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QUALITATIVE PROPERTIES OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEM INVOLVING ψ-CAPUTO FRACTIONAL DERIVATIVE

J.A. NANWARE[1,*], MADHURI N. GADSING[2]

[1]Department of Mathematics, Shrikrishna Mahavidyalaya, Gunjoti- 413606,
Dist. Osmanabad (M.S)-INDIA

[2]Department of Mathematics, Jawahar Arts,Science and Commerce College, Anadur-413 603,
Dist.Osmanabad, (M.S)-INDIA

Abstract. This paper studies qualitative properties such as existence-uniqueness of solutions for nonlinear boundary value problems involving ψ-Caputo fractional derivative. Comparison result is proved for differential equations involving ψ-Caputo fractional derivative. Monotone iterative technique coupled with method of lower-upper solutions is used to establish the existence and uniqueness of solutions for nonlinear boundary value problems involving ψ-Caputo fractional derivative.

Keywords: ψ-Caputo fractional derivative; monotone iterative technique; lower-upper solutions; minimal and maximal solutions.

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1. Introduction

Theory of fractional differential equations occur frequently in different research areas and engineering, such as Chemistry, Biology, Physics, fields of control, electromagnetic etc.(see in [12, 15, 19, 21]). Literature includes several collection of work on distinct fractional derivatives

*Corresponding author
E-mail address: jag_skmg91@rediffmail.com
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Riemann-Liouville [12, 21], the Caputo [22], the Hadamard [1] and the ψ-Caputo derivative and integrals [2, 3, 4, 5]. During last decade, many researchers paid attention to the study of existence and uniqueness of solutions of initial value problems [6, 8], boundary value problems [2, 10] for nonlinear fractional differential equations . On the other hand, monotone iterative method combined with the method of upper and lower solutions for nonlinear fractional differential equations have been used by several researchers [6, 13, 14, 16, 17, 18, 23].

In 2020, Derbazi et al. [7] developed monotone iterative technique to study the existence and uniqueness of solution for initial value problem of nonlinear fractional differential equations including ψ-Caputo derivative. Dhaigude et. al. [11] have proved the existence and uniqueness of solution of nonlinear boundary value problems for ψ-Caputo fractional differential equations by applying monotone iterative technique. Abdo et. al. [2] investigates the existence and uniqueness of solutions of boundary value problems for ψ-Caputo fractional differential equations.

Motivated by their works, we consider in this paper, the existence and uniqueness of solutions of the following nonlinear boundary value problems [BVP] involving ψ-Caputo fractional derivative

$$\begin{aligned} {}^cD^{\mu,\psi}_{a^+}z(r) &= f(r,z(r)), \quad r \in J = [a,b], \\ z(a) &= a^*, \quad z(b) = b^*, \end{aligned} \tag{1.1}$$

where ${}^cD^{\mu,\psi}_{a^+}$ is the $\psi-$Caputo fractional derivative of order $0 < \mu \leq 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a^*, b^* \in \mathbb{R}$. Monotone iterative technique combined with coupled lower-upper solutions is developed for nonlinear BVP (1.1) and qualitative properties of solutions such as existence-uniqueness are obtained.

The rest of the paper is organized as follows: In second section, some basic definitions and useful lemmas are given. In third section, lower-upper solutions of boundary value problem for ψ-Caputo fractional differential equation are introduced. Monotone technique is developed and successfully applied to obtain existence-uniqueness of solution of nonlinear BVP (1.1).

2. Preliminaries

In this section, we deduce some preliminary results required in the next section to attain existence and uniqueness results for nonlinear BVP (1.1) involving ψ-Caputo fractional derivative. Let $J=[a,b]$, where $0 \leq a < b < \infty$, be a finite interval and $\psi : J \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(r) \neq 0$,for all $r \in J$.

Definition 2.1. [4] *The left-sided ψ-Riemann-Liouville fractional integral of order $\mu > 0$ for an integrable function $z : J \to \mathbb{R}$ with respect to function ψ is defined as follows*

$$I_{a^+}^{\mu,\psi} z(r) = \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} z(s)\, ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 2.2. [4] *Let $n \in N$ and let $\psi, z \in C^n(J,\mathbb{R})$ be two functions. The left-sided ψ-Riemann-Liouville fractional derivative of function z of order $n-1 < \mu < n$ with respect to another function ψ is defined by*

$$\begin{aligned} D_{a^+}^{\mu,\psi} z(r) &= \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n I_{a^+}^{n-\mu,\psi} z(r) \\ &= \frac{1}{\Gamma(n-\mu)} \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{n-\mu-1} z(s)\, ds, \end{aligned}$$

where $n = [\mu]+1$ and $[\mu]$ denotes the integer part of the real number μ.

Definition 2.3. [4] *Let $n \in N$ and let $\psi, z \in C^n(J,\mathbb{R})$ be two functions. The left-sided ψ-Caputo fractional derivative of z of order $n-1 < \mu < n$ with respect to another function ψ is defined by*

$${}^c D_{a^+}^{\mu,\psi} z(r) = I_{a^+}^{n-\mu,\psi} \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n z(r),$$

where $n = [\mu]+1$ for $\mu \notin \mathbb{N}$, $n = \mu$ for $\mu \in \mathbb{N}$.

To simplify notation, we will use the abbreviated symbol

$$z_{\psi}^{[n]}(r) = \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n z(r).$$

From the definition, it is clear that

$$
{}^{c}D_{a^+}^{\mu,\psi}z(r) = \begin{cases} \frac{1}{\Gamma(n-\mu)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{n-\mu-1}z_{\psi}^{[n]}(s)\,ds, & \text{if} \quad \mu \notin \mathbb{N} \\ z_{\psi}^{[n]}(r), & \text{if} \quad \mu \in \mathbb{N}. \end{cases} \tag{2.1}
$$

Note that if $z \in C^n(J,\mathbb{R})$ the ψ-Caputo fractional derivative of $z(r)$ of order μ is defined in terms of left-sided ψ-Riemann-Liouville fractional derivative as

$$
{}^{c}D_{a^+}^{\mu,\psi}z(r) = D_{a^+}^{\mu,\psi}\left[z(r) - \sum_{k=0}^{n-1}\frac{z_{\psi}^{[k]}(a)}{k!}[\psi(r)-\psi(a)]^k\right].
$$

Lemma 2.1. [4] *Let $\mu,\nu > 0$, and $z \in L^1(J,\mathbb{R})$. Then*

$$
I^{\mu,\psi}I^{\nu,\psi}z(r) = I^{\mu+\nu,\psi}z(r)\,a.e.,\, r \in J.
$$

In particular, if $z \in C(J,\mathbb{R})$, then $I^{\mu,\psi}I^{\nu,\psi}z(r) = I^{\mu+\nu,\psi}z(r)$, $r \in J$.

Lemma 2.2. [4] *Let $\mu > 0$. The following holds:*

If $z(r) \in C(J,\mathbb{R})$ then

$$
{}^{c}D_{a^+}^{\mu,\psi}I^{\mu,\psi}z(r) = z(r),\, r \in J.
$$

If $z \in C^{n-1}(J,\mathbb{R})$, $n-1 < \mu < n$, then

$$
I_{a^+}^{\mu,\psi\,c}D_{a^+}^{\mu,\psi}z(r) = z(r) - \sum_{k=0}^{n-1}\frac{z_{\psi}^{[k]}(a)}{k!}[\psi(r)-\psi(a)]^k, \quad r \in J.
$$

Lemma 2.3. [12] *For $r > a$, $\mu \geq 0$, and $\nu > 0$, we have*

(1) $I_{a^+}^{\mu,\psi}[\psi(r)-\psi(a)]^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)}[\psi(r)-\psi(a)]^{\nu+\mu-1}$,

(2) ${}^{c}D_{a^+}^{\mu,\psi}[\psi(r)-\psi(a)]^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}[\psi(r)-\psi(a)]^{\nu-\mu-1}$

(3) ${}^{c}D_{a^+}^{\mu,\psi}[\psi(r)-\psi(a)]^k = 0$*, for all* $k \in \{0,1,\ldots,n-1\}$, $n \in \mathbb{N}$.

Lemma 2.4. [2] *If $\mu > 0$ and $z,\psi \in C[a,b]$, then*

(1) $I_{a^+}^{\mu,\psi}(.)$ *is linear and bounded from $C[a,b]$ to $C[a,b]$.*

(2) $I_{a^+}^{\mu,\psi}z(a) = \lim_{r\to a} I_{a^+}^{\mu,\psi}z(r) = 0$.

Definition 2.4. [12] *The one-parameter Mittag-Leffler function $E_\mu(.)$, is defined as*

$$
E_\mu(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+1)},\, (z \in \mathbb{R},\, \mu > 0),
$$

and the two-parameter Mittag-Leffler function $E_{\mu,\nu}(.)$, is defined as

$$E_{\mu,\nu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\nu)},(z\in\mathbb{R},\,\mu,\nu>0).$$

Theorem 2.1. *(Weissinger's fixed point theorem)*[9] *Assume (A,ρ) to be a non empty complete metric space and let θ_i for every $i\in\mathbb{N}$ such that $\sum_{i=0}^{\infty}\theta_i$ converges. Furthermore, let the mapping $T:A\to A$ satisfy the inequality*

$$\rho(T^ix,T^iy)\leq\theta_i\rho(x,y),$$

for every $i\in\mathbb{N}$ and every $x,y\in A$. Then T has a unique fixed point x^*. Moreover, for any $x_0\in A$, the sequence $\{T^ix_0\}_{i=1}^{\infty}$ converges to fixed point x^*.

3. Main Results

In this section, we develop monotone iterative scheme and prove the existence and uniqueness of solution of the nonlinear BVP (1.1) involving ψ-Caputo fractional derivative.

Lemma 3.1. [2] *Let $n-1<\mu<n$, $g\in C(J,\mathbb{R})$ and ψ is increasing and $\psi'(r)\neq 0$, for all $r\in J$. A function $z(r)\in C^n[a,b]$ is a solution of the fractional boundary value problem*

$$\begin{aligned}&{}^cD_{a^+}^{\mu,\psi}z(r)=g(r),\qquad r\in J,\\&z_{\psi}^{[k]}(a)=z_a^k,\quad k=0,1,2,\dots n-2;z_{\psi}^{[n-1]}(b)=z_b,\end{aligned}$$

where z_a^k, $z_b\in\mathbb{R}$, if and only if $z(r)$ satisfies the following fractional integral equation

$$\begin{aligned}z(r)=&\sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k+\left[\frac{z_b}{(n-1)!}+\frac{g(a)[\psi(b)-\psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)}\right][\psi(r)-\psi(a)]^{n-1}\\&-\frac{[\psi(r)-\psi(a)]^{n-1}}{(n-1)!\Gamma(\mu-n+1)}\int_a^b\psi'(s)[\psi(b)-\psi(s)]^{\mu-n}g(s)\,ds\\&+\frac{1}{\Gamma(\mu)}\int_a^r\psi'(s)[\psi(r)-\psi(s)]^{\mu-1}g(s)\,ds.\end{aligned}$$

Lemma 3.2. *For a given $g\in C(J,R)$ and $\mu\in(n-1,n)$, with $n\in\mathbb{N}$. A function $z(r)\in C^{n-1}[a,b]$ is the solution of the linear fractional boundary value problem*

$$\begin{aligned}&{}^cD_{a^+}^{\mu,\psi}z(r)+mz(r)=g(r),r\in J=[a,b],\\&z_{\psi}^{[k]}(a)=z_a^k,\quad k=0,1,2,\dots n-2;z_{\psi}^{[n-1]}(b)=z_b,\end{aligned}\tag{3.1}$$

where $z_a^k, z_b \in \mathbb{R}$, if and only if $z(r)$ satisfies the following fractional integral equation

$$\begin{aligned} z(r) = &\sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} \\ &+ \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}[g(s)-mz(s)]\,ds, \end{aligned} \tag{3.2}$$

where,

$$\begin{aligned} M = &\frac{z_b}{(n-1)!} + \frac{[g(a)-mz(a)][\psi(b)-\psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)} \\ &- \frac{1}{(n-1)!\Gamma(\mu-n+1)} \int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds. \end{aligned}$$

Proof. First assume that $z(r) \in C^{n-1}[a,b]$ be a solution to problem (3.1). By Lemma 2.2

$$\begin{aligned} z(r) = &c_0 + c_1[\psi(r)-\psi(a)] + c_2[\psi(r)-\psi(a)]^2 + \ldots + c_{n-1}[\psi(r)-\psi(a)]^{n-1} \\ &+ \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}[g(s)-mz(s)]\,ds. \end{aligned} \tag{3.3}$$

Using (3.3) we get,

$$z_\psi^{[0]}(a) = z_a^0 = z_a = c_0.$$

Now

$$\begin{aligned} z'(r) = &c_1\psi'(r) + 2c_2[\psi(r)-\psi(a)]\psi'(r) + \ldots + (n-1)c_{n-1}[\psi(r)-\psi(a)]^{n-2}\psi'(r) \\ &+ \frac{d}{dr}\frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}[g(s)-mz(s)]\,ds, \\ = &c_1\psi'(r) + 2c_2[\psi(r)-\psi(a)]\psi'(r) + \ldots + (n-1)c_{n-1}[\psi(r)-\psi(a)]^{n-2}\psi'(r) \\ &- \frac{1}{\Gamma(\mu)}\psi'(a)\left[[\psi(r)-\psi(a)]^{\mu-1}[g(a)-mz(a)]\right] \\ &+ \frac{1}{\Gamma(\mu-1)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-2}\psi'(r)[g(s)-mz(s)]\,ds. \end{aligned}$$

$$\begin{aligned}
\therefore z_{\psi}^{[1]}(r) &= \frac{z'(r)}{\psi'(r)} \\
&= c_1 + 2c_2[\psi(r)-\psi(a)] + \ldots + (n-1)c_{n-1}[\psi(r)-\psi(a)]^{n-2} \\
&\quad - \frac{1}{\Gamma(\mu)}\left[[\psi(r)-\psi(a)]^{\mu-1}[g(a)-mz(a)]\right] \\
&\quad + \frac{1}{\Gamma(\mu-1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-2}[g(s)-mz(s)]\,ds. \\
\therefore z_{\psi}^{[1]}(a) &= z_a^1 = c_1 \Rightarrow c_1 = \frac{z_a^1}{1!}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
z_{\psi}^{[2]}(r) &= \frac{[z_{\psi}^{[1]}(r)]'}{\psi'(r)} \\
&= 2c_2 + 6c_3[\psi(r)-\psi(a)] + \ldots + (n-1)(n-2)c_{n-1}[\psi(r)-\psi(a)]^{n-3} \\
&\quad - \frac{2}{\Gamma(\mu-1)}\left[[\psi(r)-\psi(a)]^{\mu-2}[g(a)-mz(a)]\right] \\
&\quad + \frac{1}{\Gamma(\mu-2)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-3}[g(s)-mz(s)]\,ds. \\
\therefore z_{\psi}^{[2]}(a) &= z_a^2 = 2c_2 \Rightarrow c_2 = \frac{z_a^2}{2!}
\end{aligned}$$

Repeating this process we get,

$$c_k = \frac{z_a^k}{k!}, \quad k = 0,1,2,\ldots,n-2.$$

Again,

$$\begin{aligned}
z_{\psi}^{[n-1]}(r) &= \frac{[z_{\psi}^{[n-2]}(r)]'}{\psi'(r)} \\
&= (n-1)!c_{n-1} - \frac{(n-1)}{\Gamma(\mu-n+2)}\left[[\psi(r)-\psi(a)]^{\mu-n+1}[g(a)-mz(a)]\right] \\
&\quad + \frac{1}{\Gamma(\mu-n+1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds \\
\therefore z_{\psi}^{[n-1]}(b) &= z_b = (n-1)!c_{n-1} - \frac{(n-1)}{\Gamma(\mu-n+2)}\left[[\psi(b)-\psi(a)]^{\mu-n+1}[g(a)-mz(a)]\right] \\
&\quad + \frac{1}{\Gamma(\mu-n+1)}\int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds.
\end{aligned}$$

Therefore

$$c_{n-1} = \frac{z_b}{(n-1)!} + \frac{[g(a)-mz(a)][\psi(b)-\psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)} - \frac{1}{(n-1)!\Gamma(\mu-n+1)}\int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds.$$

Hence equation (3.3) becomes

$$z(r) = \sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + \frac{z_b}{(n-1)!} + \frac{[g(a)-mz(a)][\psi(b)-\psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)}[\psi(r)-\psi(a)]^{n-1} - \frac{[\psi(r)-\psi(a)]^{n-1}}{(n-1)!\Gamma(\mu-n+1)}\int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds + \frac{1}{\Gamma(\mu)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}[g(s)-mz(s)]\,ds.$$

This equation can also be written in the form

$$z(r) = \sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} + \frac{1}{\Gamma(\mu)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}[g(s)-mz(s)]\,ds,$$

where,

$$M = \frac{z_b}{(n-1)!} + \frac{[g(a)-mz(a)][\psi(b)-\psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)} - \frac{1}{(n-1)!\Gamma(\mu-n+1)}\int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds.$$

To prove the converse, we apply ${}^cD_{a^+}^{\mu,\psi}$ to both sides of equation (3.2) and using Lemma 2.3, we obtain

$$\begin{aligned}{}^cD_{a^+}^{\mu,\psi}z(r) &= {}^cD_{a^+}^{\mu,\psi}I_{a^+}^{\mu,\psi}[g(r)-mz(r)]\\ &= g(r)-mz(r)\\ \Rightarrow {}^cD_{a^+}^{\mu,\psi}z(r)+mz(r) &= g(r).\end{aligned}$$

It is clear that $z_a^0 = z_a$. Also the direct computations leads to

$$\begin{aligned} z_\psi^{[1]}(r) &= \frac{z'(r)}{\psi'(r)} \\ &= \sum_{k=1}^{n-2} \frac{z_a^k}{(k-1)!}[\psi(r)-\psi(a)]^{k-1} + (n-1)M[\psi(r)-\psi(a)]^{n-2} \\ &\qquad + \frac{1}{\Gamma(\mu-1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-2}[g(s)-mz(s)]\,ds, \end{aligned}$$

and so $z_\psi^{[1]}(a) = z_a^1$.

$$\begin{aligned} z_\psi^{[2]}(r) &= \frac{[z_\psi^{[1]}(r)]'}{\psi'(r)} \\ &= \sum_{k=2}^{n-2} \frac{z_a^k}{(k-2)!}[\psi(r)-\psi(a)]^{k-2} + (n-1)(n-2)M[\psi(r)-\psi(a)]^{n-3} \\ &\qquad + \frac{1}{\Gamma(\mu-2)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-3}[g(s)-mz(s)]\,ds, \end{aligned}$$

and so $z_\psi^{[2]}(a) = z_a^2$. Repeating this process, we write

$$\begin{aligned} z_\psi^{[n-2]}(r) &= \frac{(z^{[n-3]})'(r)}{\psi'(r)} \\ &= z_a^{n-2} + (n-1)!M[\psi(r)-\psi(a)] \\ &\qquad + \frac{1}{\Gamma(\mu-n+2)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-n+1}[g(s)-mz(s)]\,ds. \end{aligned} \tag{3.4}$$

Taking $r \to a$ in equation (3.4), from continuity of g and using Lemma 2.4, we conclude that

$$z_\psi^{[n-2]}(a) = z_a^{n-2}.$$

Now

$$\begin{aligned} z_\psi^{[n-1]}(r) &= \frac{(z^{[n-2]})'(r)}{\psi'(r)} \\ &= (n-1)![\frac{z_b}{(n-1)!} + \frac{[g(a)-mz(a)][\psi(b)-\psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)} \\ &\qquad - \frac{1}{(n-1)!\Gamma(\mu-n+1)}\int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds] \\ &\qquad + \frac{1}{\Gamma(\mu-n+1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds \end{aligned}$$

$$
\begin{aligned}
&= z_b - \frac{1}{\Gamma(\mu-n+1)}\int_a^b \psi'(s)[\psi(b)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds \\
&\quad + \frac{1}{\Gamma(\mu-n+1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-n}[g(s)-mz(s)]\,ds. \qquad (3.5)
\end{aligned}
$$

Taking $r \to b$ in equation (3.5), from continuity of g and using Lemma 2.4, we conclude that

$$z_{\psi}^{[n-1]}(b) = z_b.$$

□

Lemma 3.3. *For a given $g(r) \in C(J,\mathbb{R})$ and $\mu \in (n-1,n]$, with $n \in \mathbb{N}$, the linear boundary value problem (3.1) has unique solution (3.2). Moreover, the explicit solution of the Volterra integral equation (3.2) can be represented by*

$$
\begin{aligned}
z(r) = \sum_{k=0}^{n-2} & z_a^k[\psi(r)-\psi(a)]^k E_{\mu,k+1}(-m(\psi(r)-\psi(a))^{\mu}) \\
&+ M\Gamma(n)[\psi(r)-\psi(a)]^{n-1}E_{\mu,n}(-m(\psi(r)-\psi(a))^{\mu}) \\
&+ \int_0^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}E_{\mu,\mu}(-m(\psi(r)-\psi(a))^{\mu})g(s)\,ds, \qquad (3.6)
\end{aligned}
$$

where $E_{\mu,\nu}(.)$ is the two-parameter Mittag-Leffer function.

Proof. By Lemma 3.2, linear BVP (3.1) has a solution (3.2).

Note that the equation (3.2) can be written in the following form

$$z(r) = T[z(r)],$$

where the operator T is defined by

$$T[z(r)] = \sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} - mI_{a^+}^{\mu,\psi}z(r) + I_{a^+}^{\mu,\psi}g(r).$$

Let $n \in \mathbb{N}$ and $x, y \in C(J,\mathbb{R})$. We have

$$
\begin{aligned}
\left|T^i(x)(r) - T^i(y)(r)\right| &= \left|-mI_{a^+}^{\mu,\psi}\left(T^{i-1}(x)(r) - T^{i-1}(y)(r)\right)\right| \\
&= \left|-mI_{a^+}^{\mu,\psi}\left(-mI_{a^+}^{\mu,\psi}\left(T^{i-2}(x)(r) - T^{i-2}(y)(r)\right)\right)\right| \\
&\;\;\vdots
\end{aligned}
$$

$$
\begin{aligned}
&= \left|(-m)^i I_{a^+}^{i\mu,\psi}((x)(r)-(y)(r))\right| \\
&\leq \frac{(m[\psi(b)-\psi(a)]^{\mu})^i}{\Gamma(i\mu+1)}\|x-y\| \\
&= \frac{(m^i[\psi(b)-\psi(a)]^{i\mu})}{\Gamma(i\mu+1)}\|x-y\|,
\end{aligned}
$$

for every $i \in N$ and $x,y \in A$. Let $\theta_i = \frac{(m^i[\psi(b)-\psi(a)]^{i\mu})}{\Gamma(i\mu+1)}$. Using generalized Mittag-Leffler functions, we have

$$\sum_{i=0}^{\infty}\theta_i = E_{\mu}(m(\psi(b)-\psi(a))^{\mu}).$$

Hence series $\sum_{i=0}^{\infty}\theta_i$ converges. Thus the mapping T^i is a contraction. Applying Weissinger's fixed point theorem, it follows that T has a unique fixed point. Hence equation (3.1) has unique solution $z(r)$. Applying method of successive approximations to prove that the integral equation (3.2) can be expressed by equation (3.6). For this, set

$$
\begin{aligned}
z_0(r) &= \sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1}, \\
z_m(r) &= z_0(r) - \frac{r}{\Gamma(\mu)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}z_{m-1}(s)\,ds \\
&\qquad + \frac{1}{\Gamma(\mu)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}g(s)\,ds.
\end{aligned}
$$

$$
\begin{aligned}
\therefore z_1(r) &= z_0(r) - mI_{a^+}^{\mu,\psi}z_0(r) + I_{a^+}^{\mu,\psi}g(r) \\
&= \sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} - m\sum_{k=0}^{n-2}\frac{z_a^k}{k!}\frac{\Gamma(k+1)}{\Gamma(\mu+k+1)}[\psi(r)-\psi(a)]^{\mu+k} \\
&\qquad - mM\frac{\Gamma(n)}{\Gamma(\mu+n)}[\psi(r)-\psi(a)]^{\mu+n-1} + I_{a^+}^{\mu,\psi}g(r) \\
&= \sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} - m\sum_{k=0}^{n-2}\frac{z_a^k}{\Gamma(\mu+k+1)}[\psi(r)-\psi(a)]^{\mu+k} \\
&\qquad - mM\frac{\Gamma(n)}{\Gamma(\mu+n)}[\psi(r)-\psi(a)]^{\mu+n-1} + I_{a^+}^{\mu,\psi}g(r).
\end{aligned}
$$

$$\begin{aligned}
\therefore z_2(r) &= z_0(t) - mI_{a^+}^{\mu,\psi} z_1(r) + I_{a^+}^{\mu,\psi} g(r) \\
&= \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} - m\sum_{k=0}^{n-2} \frac{z_a^k}{\Gamma(\mu+k+1)}[\psi(r)-\psi(a)]^{\mu+k} \\
&\quad - mM\frac{\Gamma(n)}{\Gamma(\mu+n)}[\psi(r)-\psi(a)]^{\mu+n-1} \\
&\quad + m^2\sum_{k=0}^{n-2} \frac{z_a^k\Gamma(\mu+k+1)}{\Gamma(\mu+k+1)\Gamma(2\mu+k+1)}[\psi(r)-\psi(a)]^{2\mu+k} \\
&\quad + m^2M\frac{\Gamma(n)\Gamma(\mu+n)}{\Gamma(\mu+n)\Gamma(2\mu+n)}[\psi(r)-\psi(a)]^{2\mu+k-1} - mI_{a^+}^{2\mu,\psi} g(r) + I_{a^+}^{\mu,\psi} g(r) \\
&= \sum_{l=0}^{2}\sum_{k=0}^{n-2} \frac{(-m)^l z_a^k}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu+k} + \sum_{l=0}^{2} \frac{(-m)^l M\Gamma(n)}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu+n-1} \\
&\quad + \int_a^r \psi'(s)\sum_{l=0}^{2} \frac{(-m)^l[\psi(r)-\psi(s)]^{l\mu+\mu-1}}{\Gamma(l\mu+n)} g(s)\,ds.
\end{aligned}$$

Continuing this process, we derive the following relation

$$\begin{aligned}
z_m(r) &= \sum_{l=0}^{q}\sum_{k=0}^{n-2} \frac{(-m)^l z_a^k}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu+k} + \sum_{l=0}^{q} \frac{(-m)^l M\Gamma(n)}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu+n-1} \\
&\quad + \int_a^r \psi'(s)\sum_{l=0}^{q-1} \frac{(-m)^l[\psi(r)-\psi(s)]^{l\mu+\mu-1}}{\Gamma(l\mu+n)} g(s)\,ds.
\end{aligned}$$

Taking limit as $q \to \infty$, we obtain the following explicit solution $z(r)$ to the integral equation (3.6).

$$\begin{aligned}
z(r) &= \sum_{l=0}^{\infty}\sum_{k=0}^{n-2} \frac{(-m)^l z_a^k}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu+k} + \sum_{l=0}^{\infty} \frac{(-m)^l M\Gamma(n)}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu+n-1} \\
&\quad + \int_a^r \psi'(s)\sum_{l=0}^{\infty} \frac{(-m)^l[\psi(r)-\psi(s)]^{l\mu+\mu-1}}{\Gamma(l\mu+n)} g(s)\,ds \\
&= \sum_{k=0}^{n-2} z_a^k[\psi(r)-\psi(a)]^k \sum_{l=0}^{\infty} \frac{(-m)^l}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu} \\
&\quad + M\Gamma(n)[\psi(r)-\psi(a)]^{n-1}\sum_{l=0}^{\infty} \frac{(-m)^l}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu} \\
&\quad + \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}\sum_{l=0}^{\infty} \frac{(-m)^l}{\Gamma(l\mu+\mu)}[\psi(r)-\psi(s)]^{l\mu} g(s)\,ds.
\end{aligned}$$

$$\therefore z(r) = \sum_{k=0}^{n-2} z_a^k [\psi(r)-\psi(a)]^k E_{\mu,k+1}(-m(\psi(r)-\psi(a))^{\mu})$$
$$+ M\Gamma(n)[\psi(r)-\psi(a)]^{n-1} E_{\mu,n}(-m(\psi(r)-\psi(a))^{\mu})$$
$$+ \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^{\mu}) g(s)\, ds.$$

This proves the Lemma. □

Lemma 3.4. *(Comparison Result). Let $\mu \in (0,1]$ and $m \in \mathbb{R}$. If $p \in C(J,\mathbb{R})$ satisfies the following inequalities*

$$^{c}D_{a^+}^{\mu,\psi} p(r) \geq -mp(r), p(a) \geq 0, \quad p(b) \geq 0, \quad r \in J, \tag{3.7}$$

then $p(r) \geq 0$ for all $r \in J$.

Proof. Let $g(r) = {}^{c}D_{a^+}^{\mu,\psi} p(r) + mp(r)$ and $p(a) = a^*$, $p(b) = b^*$, where $a^*, b^* \in \mathbb{R}$. Then from Equation (3.7), $g(r) \geq 0$ and $a^* \geq 0$, $b^* \geq 0$. We know that $E_{\mu,1}(z) \geq 0$, $E_{\mu,\mu}(z) \geq 0$ for all $\mu \in (0,1]$, $z \in \mathbb{R}$ (see [20]) and $M \geq 0$. Then using the integral representation (3.6), we obtain that $p(r) \geq 0$ for all $r \in J$. □

Definition 3.1. *A function $x_0 \in C(J,\mathbb{R})$ is said to be a lower solution of the nonlinear BVP (1.1), if it satisfies*

$$^{c}D_{a^+}^{\mu,\psi} x_0(r) \leq f(r,x_0), \quad x_0(a) \leq a^*, \quad x_0(b) \leq b^*, \quad r \in J,$$

Definition 3.2. *A function $y_0 \in C(J,\mathbb{R})$ is said to be a upper solution of the nonlinear BVP (1.1), if it satisfies*

$$^{c}D_{a^+}^{\mu,\psi} y_0(r) \geq f(r,y_0), \quad y_0(a) \geq a^*, \quad y_0(b) \geq b^*, \quad r \in J,$$

Theorem 3.1. *Let $f(r,x(r)) \in C(J \times \mathbb{R},\mathbb{R})$. Assume that,*

(H_1) There exist $x_0, y_0 \in C(J,\mathbb{R})$ such that x_0 and y_0 are lower and upper solutions of nonlinear BVP (1.1), respectively, with $x_0 \leq y_0$, $r \in J$.

(H_2) There exist a constant $m \in \mathbb{R}$ such that

$$f(r,y) - f(r,x) \geq -m(y-x) \quad \textit{for} \quad x_0 \leq x \leq y \leq y_0.$$

Then there exist monotone iterative sequences $\{x_n\}$ *and* $\{y_n\}$ *converging uniformly on the interval J to the extremal solutions of nonlinear BVP (1.1) in the sector* $[x_0,y_0]$*, where*

$$[x_0,y_0]=\{z\in C(J,\mathbb{R});x_0(r)\le z(r)\le y_0(r),\, r\in J\}\,.$$

Proof. For any $\omega\in[x_0,y_0]$, we consider the following linear BVP of fractional order

$$\begin{aligned} &{}^{c}D^{\mu,\psi}_{a^+}z(r)=f(r,\omega(r))-m(z(r)-\omega(r)),\quad r\in J,\\ &z(a)=a^*,\quad z(b)=b^*, \end{aligned}\tag{3.8}$$

Then the linear BVP (3.8) has unique solution $z(r)$.

Define the iterates as follows and construct the sequences $\{x_n\}$ and $\{y_n\}$:

$$\begin{aligned} &{}^{c}D^{\mu,\psi}_{a^+}x_{n+1}(r)=f(r,x_n)-m(x_{n+1}(r)-x_n(r)),\quad r\in J,\\ &x_{n+1}(a)=a^*,\quad x_{n+1}(b)=b^*, \end{aligned}\tag{3.9}$$

and

$$\begin{aligned} &{}^{c}D^{\mu,\psi}_{a^+}y_{n+1}(t)=f(r,y_n)-m(y_{n+1}(r)-y_n(r)),\quad r\in J,\\ &y_{n+1}(a)=a^*,\quad y_{n+1}(b)=b^*. \end{aligned}\tag{3.10}$$

Firstly we need to show that $x_0(r)\le x_1(r)\le y_1(r)\le y_0(r)$ for any $r\in J$.

Set $p(r)=x_1(r)-x_0(r)$ and from (3.9) with lower solution x_0, we obtain

$$\begin{aligned} {}^{c}D^{\mu,\psi}_{a^+}p(r)&={}^{c}D^{\mu,\psi}_{a^+}x_1(r)-{}^{c}D^{\mu,\psi}_{a^+}x_0(r)\\ &\ge f(r,x_0(r))-m(x_1(r)-x_0(r))-f(r,x_0(r))\\ &=-m(x_1(r)-x_0(r))\\ &=-mp(r),\\ \text{and}\, p(a)&=x_1(a)-x_0(a)=a^*-x_0(a)\ge 0,\\ p(b)&=x_1(b)-x_0(b)=b^*-x_0(b)\ge 0. \end{aligned}$$

Then by Lemma 3.4, $p(r) \geq 0$, for $r \in J$, implies that $x_0(r) \leq x_1(r)$. Similarly, set $p(r) = y_0(r) - y_1(r)$. From (3.9) and definition of upper solution, we obtain

$$\begin{aligned}
{}^{c}D_{a^+}^{\mu,\psi} p(r) &= {}^{c}D_{a^+}^{\mu,\psi} y_0(r) - {}^{c}D_{a^+}^{\mu,\psi} y_1(r) \\
&\geq f(r, y_0(r)) - f(r, y_0(r)) + m(y_1(r) - y_0(r)) \\
&= -m(y_0(r) - y_1(r)) \\
&= -mp(r), \\
\text{and } p(a) \geq 0, p(b) \geq 0.
\end{aligned}$$

Then by Lemma 3.4, $p(r) \geq 0$, for $r \in J$, implies that $y_1(r) \leq y_0(r)$ for $r \in J$.

Now to prove, $x_1(r) \leq y_1(r)$ for $r \in J$. For this, set $p(r) = y_1(r) - x_1(r)$. From (3.9), (3.10) and (H_2), we get

$$\begin{aligned}
{}^{c}D_{a^+}^{\mu,\psi} p(r) &= {}^{c}D_{a^+}^{\mu,\psi} y_1(r) - {}^{c}D_{a^+}^{\mu,\psi} x_1(r) \\
&= f(r, y_0(r)) - f(r, x_0(r)) - m(y_1(r) - y_0(r)) + m(x_1(r) - x_0(r)) \\
&\geq -m(y_0(r) - x_0(r)) - m(y_1(r) - y_0(r)) + m(x_1(r) - x_0(r)) \\
&= -mp(r), \\
\text{and } p(a) = 0,\ p(b) = 0.
\end{aligned}$$

Then by Lemma 3.4, $p(r) \geq 0$, for $r \in J$, implies that $x_1(r) \leq y_1(r)$ for $r \in J$. Thus $x_0(r) \leq x_1(r) \leq y_1(r) \leq y_0(r)$ for any $r \in J$. Assume that $n > 1$, $x_{n-1}(r) \leq x_n(r) \leq y_n(r) \leq y_{n-1}(r)$ for any $r \in J$. We claim that $x_n(r) \leq x_{n+1}(r) \leq y_{n+1}(r) \leq y_n(r)$ for any $r \in J$. To prove this, set $p(r) = x_{n+1}(r) - x_n(r)$.

$$\begin{aligned}
{}^{c}D_{a^+}^{\mu,\psi} p(r) &= {}^{c}D_{a^+}^{\mu,\psi} x_{n+1}(r) - {}^{c}D_{a^+}^{\mu,\psi} x_n(r) \\
&= f(r, x_n(r)) - m(x_{n+1}(r) - x_n(r)) - f(r, x_{n-1}(r)) + m(x_n(r) - x_{n-1}(r)) \\
&\geq -m(x_n(r) - x_{n-1}(r)) - m(x_{n+1}(r) - x_n(r)) - m(x_n(r) - x_{n-1}(r)) \\
&= -m(x_{n+1}(r) - x_n(r)) \\
&= -mp(r),
\end{aligned}$$

and $p(a) = 0$, $p(b) = 0$.

Then by Lemma 3.4, $p(r) \geq 0$, for $r \in J$, implies that $x_n(r) \leq x_{n+1}(r)$. Similarly we prove $x_{n+1}(r) \leq y_{n+1}(r)$ and $y_{n+1}(r) \leq y_n(r)$. By principle of mathematical induction, we have

$$x_0(r) \leq x_1(r) \leq \ldots \leq x_{n-1}(r) \leq x_n(r) \leq \ldots \leq y_n(r) \leq y_{n-1}(r) \leq \ldots \leq y_1(r) \leq y_0(r),\ r \in J.$$

Thus the sequences $\{x_n\}$ is monotone nondecreasing and bounded above by $y_0(r)$ and the sequences $\{y_n\}$ is monotone nonincreasing and bounded below by $x_0(r)$. Therefore the pointwise limit exist and these limits are denoted by x_*, y_*. Since sequences $\{x_n\}$, $\{y_n\}$ are of continuous functions defined on the compact set $[a,b]$. Hence by Dini's theorem, the convergence is uniform. That is

$$\lim_{n\to\infty} x_n(r) = x_*(r) \quad \text{and} \quad \lim_{n\to\infty} y_n(r) = y_*(r),$$

uniformly on $r \in J$ and the limit functions x_*, y_* satisfy BVP (1.1). Using corresponding fractional Volterra integral equations

$$\begin{aligned} x_{n+1}(r) &= a^* E_{\mu,1}(-m(\psi(r)-\psi(a))^\mu) + b^* E_{\mu,1}(-m(\psi(r)-\psi(a))^\mu) \\ &\quad + \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^\mu)(f(s,x_n(s)) + m x_n(s))\,ds, r \in J, \\ y_{n+1}(r) &= a^* E_{\mu,\mu}(-m(\psi(r)-\psi(a))^\mu) + b^* E_{\mu,\mu}(-m(\psi(r)-\psi(a))^\mu) \\ &\quad + \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^\mu)(f(s,y_n(s)) + m y_n(s))\,ds, r \in J, \end{aligned}$$

it follows that x_*, y_* are solutions of (3.9) and (3.10) respectively.

Next prove that x_* and y_* are minimal and maximal solutions of BVP (1.1) in the sector $[x_0, y_0]$. Let $w \in [x_0, y_0]$ be any solution of BVP (1.1). Assume that for some $n \in N$, $x_n(r) \leq w(r) \leq y_n(r)$, $r \in J$. Set $p(r) = w(r) - x_{n+1}(r)$. We have

$$\begin{aligned} {}^cD^{\mu,\psi}_{a^+} p(r) &= {}^cD^{\mu,\psi}_{a^+} w(r) - {}^cD^{\mu,\psi}_{a^+} x_{n+1}(r) \\ &= f(r,w(r)) - f(r,x_n(r)) + m(x_{n+1}(r) - x_n(r)) \\ &\geq -m(w(r) - x_n(r)) + m(x_{n+1}(r) - x_n(r)) \\ &= -m(w(r) - x_{n+1}(r)) \\ &= -mp(r), \end{aligned}$$

and, $p(a)=0$, $p(b)=0$. Then by Lemma 3.4, we obtain $p(r)\geq 0$, $r\in J$ implies that $x_{n+1}(r)\leq w(r)$, $r\in J$. Set $p(r)=y_{n+1}(r)-w(r)$. We have

$$\begin{aligned}
{}^{c}D^{\mu,\psi}_{a^+}p(r) &= {}^{c}D^{\mu,\psi}_{a^+}y_{n+1}(r) - {}^{c}D^{\mu,\psi}_{a^+}w(r)\\
&= f(r,y_n(r)) - m(y_{n+1}(r)-y_n(r)) - f(r,w(r))\\
&\geq -m(y_n(r)-w(r)) - m(y_{n+1}(r)-y_n(r))\\
&= -m(y_{n+1}(r)-w(r))\\
&= -mp(r),\\
\text{and}\, p(a) = 0,\ p(b)=0.
\end{aligned}$$

By Lemma 3.4, we obtain $p(r)\geq 0$, $r\in J$. Thus $w(r)\leq y_{n+1}(r)$, $r\in J$. Hence, we have

$$x_{n+1}(r)\leq w(r)\leq y_{n+1}(r), r\in J. \tag{3.11}$$

Taking the limit as $n\to\infty$ on both sides of equation (3.11), we get

$$x_*\leq w\leq y_*.$$

Therefore x_*, y_* are the minimal and maximal solutions of nonlinear BVP (1.1) in $[x_0,y_0]$. □

In the following Theorem, we establish uniqueness of solution of nonlinear BVP (1.1).

Theorem 3.2. *Assume that $[H_1]$, $[H_2]$ are satisfied.*
$[H_3]$ There exists a constant $m_*\geq -m$ such that

$$f(r,y)-f(r,x)\leq m_*(y-x),$$

for every $x_0\leq x\leq y\leq y_0$, $r\in J$. Then nonlinear BVP (1.1) has a unique solution in $[x_0,y_0]$.

Proof. By Theorem 3.1, x_* and y_* are respectively minimal and maximal solutions of the nonlinear BVP (1.1) and $x_*\leq y_*$, $r\in J$. It is sufficient to prove that $x_*\geq y_*$, $r\in J$. For this set

$p(r) = x_* - y_*$, $r \in J$. In view of $[H_3]$, we have

$$\begin{aligned} {}^cD^{\mu,\psi}_{a^+} p(r) &= {}^cD^{\mu,\psi}_{a^+} x_* - {}^cD^{\mu,\psi}_{a^+} y_* \\ &= f(r,x_*) - f(r,y_*) \\ &\geq m_*(x_* - y_*) = m_* p(r). \end{aligned}$$

Furthermore, $p(a) = x_*(a) - y_*(a) = a^* - a^* = 0$ and $p(b) = x_*(b) - y_*(b) = b^* - b^* = 0$. By Lemma 3.4, we obtain $p(r) \geq 0$, $r \in J$ implies that $x_* \geq y_*$, $r \in J$. Therefore, $x_* = y_*$ is unique solution of the nonlinear BVP (1.1) in $[x_0, y_0]$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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