_{ei}v_p \in E_3^*} \left( d(u_{ei}/G*H) + d(v_p/G*H) - 2 \right)^2
\]
= \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \left( d(v_j/H) + n_2 + 3 \right) + \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \left( d(v_j/H) + (n_2 + 1) \right)^2 \\
= 2m_1m_2 + m_1n_2^2 + 3m_1n_2 + m_1M_1(H) + 4m_1m_2(n_2 + 1) + m_1n_2(n_2 + 1)^2.

Finally, let $T_4 = \text{The contribution of } M_1(G \ast H) \text{ and } EM_1(G \ast H) \text{ in } E_4^\ast$

= \sum_{v_p \in E_4^\ast} \left( d(v_p/G \ast H) + d(v'_q/G \ast H) \right) + \sum_{v_p \in E_4^\ast} \left( d(v_p/H) + d(v'_q/G \ast H) - 2 \right)^2

= \sum_{l=1}^{m_1} \sum_{v_p \in E_4^\ast} (d(v_p/H) + d(v'_q/H) + 2) + \sum_{l=1}^{m_1} \sum_{v_p \in E_4^\ast} (d(v_p/H) + d(v'_q/H))^2

= m_1M_1(H) + 2m_1m_2 + m_1F(H) + 2m_1M_2(H).

We get the desired result by summing the above four expressions. \hfill \Box

Applying Theorem 3, we have the following results.

**Example 2.**

(i) $M_1^c(P_n \ast P_m) = (m^3n + 9m^2n + 50mn - m^3 - 9m^2 - 58m + 38n - 110)$

(ii) $M_1^c(C_n \ast C_m) = n(m^3 + 9m^2 + 54m + 88)$

(iii) $M_1^c(C_n \ast P_m) = n(m^3 + 9m^2 + 50m + 38)$.

In the following Theorem, we compute the bounds on the FEZI for RECP of two graphs.

**Theorem 4.** The bounds for the FEZI of $G \ast H$ are computed as $C \geq M_1^c(G \ast H) \geq D$, where $C = 16m_1\Delta_1^2 + 2m_1(2\Delta_1 + n_2)(2\Delta_1 + n_2 + 1) + m_1n_2(\Delta_2 + n_2)(\Delta_2 + n_2 + 1) + 2m_1m_2(2\Delta_2 + \Delta_2 + 1) - 12m_1\Delta_1 + 8m_1 + 4m_1n_2$ and $D = 16m_1\delta_1^2 + 2m_1(2\delta_1 + n_2)(2\delta_1 + n_2 + 1) + m_1n_2(\delta_2 + n_2)(\delta_2 + n_2 + 3) + 2m_1m_2(2\delta_2 + \delta_2 + 1) - 12m_1\delta_1 + 8m_1 + 4m_1n_2$. The equality holds if and only if $G$ and $H$ are regular graphs.

**Proof.** Using the Definition 1 and the Equations 1 and 3, we have $M_1^c(G \ast H)$

= \sum_{u \in V(G) \ast I(G)} \left( 2d(u/G) + d(u_k/G) \right) + \sum_{u \in V(G) \ast I(G)} \left( 2\left( d(u/G) + d(u_k/G) - 2 \right) + 2 \right)^2

+ \sum_{u \in V(G) \ast I(G)} \left( 2d(u_i/G) + n_2 + 2 \right) + \sum_{u \in V(G) \ast I(G)} \left( 2d(u_i/G) + n_2 \right)^2

+ \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \left( d(v_j/H) + n_2 + 3 \right) + \sum_{l=1}^{m_1} \sum_{v_j \in V(H)} \left( d(v_j/H) + (n_2 + 1) \right)^2

+ \sum_{l=1}^{m_1} \sum_{v_p \in E_4^\ast} (d(v_p/H) + d(v'_q/H) + 2) + \sum_{l=1}^{m_1} \sum_{v_p \in E_4^\ast} (d(v_p/H) + d(v'_q/H))^2

\leq \sum_{u \in V(G) \ast I(G)} \left( 4\Delta_1 \right) + \sum_{u \in V(G) \ast I(G)} \left( 2(\Delta_1 - 1) \right)^2

+ \sum_{u \in V(G) \ast I(G)} \left( 2\Delta_1 + n_2 + 2 \right) + \sum_{u \in V(G) \ast I(G)} \left( 2\Delta_1 + n_2 \right)^2.
\[ + \sum_{u_i \in I(G)} \sum_{v_j \in V(H)} (\Delta_2 + n_2 + 3) + \sum_{u_i \in I(G)} \sum_{v_j \in V(H)} (\Delta_2 + (n_2 + 1))^2 \]
\[ + \sum_{u_i \in I(G)} \sum_{v_j \in V(H)} (2\Delta_1 + 2) + \sum_{u_i \in I(G)} \sum_{v_j \in V(H)} (2\Delta_2)^2 \]
\[ = 4m_1(4\Delta_1^2 - 3\Delta_1 + 1) + 2m_1(2\Delta_1 + n_2 + 2) + 2m_1(2\Delta_1 + n_2)^2 \]
\[ + m_1n_2(\Delta_2 + n_2 + 3) + m_1n_2(\Delta_2 + n_2 + 1)^2 + 2m_1m_2(\Delta_2 + 1) + 4m_1m_2\Delta_2^2 \]
\[ = 16m_1\Delta_1^2 + 2m_1(2\Delta_1 + n_2)(2\Delta_1 + n_2 + 1) + m_1n_2(\Delta_2 + n_2)(\Delta_2 + n_2 + 3) + \]
\[ 2m_1m_2(2\Delta_2^2 + \Delta_2 + 1) - 12m_1\Delta_1 + 8m_1 + 4m_1n_2 = C \ (\text{say}). \]

Analogously, using the equations 1 and 3, one can calculate the following \( M_1^f (G \ast H) \geq D \).

The equality holds if and only if \( G \) and \( H \) are regular graphs. \( \square \)

### 3.3. The \( R \)-Vertex Neighborhood Corona product (RVNCP)

The RVNCP introduced by Lan et al. [9] is a one type of Corona product of two graphs based on \( R \)-graphs.

**Definition 4.** [9] The RVNCP of \( G \) and \( H \), denoted by \( G \oplus H \), is a novel graph made of one copy of \( R(G) \) graph and connects \( n_1 \) copies of \( H \), all vertex-disjoint, and joining the neighbors of the \( i \)-th vertex of \( G \) in \( R(G) \) to every vertex in the \( i \)-th copy of \( H \).

Let \( G \) and \( H \) be two simple connected graphs with \( n_1 \), \( n_2 \) vertices and \( m_1 \), \( m_2 \) edges, respectively and the vertex sets \( V(G) = \{u_1, u_2, \ldots, u_{n_1}\} \), \( I(G) = \{u_{e_1}, u_{e_2}, \ldots, u_{e_{m_1}}\} \) and \( V(H) = \{v_1, v_2, \ldots, v_{n_2}\} \). Also, let \( V(H^i) = \{v_{i_1}^j, v_{i_2}^j, \ldots, v_{i_{n_2}}^j\}, i = 1, 2, \ldots, n_1 \) be the vertex set of the \( i \)-th copy of \( H \). Then \( V(G \oplus H) = V(G) \cup I(G) \cup (V(H^1) \cup V(H^2) \cup \ldots \cup V(H^{n_1})) \)

and \( E(G \oplus H) = E_1^\oplus \cup E_2^\oplus \cup E_3^\oplus \cup E_4^\oplus \cup E_5^\oplus \), where \( E_1^\oplus = \{u_i u_k \in E(G \oplus H) | u_i, u_k \in V(G)\}, \)

\( E_2^\oplus = \{u_i u_{e_k} \in E(G \oplus H) | u_i \in V(G), u_{e_k} \in I(G)\}, E_3^\oplus = \{v_{i_p}^i v_{i_q}^i \in E(G \oplus H) | v_{i_p}^i, v_{i_q}^i \in V(H^i)\}, \)

\( E_4^\oplus = \{u_{e_k} v_{i_j}^i \in E(G \oplus H) | u_{e_k} \in I(G), v_{i_j}^i \in V(H^i)\}. \) The graph \( G \oplus H \) has \( |V(G)| + |E(G)| + |V(G)||V(H)| \) vertices and \( 3|E(G)| + |V(G)||E(H)| + 4|E(G)||V(H)| \) edges, (see Fig. 2). The degrees of the vertices of \( G \oplus H \) are given by

\[ d(u_i / G \oplus H) = (n_2 + 2)d(u_i / G) \text{ for } i = 1, 2, \ldots, n_1, \]
\[ d(u_{e_k} / G \oplus H) = 2(1 + n_2) \text{ for } k = 1, 2, \ldots, m_1, \]
\[ d(v_{i_j}^i / G \oplus H) = d(v_{i_j} / H) + 2d(u_i / G) \text{ for } i = 1, 2, \ldots, n_1 \text{ and } j = 1, 2, \ldots, n_2. \]
Theorem 5. The FEZI of $G \oplus H$ is given by

$$M^2_1(G \oplus H) = (n_1^2 + 6n_2^2 + 20m_2 + 8)F(G) + n_1F(H) + (9n_2^2 - 4n_2 + 40m_2 + 4m_2n_2 - 4)M_1(G) + (20m_1 - 3n_1)M_1(H) + 2(n_2 + 2)(5n_2 + 2)M_2(G) + 2n_1M_2(H) + 4m_2n_2^2(2n_2 + 3) + 16m_1n_2(m_2 + 1) + 4m_2n_1 - 32m_1m_2 + 8m_1.$$ 

Proof. Applying the Proposition 1 and the Equation 4, we have

$$M^2_1(G \oplus H) = M_1(G \oplus H) + EM_1(G \oplus H)$$

$$= \sum_{u_i \in V(G \oplus H)} d^2(u_i/G \oplus H) + \left\{ \sum_{u_iu_j \in E^2_1} \left( d(u_i/G \oplus H) + d(u_j/G \oplus H) - 2 \right)^2 \right\} + \sum_{u_iu_j \in E^2_3} \left( d(u_i/G \oplus H) + d(u_j/G \oplus H) - 2 \right)^2 + \left\{ \sum_{u_iu_j \in E^2_3} \left( d(u_i/G \oplus H) + d(u_j/G \oplus H) - 2 \right)^2 \right\} = C_1 + C_2 + C_3 + C_4$$

Next, we compute the above terms separately.

Firstly, $C_1 = \sum_{u \in V(G \oplus H)} d^2(u/G \oplus H)$

$$= \sum_{u \in V(G)} \left( n_2 + 2 \right)^2 \sum_{u_i \in I(G)} \left( 2(n_2 + 1) \right) = \sum_{u \in V(G)} \sum_{v \in V(H)} \left( 2d(u_i/G) + d(v_i/H) \right)^2$$

$$= (n_2 + 2)^2M_1(G) + 4(n_2 + 1)^2m_1 + 4n_2M_1(G) + n_1M_1(H) + 16m_1m_2.$$ 

$$C_2 = \sum_{u_iu_j \in E_1} \left( d(u_i/G \oplus H) + d(u_j/G \oplus H) - 2 \right)^2 + \sum_{u_iu_j \in E_2} \left( d(u_i/G \oplus H) + d(u_j/G \oplus H) - 2 \right)^2$$

$$= \sum_{u \in V(G)} \sum_{u_iu_j \in E_2} \left( (n_2 + 2) \left( d(u_i/G) + d(u_j/G) \right) - 2 \right)^2 + \sum_{u \in V(G)} \sum_{u_iu_j \in E_2} \left( (n_2 + 2) \left( d(u_i/G) + d(u_j/G) \right) - 2 \right)^2$$

$$= \sum_{u \in V(G)} \left( n_2 + 2 \right)^2 \left( d^2(u_i/G) + d^2(u_j/G) \right) + 2(n_2 + 2)^2d(u_i/G)d(u_j/G) - 4(n_2 + 2)^2d(u_i/G)d(u_j/G) + 4 \left( d(u_i/G) + d(u_j/G) \right) + 4 \right\} + \sum_{u \in V(G)} \left( (n_2 + 2)^2d^2(u_i/G) + 4n_2(n_2 + 2)d(u_i/G) + 4n_2^2 \right)d(u_i/G)$$

$$= (n_2 + 2)^2F(G) + 2(n_2 + 2)^2M_2(G) - 4(n_2 + 2)M_1(G) + 4m_1 + (n_2 + 2)^2F(G) + 4n_2(n_2 + 2)M_1(G) + 8m_1n_2^2.$$
Also, $C_3 = \sum_{v_p, v_q \in E_3} \left( d(v_p/G \oplus H) + d(v_q/G \oplus H) - 2 \right)^2$
\[= \sum_{i=1}^{n_1} \sum_{v_p, v_q \in E(H)} \left( (d(v_p/H) + d(v_q/H) - 2) + 4d(u_i/G) \right)^2 \]
\[= n_1EM_1(H) + 16m_2M_1(G) + 16m_1M_1(H) - 32m_1m_2 = n_1(F(H) - 4M_1(H) + 2M_2(H) + 4m_2) + 16m_2M_1(G) + 16m_1M_1(H) - 32m_1m_2. \]

$C_4 = \sum_{u_i \in V(G)} \sum_{v_j \in V(H)} \left\{ (n_2 + 2)d(u_i/G) + 2d(w_i/G) + d(v_j/H) - 2 \right\}^2$
\[= \sum_{u_i \in V(G)} \sum_{v_j \in V(H)} \left\{ (n_2 + 2)^2d^2(u_i/G) + 4d^2(w_i/G) + d^2(v_j/H) + 4 + 4(n_2 + 2)d(u_i/G)d(w_i/G) + 2(n_2 + 2)d(u_i/G)d(v_j/H) - 4(n_2 + 2)d(u_i/G) + 4d(w_i/G)d(v_j/H) - 8d(w_i/G) - 4d(v_j/H) \right\}
\[= n_2(n_2 + 2)^2F(G) + 4n_2F(G) + 2m_1M_1(H) + 8m_1n_2 + 8n_2(n_2 + 2)M_2(G) + 4m_2(n_2 + 2)M_1(G) - 4n_2(n_2 + 2)M_1(G) + 8m_2M_1(G) - 8n_2M_1(G) - 16m_1m_2 + 4n_2F(G) + 2m_1M_1(H) + 8m_1n_2 + 8m_2M_1(G) + 8n_2M_1(G) + 16m_1m_2. \]

Adding $C_1, C_2, C_3$ and $C_4$ and taking simple calculation, we get the desired result. \(\square\)

The following results are direct consequence of the Theorem 5.

**Example 3.** (i) $M_1^g(P_n \oplus P_m) = (16m^3n + 168m^2n + 440mn - 22m^3 - 270m^2 - 716m - 272n + 168)$

(ii) $M_1^g(C_n \oplus C_m) = n(16m^3 + 168m^2 + 472m + 88)$

(iii) $M_1^g(C_n \oplus P_m) = n(16m^3 + 168m^2 + 440m - 176)$.

**Theorem 6.** The bounds for the FEZI of $(G \oplus H)$ are given by

$K \geq M_1^g(G \oplus H) \geq L$, where $K = n_1n_2(2\Delta_1 + \Delta_2)^2 + 4m_2n_1(2\Delta_1 + \Delta_2 - 1)^2 + (n_1 + 6m_1)(n_2 + 2)^2\Delta_1^2 + 8m_1(n_2 - 1)(n_2 + 2)\Delta_1 + 4m_1n_2(3n_2 + 2) + 2m_1n_2((n_2 + 4)\Delta_1 + \Delta_2 - 2)^2 +$...
The equality holds if and only if \( G \) and \( H \) are regular graphs.

**Proof.** The proof is similar to the Theorem 2. \( \square \)

### 3.4. The \( R \)-Edge Neighborhood Corona product (RENCP)

Lan et al.\[9\] defined four new graph operations based on \( R \)-graphs. The RENCP is one of the four new graph operations.

**Definition 5.** \[9\] The RENCP of \( G \) and \( H \), denoted by \( G \otimes H \), is a new graph which is achieved from one copy of \( R(G) \) graph and also it adjoins \(|I(G)|\) copies of \( H \) and joining the neighbors of the \( i \)th vertex of \( I(G) \) in \( R(G) \) to every vertex in the \( i \)th copy of \( H \).

Let \( V(G) = \{u_1, u_2, \ldots, u_{n_1}\} \), \( I(G) = \{u_{e_1}, u_{e_2}, \ldots, u_{e_{m_1}}\} \), and \( V(H) = \{v_1, v_2, \ldots, v_{n_2}\} \). For \( i = 1, 2, \ldots, m_1 \), let \( V(H^i) = \{v_{i1}^1, \ldots, v_{i_{n_2}}^i\} \) be the vertex set of the \( i \)th copy of \( H \). So, \( V(G) \cup I(G) \cup \left( \bigcup_{i=1}^{m_1} V(H^i) \right) \) is the partition of \( V(R(G) \otimes H) \) and \( E(G \otimes H) = E_1^\otimes \cup E_2^\otimes \cup E_3^\otimes \cup E_4^\otimes \), where

\[
E_1^\otimes = \{u_iu_k \in E(G \otimes H) | u_i, u_k \in V(G)\}, \quad E_2^\otimes = \{u_iu_{e_k} \in E(G \otimes H) | u_i \in V(G), u_{e_k} \in I(G)\}, \quad E_3^\otimes = \{v_{ip}v_{iq} \in E(G \otimes H) | v_{ip}, v_{iq} \in V(H^i)\} \quad \text{and} \quad E_4^\otimes = \{u_{ek}v_j \in E(G \otimes H) | u_{ek} \in V(G), v_j \in V(H^i)\}.
\]

Thus the graph \( G \otimes H \) has \(|V(G)| + |E(G)| + |E(G)||V(H)|\) vertices and \(3|E(G)| + |E(G)| |E(H)| + 2|E(G)||V(H)|\) edges, respectively (see Fig. 2).

From definition, the degrees of the vertices of \( G \otimes H \) are follows as

\[
\begin{align*}
d(u_i/G \otimes H) &= (n_2 + 2)d(u_i/G) \quad \text{for} \quad i = 1, 2, \ldots, n_1, \quad \\
(5) \quad d(u_{e_k}/G \otimes H) &= 2 \quad \text{for} \quad k = 1, 2, \ldots, m_1, \quad \\
d(v_j/G \otimes H) &= d(v_j/H) + 2 \quad \text{for} \quad i = 1, 2, \ldots, m_1 \quad \text{and} \quad j = 1, 2, \ldots, n_2.
\end{align*}
\]

Here we calculate the FEZI for RENCP of two graphs.

**Theorem 7.** The FEZI for \( G \otimes H \) is given by

\[
M_1^f(G \otimes H) = (n_2 + 2)^3F(G) + m_1F(H) + (n_2 + 2)(n_2 + 4m_2 - 2)M_1(G) + 7m_1M_1(H) + 2(n_2 + 2)^2M_2(G) + 2m_1M_2(H) + 12m_1m_2 + 4m_1n_2 + 8m_1.
\]

**Proof.** With the help of the Definition 1 as well as the Proposition 1 and also the Equation 5, we have
Proof. The proof is similar to the Theorem 4. □

From the Theorem 4, we have the following results.

**Example 4.** (i) $M^F_1(P_n \otimes P_m) = 2(4m^3n + 38m^2n + 102mn − 7m^3 − 65m^2 − 158m − 14n − 14)$

(ii) $M^F_1(C_n \otimes C_m) = 4n(2m^3 + 19m^2 + 55m + 22)$

(iii) $M^F_1(C_n \otimes P_m) = 4n(2m^3 + 19m^2 + 51m − 7)$.

**Theorem 8.** The bounds for the FEZI of $G \otimes H$ are given by

$$U \geq M^F_1(G \otimes H) \geq V,$$

where $U = 2m_1Δ_1(n_2 + 2)\left(3(n_2 + 2)Δ_1 - 2\right) + 2m_1m_2(2Δ^2_1 + 5Δ_2 + 4) + 2m_1\sum (\left((n_2 + 2)Δ_1 + Δ_2\right)^2 + (n_2 + 2)Δ_1 + Δ_2 + 2) + 8m_1$ and $V = 2m_1Δ_1(n_2 + 2)\left(3(n_2 + 2)Δ_1 - 2\right) + 2m_1m_2(2Δ^2_1 + 5Δ_2 + 4) + 2m_1\sum (\left((n_2 + 2)Δ_1 + Δ_2\right)^2 + (n_2 + 2)Δ_1 + Δ_2 + 2) + 8m_1$.

The equality holds if and only if $G$ and $H$ are regular graphs.

Proof. The proof is similar to the Theorem 4. □
3.5. The Subdivision-Vertex Corona product (SVCP). The SVCP of two graphs was introduced by Lu et al. [11].

Definition 6. [11] The SVCP of G and H, denoted by G ⊙ H, is a new graph. It is obtained from S(G) and n₁ copies of H, all vertex-disjoint, by joining the iᵗʰ vertex of V(G) to every vertex in the iᵗʰ copy of H.

From definition it is clear that the G ⊙ H has \((m₁ + n₁ + n₁n₂ + n₁m₂)\) vertices and \((2m₁ + n₁n₂ + n₁m₂)\) edges, (see Fig.3). Also, let \(V(G) = \{u₁,u₂,\ldots,u_m\}\), \(I(G) = V(S(G)) \setminus V(G) = \{u₁, u₂,\ldots,u_{m₁}\}\) and \(V(H) = \{v₁,v₂,\ldots,v_n\}\), so that \(V(S(G)) = V(G) \cup I(G)\). Let \(V(H^i) = \{v₁^i,v₂^i,\ldots,v_n^i\}\) be the vertex set of the iᵗʰ copy of H, \(i = 1,2,\ldots,n₁\), so that \(V(G \odot H) = V(G) \cup I(G) \cup \left( \bigcup_{i=1}^{n₁} V(H^i) \right)\) and \(E(G \odot H) = E₁ \cup E₂ \cup E₃\) where \(E₁ = \{uᵢvᵢ^j \in E(G \odot H) | uᵢ \in V(G), vᵢ^j \in V(H^i)\}\) for \(i = 1,2,\ldots,n₁\) and \(j = 1,2,\ldots,n₂\), \(E₂ = \{uᵢu₁ \in E(G \odot H) | uᵢ \in V(G), u₁ \in I(G)\}\) for \(i = 1,2,\ldots,n₁\) and \(k = 1,2,\ldots,m₁\) and \(E₃ = \{vᵢ^ivᵢ^m \in E(G \odot H) | vᵢ^i,vᵢ^m \in V(H^i)\}\) for \(i = 1,2,\ldots,n₁\) and \(l,m = 1,2,\ldots,n₂\). The degrees of the vertices of G ⊙ H are:

\[
\begin{align*}
\text{d}(uᵢ/G \odot H) &= \text{d}(uᵢ/G) + n₂ \quad \text{for} \quad i = 1,2,\ldots,n₁, \\
\text{d}(uᵢ/G \odot H) &= 2 \quad \text{for} \quad k = 1,2,\ldots,m₁, \\
\text{d}(vᵢ^j/G \odot H) &= \text{d}(vᵢ^j/H) + 1 \quad \text{for} \quad i = 1,2,\ldots,n₁ \quad \text{and} \quad j = 1,2,\ldots,n₂.
\end{align*}
\]

In the following theorem, the FEZI for SVCP of two graphs is computed.

Theorem 9. The FEZI of G ⊙ H is given by

\[
M₁^E(G \odot H) = F(G) + n₁F(H) + (3n₂ + 1)M₁(G) + 2n₁M₁(H) + 2n₁M₂(H) + 4n₂(m₁n₂ + m₂n₁) + n₁n₂(n₂^2 - n₂ + 2) + 2m₁n₂^2 + 8m₁m₂ + 4m₁.
\]

Proof. Using the Definition 1 and the Equation 6, the FEZI for G ⊙ H is

\[
M₁^E(G \odot H) = M₁(G \odot H) + EM₁(G \odot H) = S₁ + S₂ \quad \text{(say)}, \quad \text{respectively.}
\]

Now, \(S₁ = M₁(G \odot H)\)

\[
\begin{align*}
&= \sum_{uᵢvᵢ^j ∈ E₁} \left( \text{d}(uᵢ/(G \odot H)) + \text{d}(vᵢ^j/(G \odot H)) \right) + \sum_{uᵢuᵢ \in E₂} \left( \text{d}(uᵢ/(G \odot H)) + \text{d}(uᵢ/(G \odot H)) \right) \\
&+ \sum_{vᵢ^ivᵢ^m \in E₃} \left( \text{d}(vᵢ^j/(G \odot H)) + \text{d}(vᵢ^m/(G \odot H)) \right) \\
&= \sum_{uᵢ \in V(G)} \sum_{vᵢ \in V(H)} \left( \text{d}(uᵢ/G) + n₂ + \text{d}(vᵢ/H) + 1 \right) + \sum_{uᵢ \in V(G)} \left( \text{d}(uᵢ/G) + n₂ + 2 \right) \text{d}(uᵢ/G)
\end{align*}
\]
\[ + \sum_{i=1}^{n_1} \sum_{v_m \in E(H)} \left( d(u_i/H) + d(v_m/H) + 2 \right) \]
\[ = 2(m_1n_2 + m_2n_1) + n_1n_2(n_2 + 1) + M_1(G) + 2m_1(n_2 + 2) + n_1M_1(H) + 2n_1m_2. \]

Finally, \( S_2 = EM_1(G \odot H) \)
\[ = \sum_{u \in V(G)} \sum_{v \in V(H)} \left( d(u/(G \odot H)) + d(v/(G \odot H)) - 2 \right)^2 + \sum_{u \in V(G)} \left( d(u/(G \odot H)) + d(v/(G \odot H)) - 2 \right)^2 \]
\[ = n_2M_1(G) + n_1M_1(H) + n_1n_2(n_2 - 1)^2 + 8m_1m_2 + 4(n_2 - 1)(m_1n_2 + m_2n_1) + F(G) + 2n_2M_1(G) + 2m_1n_2^2 + n_1F(H) + 2n_1M_2(H). \]

By adding \( S_1 \) and \( S_2 \) with simple calculation, we have the desired result. \( \square \)

Using Theorem 9, we calculate the FEZI of several chemically interesting molecular graphs

**Example 5.** (i) \( M^F_1(P_n \odot P_m) = (m^3n + 9m^2n + 42mn - 6m^2 - 26m - 34n - 16). \)
(ii) \( M^F_1(C_n \odot C_m) = n(m^3 + 9m^2 + 46m + 16). \)
(iii) \( M^F_1(C_n \odot P_m) = n(m^3 + 9m^2 + 42m - 34). \)

**Theorem 10.** The bounds of the FEZI for \( G \odot H \) are determined as
\[ W_1 \geq M^F_1(G \odot H) \geq W_2, \text{ where } W_1 = n_1n_2(\Delta_1 + \Delta_2 + n_2)(\Delta_1 + \Delta_2 + n_2 - 1) + 2m_1(\Delta_1 + n_2)(\Delta_1 + n_2 + 1) + 2m_2n_1(2\Delta_2^2 + \Delta_2 + 1) + 2n_1n_2 + 4m_1 \text{ and } W_2 = n_1n_2(\delta_1 + \delta_2 + n_2)(\delta_1 + \delta_2 + n_2 - 1) + 2m_1(\delta_1 + n_2)(\delta_1 + n_2 + 1) + 2m_2n_1(2\delta_2^2 + \delta_2 + 1) + 2n_1n_2 + 4m_1. \]

The equality holds if and only if \( G \) and \( H \) are regular graphs.

**Proof.** The proof is analogous to that of Theorem 2. \( \square \)

**3.6. The Subdivision-Edge Corona product (SECP).** The SECP of two graphs is a new type graph operation among different types of Corona product and it was introduced by Lu et al. [11].
Definition 7. [11] The SECP for G and H, denoted by $G \ominus H$, is a novel graph based on the subdivision graph $S(G)$ and $|I(G)|$ copies of H and by connecting the $i^{th}$ vertex of $I(G)$ to every vertex in the $i^{th}$ copy of H.

The graph $G \ominus H$ has $(|V(G)| + |E(G)||V(H)| + |E(G)|)$ vertices and $|E(G)|(|E(H)| + |V(H)| + 2)$ edges, (see Fig. 3). Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, $I(G) = V(S(G)) \setminus V(G) = \{u_{e_1}, u_{e_2}, \ldots, u_{e_{m_1}}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$, so that $V(G) = V(G) \cup I(G)$. Let $V(H_i) = \{v_{1}^i, v_{2}^i, \ldots, v_{n_2}^i\}$ be the vertex set of the $i^{th}$ copy of H for $i = 1, 2, \ldots, n_2$, so that $V(G \ominus H) = V(G) \cup I(G) \cup \left( \bigcup_{i=1}^{n_2} V(H_i) \right)$ and $E(G \ominus H) = E(G) \cup \bigcup_{i=1}^{n_1} E(H_i) \cup \{u_{e_i}v_j : u_{e_i} \in I(G), v_j \in V(H_i) \text{ for } i = 1, 2, \ldots, n_1 \text{ and } j = 1, 2, \ldots, n_2\}$. The degree distributions of the vertices of $G \ominus H$ are given by

\[
\begin{align*}
d(u_i / G \ominus H) &= d(u_i / G) & \text{for } i = 1, 2, \ldots, n_1, \\
d(u_{e_i} / G \ominus H) &= (2 + n_2) & \text{for } i = 1, 2, \ldots, m_1, \\
d(v_j^i / G \ominus H) &= d(v_j / H) + 1 & \text{for } i = 1, 2, \ldots, m_1 \text{ and } j = 1, 2, \ldots, n_2.
\end{align*}
\]

Now we reckon the FEZI for SECP of two graphs.

Theorem 11. The FEZI for $G \ominus H$ is given by

\[
M_1^c(G \ominus H) = F(G) + m_1 F(H) + (2n_2 + 1)M_1(G) + 2m_1 M_1(H) + 2m_1 M_2(H) + m_1 \left( n_2^3 + 5n_2^2 + 4n_2m_2 + 6n_2 + 8m_2 + 4 \right).
\]

Proof. From the Proposition 1, we get $M_1^c(G \ominus H) = M_1(G \ominus H) + EM_1(G \ominus H) = R_1 + R_2$ (say), respectively.

Now, $R_1 = M_1(G \ominus H)$

\[
\begin{align*}
&= \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \left( d(u_j / G) + n_2 + 3 \right) + \sum_{i=1}^{m_1} \sum_{v_j \in V(H)} \left( d(v_j / H) + d(v_k / H) + 2 \right) \\
&+ \sum_{u_i \in V(G)} \left( d(u_i / G) + n_2 + 2 \right) d(u_i / G) \text{ (Using the Equation 7)} \\
&= M_1(G) + m_1 \left( M_1(H) + 4m_2 + n_2 \right) + m_1(n_2 + 2)^2.
\end{align*}
\]

Finally, $R_2 = EM_1(G \ominus H)$

\[
\begin{align*}
&= \sum_{i=1}^{m_1} \sum_{j=1}^{n_2} \left( d(v_j / H) + 1 + n_2 \right)^2 + \sum_{i=1}^{m_1} \sum_{v_j \in V(H)} \left( d(v_j / H) + d(v_k / H) \right)^2 + \sum_{u_i \in V(G)} \left( d(u_i / G) + n_2 \right)^2 d(u_i / G) \\
&= m_1 \left( \sum_{j=1}^{n_2} \left( d(v_j / H) + (n_2 + 1)^2 + 2(n_2 + 1)d(v_j / H) \right) + \sum_{v_j \in V(H)} \left( d^2(v_j / H) + d^2(v_k / H) \right) \right)
\end{align*}
\]
\[ 2d(v_j/H)d(v_k/H)) + \sum_{u_i \in V(G)} (d^3(u_i/G) + n_2^2d(u_i/G) + 2n_2d^2(u_i/G)). \]

\[ = m_1 \left( M_1(H) + n_2(n_2 + 1)^2 + 4m_2(n_2 + 1) + F(H) + 2M_2(H) \right) + F(G) + 2m_1n_2 + 2n_2M_1(G). \]

Adding \( R_1 \) and \( R_2 \), we get the desired result. \( \square \)

Using Theorem 11, we obtain the following results.

**Example 6.**

(i) \( M_1^e(P_n \Join P_m) = (m^3n + 13m^2n + 38mn - m^3 - 13m^2 - 42m - 30n + 22) \)

(ii) \( M_1^e(C_n \Join C_m) = n(m^3 + 9m^2 + 46m + 16) \)

(iii) \( M_1^e(C_n \Join P_m) = n(m^3 + 9m^2 + 42m - 42) \).

**Theorem 12.** The upper and lower bound of the FEZI for \( G \Join H \) are given by

\[ X_1 \geq M_1^e(G \Join H) \geq X_2, \text{ where } X_1 = m_1n_2(\Delta_2 + n_2)(\Delta_2 + n_2 + 3) + 2m_1m_2(2\Delta_2^2 + \Delta_2 + 1) + 2m_1(\Delta_1 + n_2)(\Delta_1 + n_2 + 1) + 4m_1(n_2 + 1), \text{ and } X_2 = m_1n_2(\delta_2 + n_2)(\delta_2 + n_2 + 3) + 2m_1m_2(2\delta_2^2 + \delta_2 + 1) + 2m_1(\delta_1 + n_2)(\delta_1 + n_2 + 1) + 4m_1(n_2 + 1). \]

The equality holds if and only if \( G \) and \( H \) are regular graphs.

**Proof.** Similar proof to the Theorem 4. \( \square \)

![Figure 3. An example of various types of Corona products based on S-graphs of two graphs \( P_4 \) and \( P_3 \) like as (i) SVCP, (ii) SECP, (iii) SVNCP and (iv) SENCP.](image-url)

**Definition 8.** [10] The SVNCP of G and H, denoted by $G \boxtimes H$, is the graph obtained from $S(G)$ and $n_1$ copies of $H$, all vertex-disjoint, and joining the neighbors of the $i^{th}$ vertex of $V(G)$ to every vertex in the $i^{th}$ copy of $H$.

The graph $G \boxtimes H$ has $(n_1 + m_1 + n_1 n_2)$ vertices and $(2m_1 + n_1 m_2 + 2m_1 n_2)$ edges (see Fig. 3).

Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}, I(G) = \{v_{e_1}, v_{e_2}, \ldots, v_{e_{m_1}}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$. Also, let $V(H^i) = \{v_i^1, v_i^2, \ldots, v_i^{n_2}\}$ be the vertex set of the $i^{th}$ copy of $H$, for $i = 1, 2, \ldots, n_1$.

So, $V(G \boxtimes H) = V(G) \cup I(G) \cup \{V(H^1) \cup V(H^2) \cup \ldots \cup V(H^{n_1})\}$ and $E(G \boxtimes H) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{v_i^j v_j^k \in E(G \boxtimes H) | v_i^j, v_j^k \in V(H^i)\}$, $E_2 = \{u_i u_{e_k} \in E(G \boxtimes H) | u_i \in V(G), u_{e_k} \in I(G)\}$ and $E_3 = \{v_i^j v_j^k \in E(G \boxtimes H) | u_{e_k} \in I(G), v_j^i \in V(H^i)\}$.

The degrees of the vertices of $G \boxtimes H$ are given by

$$
\begin{align*}
&d(u_i/G \boxtimes H) = d(u_i/G) \quad \text{for } i = 1, 2, \ldots, n_1, \\
&d(u_{e_k}/G \boxtimes H) = 2(n_2 + 1) \quad \text{for } i = 1, 2, \ldots, m_1, \\
&d(v_j^i/G \boxtimes H) = d(v_j^i/H) + d(u_i/G) \quad \text{for } i = 1, 2, \ldots, n_1 \text{ and } j = 1, 2, \ldots, n_2.
\end{align*}
$$

(8)

In the following Theorem, we obtain the FEZI for SVNCP of two graphs.

**Theorem 13.** The FEZI of $G \boxtimes H$ is given by

$$
M_1^f(G \boxtimes H) = n_1 EM_1(H) + (n_2 + 1) F(G) + (4n_2^2 + 12m_2 + n_2 + 1) M_1(G) + (10m_1 + n_1) M_1(H) + 4m_1 n_2^2(2n_2 + 3) + 4m_1(2n_2 + 1) + 8m_1 m_2(2n_2 - 1).
$$

**Proof.** From the Definition 1 and the Equation 8, the first entire Zagreb index of $G \boxtimes H$ is

$$
M_1^f(G \boxtimes H) = M_1(G \boxtimes H) + EM_1(G \boxtimes H) = \sum_{v_j^i v_j^k \in E_1} \left( d(v_j^i/(G \boxtimes H)) + d(v_j^k/(G \boxtimes H)) \right) + \sum_{u_i u_{e_k} \in E_2} \left( d(u_i/(G \boxtimes H)) + d(u_{e_k}/(G \boxtimes H)) \right) + \sum_{u_{e_k} v_j^i \in E_3} \left( d(u_{e_k}/(G \boxtimes H)) + d(v_j^i/(G \boxtimes H)) \right) - 2^2 + \sum_{u_i u_{e_k} \in E_2} \left( d(u_i/(G \boxtimes H)) + d(u_{e_k}/(G \boxtimes H)) - 2 \right)^2 + \sum_{u_{e_k} v_j^i \in E_3} \left( d(u_{e_k}/(G \boxtimes H)) + d(v_j^i/(G \boxtimes H)) - 2 \right)^2
$$
Theorem 14. The bounds for the FEZI of $G \square H$ are given by $Y_1 \geq M_1^e(G \square H) \geq Y_2$, where

$$Y_1 = 2m_2n_1(\Delta_1 + \Delta_2) \left( 2(\Delta_1 + \Delta_2) - 3 \right) + 2m_1n_2(\Delta_1 + \Delta_2 + 2n_2)(\Delta_1 + \Delta_2 + 2n_2 + 1) + 2m_1(\Delta_1 + 2n_2)(\Delta_1 + 2n_2 + 1) + 4(m_1n_2 + m_2n_1) + 4m_1 \quad \text{and} \quad Y_2 = 2m_2n_1(\delta_1 + \delta_2) \left( 2(\delta_1 + \delta_2) - 3 \right) + 2m_1n_2(\delta_1 + \delta_2 + 2n_2)(\delta_1 + \delta_2 + 2n_2 + 1) + 2m_1(\delta_1 + 2n_2)(\delta_1 + 2n_2 + 1) + 4(m_1n_2 + m_2n_1) + 4m_1.$$

Proof. In a manner analogous to the proof of the Theorem 4, we can establish the above Theorem.

3.8. The Subdivision-Edge Neighborhood Corona product(SENCP). Liu et al. [10] defined four new graph operations based on $S$-graphs. The SENCP of two graphs is one out of four operations.

Definition 9. [10] The SENCP of $G$ and $H$, denoted by $G \square H$, is a new graph achieved from $S(G)$ and $|I(G)|$ copies of $H$, all vertex-disjoint, and by adjoining the neighbors of the $i^{th}$ vertex of $I(G)$ to every vertex in the $i^{th}$ copy of $H$. 
It is clear that the graph $G \Box H$ has $|V(G)| + |E(G)| + |E(G)||V(H)|$ vertices and $|E(G)||E(H)| + 2|E(G)| + 2|E(G)||V(H)|$ edges, (see Fig. 3).

Let $V(G) = \{u_1, u_2, \ldots, u_m\}, I(G) = \{e_1, e_2, \ldots, e_m\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Also, for $i = 1, 2, \ldots, m_1$ let $V(H^i) = \{v^i_1, v^i_2, \ldots, v^i_{n_2}\}$ be the vertex set of the $i^{th}$ copy of $H$.

So, $V(G) = V(G) \cup I(G)$ and $V(G \Box H) = V(G) \cup I(G) \cup (V(H^1) \cup V(H^2) \cup \ldots \cup V(H^{m_1}))$ and $E(G \Box H) = E_1^\Box \cup E_2^\Box \cup E_3^\Box$ where $E_1^\Box = \{v^i_jv^i_k \in E(G \Box H) | v^i_j, v^i_k \in V(H^i) \} \text{ for } i = 1, 2, \ldots, m_1$ and $j, k = 1, 2, \ldots, n_2$.

$E_2^\Box = \{u_iu_{e_k} \in E(G \Box H) | u_i \in V(G), u_{e_k} \in I(G) \text{ for } i = 1, 2, \ldots, n_1 \text{ and } k = 1, 2, \ldots, m_1\}$ and $E_3^\Box = \{u_kv^i_j \in E(G \Box H) | u_k \in V(G), v^i_j \in V(H^i) \text{ for } i = 1, 2, \ldots, n_1 \text{ and } j = 1, 2, \ldots, n_2\}$.

The degrees of the vertices of $G \Box H$ are given by

\begin{equation}
\begin{align*}
\sum d(u_i/G \Box H) &= (n_2 + 1)d(u_i/G) \text{ for } i = 1, 2, \ldots, n_1, \\
\sum d(e_i/G \Box H) &= 2 \text{ for } i = 1, 2, \ldots, m_1, \\
\sum d(v^i_j/G \Box H) &= d(v^i_j/H) + 2 \text{ for } i = 1, 2, \ldots, n_1 \text{ and } j = 1, 2, \ldots, n_2.
\end{align*}
\end{equation}

We obtain the FEZI for the SENCPS of two graphs.

**Theorem 15.** The FEZI for the SENCPS of $G$ and $H$ is given by

\[ M'_1(G \Box H) = (n_2 + 1)^3F(G) + m_1EM_1(H) + (n_2 + 1)(n_2 + 4m_2 + 1)M_1(G) + 11m_1M_1(H) + 8m_1m_2 + 4m_1n_2 + 4m_1. \]

**Proof.** Using the Proposition 1, we obtain

\[
M'_1(G \Box H) = M_1(G \Box H) + EM_1(G \Box H)
\]

\[
= \sum_{v_j^i k^i \in E_1^\Box} \left( d(v_j^i/(G \Box H)) + d(v_k^i/(G \Box H)) \right) + \sum_{u_i u_{e_k} \in E_2^\Box} \left( d(u_i/(G \Box H)) + d(u_{e_k}/(G \Box H)) \right)
\]

\[
+ \sum_{u_i u_{e_k} \in E_3^\Box} \left( d(u_i/(G \Box H)) + d(u_{e_k}/(G \Box H)) \right) + \sum_{v_j^i k^i \in E_1^\Box} \left( d(v_j^i/(G \Box H)) + d(v_k^i/(G \Box H)) - 2 \right)^2
\]

\[
+ \sum_{u_i u_{e_k} \in E_2^\Box} \left( d(u_i/(G \Box H)) + d(u_{e_k}/(G \Box H)) - 2 \right)^2 + \sum_{u_i u_{e_k} \in E_3^\Box} \left( d(u_i/(G \Box H)) + d(u_{e_k}/(G \Box H)) - 2 \right)^2
\]

\[
= \sum_{i=1}^{m_1} \sum_{v_j^i k^i \in E(H)} \left( d(v_j/H) + d(v_k/H) + 4 \right) d(u_i/G) + \sum_{i=1}^{n_1} \left( (n_2 + 1)d(u_i/G) + 2 \right) d(u_i/G)
\]

\[
+ \sum_{k=1}^{n_2} \sum_{j=1}^{n_2} \left( n_2 + 1)d(u_k/G) + d(v_j/H) + 2 \right) d(u_k/G)
\]
The bounds for $M_n$ (Theorem 16).

Example 8. (i) $M^f_1(P_n \square P_m) = (8m^3n + 44m^2n + 92mn - 14m^3 - 72m^2 - 114m - 84n + 84)$

(ii) $M^f_1(C_n \square C_m) = 4n(2m^3 + 11m^2 + 27m + 4)$

(iii) $M^f_1(C_n \square P_m) = 4n(2m^3 + 11m^2 + 23m - 21).

Theorem 16. The bounds for $M^f_1(G \square H)$ are calculated as

$Z_1 \geq M^f_1(G \square H) \geq Z_2$, where $Z_1 = 2m_1m_2(2\Delta^2 + 5\Delta_2 + 4) + 2m_1(n_2 + 1)(n_2 + 1)\Delta_1 + 1)\Delta_1 + 2m_1n_2\left((n_2 + 1)\Delta_1 + \Delta_2\right)\left((n_2 + 1)\Delta_1 + \Delta_2 + 1\right) + 4m_1(n_2 + 1)$, and for the lower bound

$Z_2 = 2m_1m_2(2\Delta^2 + 5\Delta_2 + 4) + 2m_1(n_2 + 1)\left((n_2 + 1)\Delta_1 + \Delta_2\right)\left((n_2 + 1)\Delta_1 + \Delta_2 + 1\right) + 4m_1(n_2 + 1)$. The equality hold if and only if $G$ and $H$ are regular graphs.

Proof. The proof is analogous to that of the Theorem 2. \hfill \square


Definition 10. [14] The VECP of two graphs $G$ and $H$ is denoted by $G \bullet H$, is a new graph created by taking one copy of $G$, $|V(G)|$ copies of $H$ and $|E(G)|$ copies of $H$, along with joining $i^{th}$ vertex of $G$ to every vertex in the $i^{th}$ vertex copy of $H$ and also joining end vertices of $j^{th}$ edge of $G$ to every vertex in the $j^{th}$ edge copy of $H$, where $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, m_1$.

Let $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$, $E(G) = \{e_1, e_2, \ldots, e_{m_1}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$ and $E(H) = \{e_1^*, e_2^*, \ldots, e_{m_2}^*\}$. We denote, the vertex set of the $i^{th}$ vertex copy of $H$ by $V_{Vi}(H) = \{v^i_1, v^i_2, \ldots, v^i_{n_2}\}$ and the vertex set of the $j^{th}$ edge copy of $H$ by
\(V_e(H) = \{v_{j_1}^{e_j}, v_{j_2}^{e_j}, \ldots, v_{j_m}^{e_j}\}\). Also, we denote by \(E_{e_j}(H)\) and \(E_{v_i}(H)\), the edge set of \(j\)th edge and \(i\)th vertex copy of \(H\), respectively. Then \(V(G \bullet H) = \bigcup_{i=1}^{n_i} V_i(H) \cup \bigcup_{j=1}^{m_j} V_{e_j}(H)\) and \(E(G \bullet H) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5\), where \(E_1 = \{u_i \in E(G \bullet H) | u_i, u_j \in V(G); i \neq j\}\) and \(i, j = 1, 2, \ldots, n_1\), \(E_2 = \{v_{j_k}^{e_j} \in E(G \bullet H) | v_{j_k}^{e_j} \in V_{e_j}(H)\} or \{v_{j_k}^{e_j} \in E(G \bullet H) | v_{j_k}^{e_j} \in V_{e_j}(H)\}\) and \(j_k = 1, 2, \ldots, m_1\) and \(k \neq l; k, l = 1, 2, \ldots, n_2\), \(E_3 = \{u_i \in V(G) \cup V(\cdot) | u_i \in V(G), v_{j_k}^{e_j} \in V_{e_j}(H)\}\) for \(i = 1, 2, \ldots, n_1; j = 1, 2, \ldots, m_1\) and \(k = 1, 2, \ldots, n_2\), \(E_4 = \{v_{j_k}^{e_j} v_{l_k}^{e_j} \in E(G \bullet H) | v_{j_k}^{e_j} v_{l_k}^{e_j} \in V_{v_i}(H)\}\) or \{\(v_{j_k}^{e_j} v_{l_k}^{e_j} \in E(G \bullet H) | v_{j_k}^{e_j} v_{l_k}^{e_j} \in V_{v_i}(H)\)\} and \(j_k = 1, 2, \ldots, m_1; j = 1, 2, \ldots, n_2\) and 

\(E_5 = \{u_i v_{j_k}^{e_j} \in E(G \bullet H) | u_i \in V(G), v_{j_k}^{e_j} \in V_{v_i}(H)\}\). The graph \(G \bullet H\) has \((|V(G)| + |E(G)|||V(H)| + |V(G)||V(H)|)\) vertices and \((|E(G)| + |V(G)||(|E(H)| + |V(H)|) + |E(G)||(|E(H)| + 2|V(H)|))\) edges, (see Fig. 4).

The degrees of the vertices of \(G \bullet H\) are given by

\[
\begin{align*}
d(u_i / G \bullet H) &= (n_2 + 1)d(u_i / G) + n_2, \forall u_i \in V(G) \text{ for } i = 1, 2, \ldots, n_1, \\
d(v_{j_k}^{e_j} / G \bullet H) &= d(v_k / H) + 2, \forall v_{j_k}^{e_j} \in V_{e_j}(H) \text{ for } j = 1, 2, \ldots, m_1 \text{ and } k = 1, 2, \ldots, n_2, \\
d(v_{j_k}^{e_j} / G \bullet H) &= d(v_k / H) + 1, \forall v_{j_k}^{e_j} \in V_{v_i}(H) \text{ for } i = 1, 2, \ldots, n_1 \text{ and } k = 1, 2, \ldots, n_2.
\end{align*}
\]

In the following Theorem, the explicit expression of FEZI for \((G \bullet H)\) is computed.
Theorem 17. The FEZI of $G \bullet H$ is given by

$$M_1^e(G \bullet H) = (n_2 + 1)^3F(G) + (m_1 + n_1)F(H) + (n_2 + 1)(3n_2^2 + 6n_2 + 4m_2 - 3)M_1(G) + (7m_1 + 2n_1)M_1(H) + 2(n_2 + 1)^2M_2(G) + 2(m_1 + n_1)M_2(H) + 8m_1n_2^2 - n_1n_2^2 + 6m_1n_2^3 + n_1n_2^3 - 4m_1n_2 + 2n_1n_2 + 4m_2n_2 + 16m_1m_2n_2 + 20m_1m_2 + 4m_1.

Proof. By the Proposition 1 and also from the Equation 10, we have

$$M_1^e(G \bullet H) = M_1(G \bullet H) + EM_1(G \bullet H)$$

$$= \left( \sum_{u_i \in V(G)} d^2(u_i/(G \bullet H)) + \sum_{j=1}^{m_1} \sum_{v_{jk} \in V_{e_j}(H)} d^2(v_{jk}^e/(G \bullet H)) + \sum_{u_i \in V(G)} \sum_{v_{ik} \in V_{e_i}(H)} d^2(v_{ik}^e/(G \bullet H)) \right)$$

$$+ \sum_{u_i, v_j \in E_1} \left( d(u_i/(G \bullet H)) + d(v_j/(G \bullet H)) - 2 \right)^2 + \sum_{v_{jk} \in E_2} \left( d(v_{jk}/(G \bullet H)) + d(v_{jk}^e/(G \bullet H)) - 2 \right)^2$$

$$+ \sum_{u_i, u_j, v_{ik} \in E_5} \left( d(u_i/(G \bullet H)) + d(v_{ij}^n/(G \bullet H)) - 2 \right)^2$$

$$+ J_1 + J_2 + J_3 + J_4 + J_5 + J_6$$

$$J_1 = \left( \sum_{u_i \in V(G)} \left( (n_2 + 1)d(u_i/G) + n_2 \right)^2 + \sum_{j=1}^{m_1} \sum_{v_{ik} \in V_{e_i}(H)} \left( d(v_{ik}/H) + 2 \right)^2 + \sum_{u_i \in V(G)} \sum_{v_{ik} \in V_{e_i}(H)} \left( d(v_{ik}/H) + 1 \right)^2 \right)$$

$$= \sum_{u_i \in V(G)} \left( (n_2 + 1)^2d^2(u_i/G) + 2n_2(n_2 + 1)d(u_i/G) + n_2^2 \right) + \sum_{j=1}^{m_1} \sum_{v_{ik} \in V_{e_i}(H)} \left( d^2(v_{ik}/H) + 4d(v_{ik}/H) + 4 \right)$$

$$+ \sum_{u_i \in V(G)} \sum_{v_{ik} \in V_{e_i}(H)} \left( d^2(v_{ik}/H) + 2d(v_{ik}/H) + 1 \right)$$

$$= (n_2 + 1)^2M_1(G) + 4m_1n_2(n_2 + 1) + n_1n_2^2 + m_1M_1(H) + 4m_1n_2 + 8m_1m_2 + n_1M_1(H) + n_1n_2 + 4m_2n_1.$$

$$J_2 = \sum_{u_i, u_j \in E_1} \left( d(u_i/(G \bullet H)) + d(v_j/(G \bullet H)) - 2 \right)^2$$

$$= \sum_{u_i, u_j \in E(G)} \left( (n_2 + 1)d(u_i/(G)) + (n_2 + 1)d(v_j/(G)) + 2n_2 - 2 \right)^2$$

$$= \sum_{u_i, u_j \in E(G)} \left( (n_2 + 1)^2(d(u_i/G) + d(u_j/G) - 2)^2 + 8n_2(n_2 + 1)(d(v_i/(G) + d(v_j/G) - 2)^2 +$$
Also, \( J_3 = \sum_{v_jkE_j(H)} \left\{ d(v_k/H) + d(v_j/H) + 2 \right\}^2 \)
\[ = \sum_{j=1}^{m_1} \sum_{v_jkE_j(H)} \left\{ d(v_k/H) + d(v_j/H) + 2 \right\}^2 \]
\[ = \sum_{j=1}^{m_1} \left\{ \left( d^2(v_k/H) + d^2(v_j/H) + 4d(v_k/H) + 2d(v_k/H)d(v_j/H) + 4 \right) \right\} \]
\[ = m_1 F(H) + 2m_1 M_2(H) + 4m_1 M_1(H) + 4m_1 m_2. \]

Similarly, \( J_4 = \sum_{u_iV_jkE_j(H)} \left\{ d(u_i/G) + d(v_jk/H) - 2 \right\}^2 \)
\[ = \sum_{u_i \in V(G)} \sum_{v_jkE_j(H)} \left\{ (n_2 + 1)d(u_i/G) + d(v_k/H) + n_2 \right\}^2 d(u_i/G) \]
\[ = \sum_{u_i \in V(G)} \sum_{v_jkE_j(H)} \left\{ (n_2 + 1)^2d^2(u_i/G) + d^2(v_k/H) + n_2^2 + 2n_2(n_2 + 1)d(u_i/G) + 2(n_2 + 1)d(u_i/G)d(v_k/H) + 2n_2d(v_k/H) \right\} d(u_i/G) \]
\[ = n_2(n_2 + 1)^2 F(G) + 2m_1 n_2^2 + 2m_1 M_1(H) + 2n_2(n_2 + 1)M_1(G) + 8m_1 m_2 n_2 + 4m_2(n_2 + 1)M_1(G). \]

Next, \( J_5 = \sum_{v_jkE_j(H)} \left\{ d(v_jk/H) - 2 \right\}^2 \)
\[ = \sum_{u_i \in V(G)} \sum_{v_jkE_j(H)} \left\{ d(v_jk/H) + d(v_k/H) \right\}^2 \]
\[ = n_1 F(H) = 2n_1 M_2(H). \]

Finally, \( J_6 = \sum_{u_iV_jkE_j(H)} \left\{ d(u_i/G) + d(v_jk/H) - 2 \right\}^2 \)
\[ = \sum_{u_i \in V(G)} \sum_{v_jkE_j(H)} \left\{ (n_2 + 1)d(u_i/G) + d(v_j/H) + n_2 - 1 \right\}^2 \]
\[ = \sum_{u_i \in V(G)} \sum_{v_jkE_j(H)} \left\{ (n_2 + 1)^2d^2(u_i/G) + d^2(v_j/H) + (n_2 - 1)^2 + 2(n_2 + 1)d(u_i/G)d(v_j/H) + 2(n_2 - 1)d(u_i/G) + 2(n_2 - 1)d(v_j/H) \right\} \]
\[ = n_2(n_2 + 1)^2 M_1(G) + n_1 M_1(H) + n_1 n_2(n_2 - 1)^2 + 8m_1 m_2(n_2 + 1) + 4m_1 n_2(n_2^2 - 1) + 4n_1 m_2(n_2 - 1). \]
By adding $J_1, J_2, J_3, J_4, J_5$ and $J_6$, we get the desired result.

From the above Theorem 17, we get the following results.

**Example 9.**

(i) $M_1^G(C_n \bullet C_m) = n(27m^3 + 111m^2 + 154m + 8)$

(ii) $M_1^G(P_n \bullet P_m) = (27m^3n + 111m^2n + 118mn - 38m^3 - 160m^2 - 136m - 142n + 100)$

(iii) $M_1^G(C_n \bullet P_m) = n(27m^3 + 111m^2 + 118m - 142)$.

**Theorem 18.** The bounds for the FEZI of $G \bullet H$ are given by

\[
\alpha \geq M_1^G(G \bullet H) \geq \beta,
\]

where \( \alpha = 2m_1\left( (n_2 + 1)\Delta_1 + n_2 \right) \left( 2(n_2 + 1)\Delta_1 + 2n_2 - 3 \right) + 2m_1m_2 \left( 2\Delta_2^2 + 5\Delta_2 + 4 \right) + 2m_1n_2 \left( (n_2 + 1)\Delta_1 + \Delta_2 + n_2 \right) \left( (n_2 + 1)\Delta_1 + \Delta_2 + n_2 + 1 \right) + 2m_2n_1 \left( 2\Delta_2^2 + \Delta_2 + 1 \right) + n_1n_2 \left( (n_2 + 1)\Delta_1 + \Delta_2 + n_2 \right) \left( (n_2 + 1)\Delta_1 + \Delta_2 + n_2 - 1 \right) + 2n_1n_2 + 4m_1(n_2 + 1) \), and \( \beta = 2m_1 \left( (n_2 + 1)\delta_1 + n_2 \right) \left( 2(n_2 + 1)\delta_1 + 2n_2 - 3 \right) + 2m_1m_2 \left( 2\delta_2^2 + 5\delta_2 + 4 \right) + 2m_1n_2 \left( (n_2 + 1)\delta_1 + \delta_2 + n_2 \right) \left( (n_2 + 1)\delta_1 + \delta_2 + n_2 + 1 \right) + 2m_2n_1 \left( 2\delta_2^2 + \delta_2 + 1 \right) + n_1n_2 \left( (n_2 + 1)\delta_1 + \delta_2 + n_2 \right) \left( (n_2 + 1)\delta_1 + \delta_2 + n_2 - 1 \right) + 2n_1n_2 + 4m_1(n_2 + 1) \).

The equality holds if and only if $G$ and $H$ are regular graphs.

**4. Conclusion**

In this paper, we have established some exact formulas with examples for the FEZI of several types of Corona product of two graphs based on $R$-graphs and $S$-graphs such as RVCP, RECP, RVNCP, RENCP and SVCP, SECP, SVNCP and SENC, respectively. Additionally, we have determined the FEZI of VECP of two graphs. As an application, we have obtained the lower and upper bounds for the FEZI of each Corona product of the two graphs. Also, we have applied our results to find the FEZI of several chemically interesting molecular graphs. We have proposed some possible directions for future research.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

**References**