ON AN UPPER BOUND FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Liman, Mohapatra and Shah proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \geq 1, |\beta| \leq 1 \) and \( |z| = 1 \),
\[
\left| zD_\alpha p(z) + np\left( \frac{|\alpha|-1}{2} \right)p(z) \right| \leq \left\{ \left| \alpha + \beta \left( \frac{|\alpha|-1}{2} \right) \right| + \left| z + \beta \left( \frac{|\alpha|-1}{2} \right) \right| \right\}_\max |p(z)|
- \left\{ \left| \alpha + \beta \left( \frac{|\alpha|-1}{2} \right) \right| - \left| z + \beta \left( \frac{|\alpha|-1}{2} \right) \right| \right\}_\min |p(z)|,
\]
where \( D_\alpha p(z) = np(z) + (\alpha - z)p'(z) \) is the polar derivative of \( p(z) \) with respect to the point \( \alpha \). We extend and generalize this inequality for the polynomial \( p(z) \) which does not vanish in \( |z| < k, k \leq 1 \). Our result also generalizes other known inequalities as well.

Keywords: Bernstein inequality; polar derivative; polynomial; zero.

2010 AMS Subject Classification: 30C10, 30C15.

1. INTRODUCTION

Bernstein [4] established an estimate of the derivative of a polynomial \( p(z) \) of degree \( n \) in terms of the maximum modulus of \( p(z) \) on the unit circle by proving
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.
\]
In (1.1) equality is attained if \( p(z) \) is of the form \( \alpha z^n \), where \( \alpha \) is a non zero constant. Erdös conjectured that if we restrict \( p(z) \) to the polynomials of degree \( n \) having no zero in \( |z| < 1 \), then (1.1) can be sharpened and replaced by

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.
\]

Inequality (1.2) was proved later by Lax [9]. Equality is attained in (1.2) for \( p(z) = \alpha z^n + \beta \), where \( |\alpha| = |\beta| \). For the same class of polynomials as considered by Erdös and Lax, Aziz and Dawood [1] involved \( \min |p(z)| \) on the unit circle and proved a refinement of (1.2). In fact, they proved

\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left( \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right).
\]

Dewan and Hans [6] improved (1.3) by proving that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for any \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \) and \( |z| = 1 \)

\[
\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \leq \frac{n}{2} \left\{ \left( |1 + \frac{\beta}{2}| + \left| \frac{\beta}{2} \right| \right) \max_{|z|=1} |p(z)| - \left( |1 + \frac{\beta}{2}| - \left| \frac{\beta}{2} \right| \right) \min_{|z|=1} |p(z)| \right\}.
\]

Let \( \alpha \) be any real or complex number and let \( p(z) \) be a polynomial of degree \( n \). We define the polar derivative [11] of \( p(z) \) with respect to \( \alpha \), denoted by \( D_{\alpha} p(z) \), as

\[
D_{\alpha} p(z) = np(z) + (\alpha - z)p'(z).
\]

\( D_{\alpha} p(z) \) is a polynomial of degree at most \( n - 1 \). Since,

\[
\lim_{\alpha \to \infty} \frac{D_{\alpha} p(z)}{\alpha} = p'(z),
\]

therefore, \( D_{\alpha} p(z) \) is considered as a generalized form of the ordinary derivative of \( p(z) \).

Aziz and Shah [2] extended (1.1) to polar derivative and proved that if \( p(z) \) is a polynomial of degree \( n \), then for every \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\max_{|z|=1} |D_{\alpha} p(z)| \leq n|\alpha| \max_{|z|=1} |p(z)|.
\]
Aziz and Shah [3] refined and extended their result (1.5) by considering that the polynomial $p(z)$ of degree $n$ having no zeros in $|z| < 1$ and for every real or complex number $\alpha$ satisfying $|\alpha| \geq 1$,

\begin{equation}
|zD_\alpha p(z) + n\beta |\alpha| - 1 \frac{1}{2} p(z)| \leq n \left\{ \left( |\alpha + \beta| |\alpha| - 1 \frac{1}{2} + |z + \beta| |\alpha| - 1 \frac{1}{2} \right) \max_{|z|=1} |p(z)| \right. \left. - \left( |\alpha + \beta| |\alpha| - 1 \frac{1}{2} - |z + \beta| |\alpha| - 1 \frac{1}{2} \right) \min_{|z|=1} |p(z)| \right\}.
\end{equation}

Theorem 1.1. If $p(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then for all $\alpha$, $\beta$ with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $|z| = 1$,

\begin{equation}
\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2} \left\{ (|\alpha| + 1) \max_{|z|=1} |p(z)| - (|\alpha| - 1) \min_{|z|=1} |p(z)| \right\}.
\end{equation}

2. Main Results

In this paper, by involving some coefficients of the polynomial $p(z)$, we generalize and extend inequality (1.7). The result also generalizes other inequalities mentioned in the preceding section. More precisely, we prove the following result.

Theorem 2.1. Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree $n$ which does not vanish in $|z| < k$, $k \leq 1$. Then, for all $\alpha$, $\beta \in \mathbb{C}$ with $|\alpha| \geq A$, $|\beta| \leq 1$ and $|z| = 1$,

\begin{equation}
|zD_\alpha p(z) + n\beta |\alpha| - A \frac{1}{1 + A} p(z)| \leq n \left\{ \left( k^{-n} |\alpha + \beta| |\alpha| - A \frac{1}{1 + A} + |z + \beta| |\alpha| - A \frac{1}{1 + A} \right) \max_{|z|=1} |p(z)| \right. \left. - \left( k^{-n} |\alpha + \beta| |\alpha| - A \frac{1}{1 + A} - |z + \beta| |\alpha| - A \frac{1}{1 + A} \right) \min_{|z|=k} |p(z)| \right\},
\end{equation}

where

\begin{equation}
A = \frac{\mu |a_{n-\mu}| k^{\mu-1} + n |a_n| k^{2\mu}}{\mu |a_{n-\mu}| + n |a_n| k^{\mu-1}}.
\end{equation}
Remark 2.2. Under the assumptions of Theorem 2.1, we can verify that $A = 1$ when $k = 1$, whereas, for $k < 1$ we can verify that $A \leq k$ as shown below. Using inequality (3.4), we have

$$\mu |a_{n-\mu}|k^\mu \leq n|a_n|k^{2\mu}$$

$$\implies \mu |a_{n-\mu}|k^\mu \left(\frac{1}{k} - 1\right) \leq n|a_n|k^{2\mu} \left(\frac{1}{k} - 1\right),$$

or $$\mu |a_{n-\mu}|k^{\mu-1} + n|a_n|k^{2\mu} \leq n|a_n|k^{2\mu-1} + \mu |a_{n-\mu}|k^\mu,$$

i.e. $$\frac{\mu |a_{n-\mu}|k^{\mu-1} + n|a_n|k^{2\mu}}{\mu |a_{n-\mu}| + n|a_n|k^{\mu-1}} \leq k^\mu \leq k \text{ as } k \leq 1 \text{ and } \mu \geq 1,$$

i.e. $A \leq k$.

Remark 2.3. Taking $k = 1$ (so that $A = 1$) in Theorem 2.1, inequality (2.1) for $\mu = 1$ reduces to (1.7) due to Liman[10]. Thus, Theorem 2.1 is an extension and a generalization of Theorem 1.1 for the lacunary polynomial $p(z) = a_nz^n + \sum_{\nu=\mu}^n a_{n-\nu}z^{-\nu}, 1 \leq \mu \leq n$.

If we take $\beta = 0$ in Theorem 2.1, it takes the following simplified form.

Corollary 2.4. If $p(z) = a_nz^n + \sum_{\nu=\mu}^n a_{n-\nu}z^{-\nu}, 1 \leq \mu \leq n$, is a polynomial of degree $n$ and $p(z) \neq 0$ in $|z| < k$, $k \leq 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$

$$\max_{|z|=1} |zD_\alpha p(z)| \leq \frac{n}{2} \left(\left(k^{-n}|\alpha| + 1\right) \max_{|z|=1} |p(z)| - \left(k^{-n}|\alpha| - 1\right) \min_{|z|=k} |p(z)| \right),$$

(2.3) where $A$ is given by (2.2).

Remark 2.5. If we take $\mu = 1$ and $k = 1$ in Corollary 2.4, then (2.3) reduces to (1.6) due to Aziz and Shah [3] and therefore Theorem 2.1 extends and generalizes (1.6) to lacunary polynomials of the type $p(z) = a_nz^n + \sum_{\nu=\mu}^n a_{n-\nu}z^{-\nu}, 1 \leq \mu \leq n$.

Dividing both sides of (2.1) by $|\alpha|$ and taking the limit as $|\alpha| \to \infty$, we have the following result.
Corollary 2.6. If \( p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu} \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) and \( p(z) \neq 0 \) in \( |z| < k, k \leq 1 \), then for any \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \) and \( |z| = 1 \),

\[
\left| z p'(z) + \frac{n \beta}{1 + A} p(z) \right| \leq \frac{n}{2} \left\{ \left( k^n \left| 1 + \frac{\beta}{1 + A} \right| + \left| \frac{\beta}{1 + A} \right| \right) \max_{|z|=1} |p(z)| - \left( k^n \left| 1 + \frac{\beta}{1 + A} \right| - \left| \frac{\beta}{1 + A} \right| \right) \min_{|z|=k} |p(z)| \right\},
\]

(2.4)

where \( A \) is given by (2.2).

Remark 2.7. If we take \( k = 1 \) (so that \( A = 1 \)) and \( \mu = 1 \) in Corollary 2.6, inequality (2.4) reduces to (1.4) due to Dewan and Hans [6]. Furthermore, if \( \beta = 0 \) along with \( k = 1 \) and \( \mu = 1 \), inequality (2.4) becomes (1.3) due to Aziz and Dawood [1].

3. Lemmas

For the proof of Theorem 2.1, we require the following lemmas. The first lemma is due to Laguerre [8, 11]

Lemma 3.1. If all the zeros of an \( n \)th degree polynomial \( p(z) \) lie in a circular region \( C \), and \( \omega \) is any zero of \( D \alpha p(z) \), where \( \alpha \) is any real or complex number, then at most one of the points \( \omega \) and \( \alpha \) may lie outside \( C \).

Lemma 3.2. If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu} \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having no zeros in \( |z| < k, k \geq 1 \), then on \( |z| = 1 \)

\[
|q'(z)| \geq k^{\mu+1} \left( \frac{\mu |a_\mu| k^{\mu-1} + 1}{n |a_0| k^{\mu+1} + 1} \right) |p'(z)|
\]

(3.1)

and

\[
\frac{\mu |a_\mu|}{n |a_0|} k^{\mu} \leq 1.
\]

(3.2)

This lemma is due to Qazi [12].

Lemma 3.3. If \( p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu} \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \) having all its zeros in the closed disk \( |z| \leq k, k \leq 1 \), then for any real or complex number \( \alpha \) with \( |\alpha| \geq A \)
and $|z| = 1$

$$|D_\alpha p(z)| \geq n \frac{\alpha - A}{1 + A} |p(z)|.$$  \hspace{1cm} (3.3)

where $A$ is given by (2.2).

Before proving Lemma 3.3, we take note of an important consequence of Lemma 3.2 and Lemma 3.3. If $p(z)$ is polynomial assumed as in Lemma 3.3 and $q(z) = z^n p\left(\frac{1}{z}\right)$, then $q(z)$ has no zero in $|z| < \frac{1}{k}, \frac{1}{k} \geq 1$. Thus, on applying Lemma 3.2 to $q(z)$, by inequality (3.2) we obtain

$$\frac{\mu}{n} \frac{|a_n - \mu|}{|a_n|} \frac{1}{k^\mu} \leq 1.$$  \hspace{1cm} (3.4)

**Proof of Lemma 3.3.** Let $q(z) = z^n p\left(\frac{1}{z}\right) = \bar{a}_n + \sum_{v=\mu}^n \bar{a}_{n-v} z^v$. Then, it can be easily verified that

$$|q'(z)| = |np(z) - zp'(z)| \quad \text{for} \quad |z| = 1.$$  \hspace{1cm} (3.5)

Since $p(z)$ has all its zeros in $|z| \leq k, k \leq 1$, therefore, the polynomial $q(z)$ has no zero in $|z| < \frac{1}{k}, \frac{1}{k} \geq 1$. Thus, applying Lemma 3.2 to $q(z)$, we have by (3.1) for $|z| = 1$

$$|p'(z)| \geq \frac{1}{k^\mu + 1} \frac{\mu}{n} \frac{|a_n - \mu|}{|a_n|} \frac{1}{k^{\mu-1}} + 1 |q'(z)|$$

$$= \frac{\mu}{n} |a_n - \mu| + n |a_n| k^{\mu-1} |q'(z)|,$$

therefore,

$$|q'(z)| \leq \frac{n |a_n| k^{\mu-1} + \mu |a_n - \mu| k^{\mu-1}}{\frac{\mu}{n} |a_n - \mu| + n |a_n| k^{\mu-1}} |p'(z)|.$$  \hspace{1cm} (3.6)

Equivalently, for $|z| = 1$

$$|q'(z)| \leq A |p'(z)|.$$  \hspace{1cm} (3.7)

$$|p'(z)| + |q'(z)| \leq (1 + A) |p'(z)|,$$
where \( A = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{\mu|a_{n-\mu}| + n|a_n|k^{\mu-1}}. \)
Now,
\[
n|p(z)| = |np(z) - zp'(z) +zp'(z)| \\
\leq |np(z) - zp'(z)| + |p'(z)| & \text{ on } |z| = 1,
\]
which on using inequality (3.5) gives for \(|z| = 1\)
\[(3.8) \quad n|p(z)| \leq |p'(z)| + |q'(z)|.\]
combining (3.7) and (3.8), we have for \(|z| = 1\)
\[(3.9) \quad n|p(z)| \leq (1 + A)|p'(z)|.\]
i.e. \(|p'(z)| \geq \frac{n}{1 + A}|p(z)|.\]
By definition, if \( \alpha \in \mathbb{C} \), particularly for \(|\alpha| \geq A\), we have
\[
D_{\alpha}p(z) = np(z) + (\alpha - z)p'(z).
\]
Then,
\[
|D_{\alpha}p(z)| = |np(z) - zp'(z) + \alpha p'(z)| \\
\geq |\alpha||p'(z)| - |np(z) - zp'(z)|,
\]
which on using inequality (3.5) gives for \(|z| = 1\)
\[(3.10) \quad |D_{\alpha}p(z)| \geq |\alpha||p'(z)| - |q'(z)|.\]
Using (3.6) to (3.10), we have for \(|z| = 1\)
\[
|D_{\alpha}p(z)| \geq |\alpha||p'(z)| - A|p'(z)| \\
= (|\alpha| - A)|p'(z)|,
\]
which in conjunction with (3.9) gives for \(|z| = 1\)
\[
|D_{\alpha}p(z)| \geq n\frac{|\alpha| - A}{1 + A}|p(z)|.
\]
\(\square\)
Lemma 3.4. Let \( p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu} \), \( 1 \leq \mu \leq n \), be a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), \( k \leq 1 \), then for every \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \geq A \), \( |\beta| \leq 1 \) and \( |z| = 1 \), we have
\[
|zD_\alpha p(z) + n\beta \frac{|\alpha|-A}{1+A} p(z)| \geq \frac{n}{k^n} \left| \alpha + \beta \frac{|\alpha|-A}{1+A} \right| \min_{|z|=k} |p(z)|,
\]
where \( A \) is given by (2.2).

Proof of Lemma 3.4. If \( p(z) \) has a zero on \( |z| = k \), then (3.11) follows trivially. Therefore, we assume that \( p(z) \) has all its zeros in \( |z| < k \). Let \( m = \min_{|z|=k} |p(z)| \), then \( m > 0 \) and \( |p(z)| \geq m \), where \( |z| = k \). Therefore, for every \( \lambda \) with \( |\lambda| < 1 \), it follows by Rouche’s theorem that the polynomial \( G(z) = p(z) - \lambda m \left( \frac{z}{k} \right)^n \) has all its zeros in \( |z| < k \). By lemma 3.1, \( D_\alpha G(z) \) has all its zeros in \( |z| < k \), where
\[
D_\alpha G(z) = D_\alpha p(z) - D_\alpha \left( \lambda m \frac{z^n}{k^n} \right)
= D_\alpha p(z) - \left\{ n\lambda m \frac{z^n}{k^n} - (\alpha - z) n\lambda m \frac{z^{n-1}}{k^n} \right\}
= D_\alpha p(z) - \alpha \lambda mn
\]
with \( |\alpha| \geq A \).

Applying Lemma 3.3 to the polynomial \( G(z) \), we have for \( |z| = 1 \)
\[
|D_\alpha G(z)| \geq n \frac{|\alpha|-A}{1+A} |G(z)|,
\]
which is equivalent to
\[
|zD_\alpha G(z)| \geq n \frac{|\alpha|-A}{1+A} |G(z)| \quad \text{on} \quad |z| = 1.
\]

Since \( zD_\alpha G(z) \) has all its zeros in \( |z| < k \leq 1 \), by using Rouche’s theorem, it can be easily verified from (3.12) that the polynomial \( zD_\alpha G(z) + \beta n \frac{|\alpha|-A}{1+A} G(z) \) has all its zeros in \( |z| < 1 \), where \( |\beta| < 1 \). Then,
\[
T(z) = zD_\alpha p(z) - \alpha \lambda mn \frac{z^n}{k^n} + \beta n \frac{|\alpha|-A}{1+A} \left( p(z) - \lambda m \frac{z^n}{k^n} \right)
= zD_\alpha p(z) + n\beta \frac{|\alpha|-A}{1+A} p(z) - \alpha \lambda mn \frac{z^n}{k^n} \left( \alpha + \beta \frac{|\alpha|-A}{1+A} \right)
\]
(3.13)
will have no zeros in $|z| \geq 1$. This implies for every $\beta$ with $|\beta| < 1$ and $|z| \geq 1$,

\[(3.14) \quad |zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z)| \geq nm \frac{|z|}{k^n} \alpha + \beta \frac{|\alpha| - A}{1 + A} .\]

If (3.14) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

\[|z_0D_\alpha p(z_0) + n\beta \frac{|\alpha| - A}{1 + A} p(z_0)| < nm \left| \frac{z_0}{k^n} \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| .\]

Take

\[\lambda = \frac{z_0D_\alpha p(z_0) + n\beta \frac{|\alpha| - A}{1 + A} p(z_0)}{nm \left( \frac{z_0}{k^n} \alpha + \beta \frac{|\alpha| - A}{1 + A} \right)} ,\]

then $|\lambda| < 1$ and with this choice of $\lambda$, we have $T(z_0) = 0$ from (3.13). But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. Thus, for $\beta \in \mathbb{C}$ with $|\beta| < 1$ inequality (3.14) holds and for $|\beta| = 1$, it follows by continuity. Hence,

\[|zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z)| \geq n \left| \frac{z^n}{k^n} \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| \min |p(z)| .\]

Lemma 3.5. If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|\alpha| \geq k \geq A$, where $k \leq 1$, we have for $|z| = 1$

\[(3.15) \quad |zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z)| \leq n \left| \frac{k^n}{k^n} \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| \max |p(z)| ,\]

where $A$ is given by (2.2).

Proof of Lemma 3.5. Let $M = \max_{|z|=k} |p(z)|$. If $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$, then $|\lambda p(z)| < |M \left( \frac{z^n}{k^n} \right)|$ for $|z| = k$. Therefore, it follows by Rouche’s Theorem that $G(z) = M \frac{z^n}{k^n} - \lambda p(z)$ has all its zeros in $|z| < k$. Thus, by using Lemma 3.1,

\[D_\alpha G(z) = \alpha M n \left( \frac{z^{n-1}}{k^n} \right) - \lambda D_\alpha p(z) \]

has all its zeros in $|z| < k$ for $|\alpha| \geq A$.

On applying Lemma 3.3 to the polynomial $G(z)$, we have for $|z| = 1$

\[(3.16) \quad |zD_\alpha G(z)| \geq n \left| \frac{|\alpha| - A}{1 + A} \right| |G(z)| .\]
Following a similar argument as used in the proof of Lemma 3.4, the result follows. □

**Lemma 3.6.** If \( p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v} \), \( 1 \leq \mu \leq n \), is a polynomial of degree \( n \), then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \) and \( |\alpha| \geq k \geq A \), where \( k \leq 1 \), we have for \( |z| = 1 \)

\[
|zD_\alpha p(z) + n \beta \frac{|\alpha| - A}{1 + A} p(z) + |zD_\alpha Q(z) + n \beta \frac{|\alpha| - A}{1 + A} Q(z)| \leq n \left\{ k^{-n} \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| + n \left| |z + \beta \frac{|\alpha| - A}{1 + A} | \right| \right\} \max_{|z|=1} |p(z)|,
\]

(3.17)

where \( A \) is given by (2.2) and \( Q(z) = \left( \frac{z^n}{k} \right) p \left( \frac{k^2}{z} \right) \).

**Proof of Lemma 2.6.** Let \( M = \max_{|z|=k} |p(z)| \). For \( \lambda \) with \( |\lambda| > 1 \), it follows by Rouche’s theorem that the polynomial \( G(z) = p(z) - \lambda M \) has no zeros in \( |z| < k \). Consequently, the polynomial

\[
H(z) = \left( \frac{z}{k} \right)^n G \left( \frac{k^2}{z} \right)
\]

(3.18)

has all its zeros in \( |z| \leq k \), also \( |G(z)| = |H(z)| \) for \( |z| = k \). Since all the zeros of \( H(z) \) lie in \( |z| \leq k \), therefore, for \( \delta \) with \( |\delta| > 1 \), by Rouche’s Theorem all the zeros of \( G(z) + \delta H(z) \) lie in \( |z| \leq k \). Hence, by Lemma 3.3 for every \( \alpha \) with \( |\alpha| \geq A \), and \( |z| = 1 \), we have

\[
n \left| \alpha - A \right| G(z) + \delta H(z) \leq |zD_\alpha(G(z) + \delta H(z))|
\]

(3.19)

On the other hand, by Lemma 3.1, all the zeros of \( D_\alpha(G(z) + \delta H(z)) \) lie in \( |z| < k < 1 \), where \( |\alpha| \geq A \). Therefore, for any \( \beta \) with \( |\beta| \leq 1 \), Rouche’s theorem implies that all the zeros of \( zD_\alpha(G(z) + \delta H(z)) + n\beta \frac{\alpha - A}{1 + A} (G(z) + \delta H(z)) \) lie in \( |z| < 1 \). This means that the polynomial

\[
T(z) = zD_\alpha G(z) + n \beta \frac{\alpha - A}{1 + A} G(z) + \delta(zD_\alpha H(z) + n \beta \frac{\alpha - A}{1 + A} H(z))
\]

(3.20)

will have no zeros in \( |z| \geq 1 \). Now, using a similar argument as used in the proof of Lemma 3.4, we get for \( |z| \geq 1 \),

\[
|zD_\alpha G(z) + n \beta \frac{\alpha - A}{1 + A} G(z) | \leq |zD_\alpha H(z) + n \beta \frac{\alpha - A}{1 + A} H(z) |.
\]

(3.21)

Therefore, by the equalities

\[
H(z) = \left( \frac{z}{k} \right)^n G \left( \frac{k^2}{z} \right) = \left( \frac{z}{k} \right)^n p \left( \frac{k^2}{z} \right) - \lambda M \left( \frac{z}{k} \right)^n = Q(z) - \lambda M \left( \frac{z}{k} \right)^n
\]

(3.22)
and substituting for $G(z)$ and $H(z)$ in (3.21), we get

\[ \left| \left( zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right) - \lambda nM \left( z + \beta \frac{|\alpha| - A}{1 + A} \right) \right| \]

\[ \leq \left| \left( zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right) - \lambda nM \left( z + \beta \frac{|\alpha| - A}{1 + A} \right) \right|. \tag{3.23} \]

This implies that

\[ \left| \left( zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right) - \lambda nM \left( z + \beta \frac{|\alpha| - A}{1 + A} \right) \right| \]

\[ \leq \left| \left( zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right) - \lambda nM \left( z + \beta \frac{|\alpha| - A}{1 + A} \right) \right|. \tag{3.24} \]

As $|p(z)| = |Q(z)|$ for $|z| = k$, that is, $\max_{|z|=k} |p(z)| = \max_{|z|=k} |Q(z)| = M$, by Lemma 3.5 for $Q(z)$, we obtain

\[ \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right| < |\lambda| nM k^{-n} \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right|. \tag{3.25} \]

Thus, taking a suitable choice of the argument of $\lambda$,

\[ \left| \left( zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right) - \lambda nM \left( z + \beta \frac{|\alpha| - A}{1 + A} \right) \right| \]

\[ = |\lambda| nM k^{-n} \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| - \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right|. \tag{3.26} \]

By combining the right hand sides of (3.24) and (3.26) for $|z| = 1$ and $|\beta| \leq 1$, we get

\[
\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right| - \lambda nM \left( z + \beta \frac{|\alpha| - A}{1 + A} \right) \\
\leq |\lambda| nM k^{-n} \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| - \left| zD_\alpha q(z) + n\beta \frac{|\alpha| - A}{1 + A} q(z) \right|.
\]

i.e.,

\[
\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right| + \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right| \\
\leq |\lambda| \left\{ \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| k^{-n} + |z + \beta \frac{|\alpha| - A}{1 + A}| \right\} nM.
\]

Taking $|\lambda| \to 1$, we have

\[
\left| zD_\alpha p(z) + n\beta \frac{|\alpha| - A}{1 + A} p(z) \right| + \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - A}{1 + A} Q(z) \right| \\
\leq n \left\{ \left| \alpha + \beta \frac{|\alpha| - A}{1 + A} \right| k^{-n} + |z + \beta \frac{|\alpha| - A}{1 + A}| \right\} M.
\]
Lemma 3.7. Let \( H(z) = a_n z^n + \sum_{\nu=\mu}^n a_{n-\nu} z^{n-\nu}, \) 1 \( \leq \mu \leq n, \) be a polynomial of degree \( n \) having all its zeros in \( |z| \leq k, k \leq 1, \) and \( G(z) \) be a polynomial of degree not exceeding that of \( H(z). \) If \( |G(z)| \leq |H(z)| \) for \( |z| = k, \) then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \geq A, \) \( |\beta| \leq 1 \) and \( |z| = 1, \) we have

\[
(3.27) \quad |zD_\alpha G(z) + n\beta \left| \frac{\alpha - A}{1 + A} G(z) \right| \leq |zD_\alpha H(z) + n\beta \left| \frac{\alpha - A}{1 + A} H(z) \right|,
\]

where \( A \) is given by (2.2).

Proof of Lemma 2.7. Since \( |\lambda G(z)| \leq |G(z)| \leq |H(z)| \) for \( |z| = 1 \) and \( \lambda \in \mathbb{C} \) with \( |\lambda| < 1, \) then it follows by Rouche’s theorem that the polynomials \( H(z) \) and \( H(z) - \lambda G(z) \) have the same number of zeros in the open disk \( |z| < k. \) Also, the inequality \( |G(z)| \leq |H(z)| \) for \( |z| = k \) implies that any zero of \( H(z) \) on \( |z| = k \) is also a zero of \( G(z). \) Therefore, \( H(z) - \lambda G(z) \) has all its zeros in the closed disk \( |z| \leq k, k \leq 1. \) Thus, applying Lemma 3.3, we have for all real or complex \( \alpha \) with \( |\alpha| \leq A \) and \( |z| = 1 \)

\[
|zD_\alpha (H(z) - \lambda G(z))| \geq n \left| \frac{\alpha - A}{1 + A} H(z) - \lambda G(z) \right|.
\]

Following a similar argument as used in the proof of Lemma 3.4, we have for any \( \beta \) with \( |\beta| < 1 \) and \( |z| = 1 \)

\[
|zD_\alpha (H(z) - \lambda G(z))| \geq n \left| \frac{\alpha - A}{1 + A} H(z) - \lambda G(z) \right|.
\]

\[
> n|\beta| \left| \frac{\alpha - A}{1 + A} H(z) - \lambda G(z) \right|.
\]

Thus, for \( |z| = 1 \)

\[
(3.28) \quad T(z) = [zD_\alpha H(z) - \lambda zD_\alpha G(z)] + n|\beta| \left| \frac{\alpha - A}{1 + A} H(z) - \lambda G(z) \right| \neq 0,
\]

which implies that for \( |z| = 1 \)

\[
(3.29) \quad \left\{ zD_\alpha H(z) + n|\beta| \left| \frac{\alpha - A}{1 + A} H(z) \right| - \lambda \left\{ zD_\alpha G(z) + n|\beta| \left| \frac{\alpha - A}{1 + A} G(z) \right| \right\} \neq 0.
\]

Hence, we can conclude that for \( |z| = 1 \)

\[
(3.30) \quad |zD_\alpha H(z) + n|\beta| \left| \frac{\alpha - A}{1 + A} H(z) \right| \geq |zD_\alpha G(z) + n|\beta| \left| \frac{\alpha - A}{1 + A} G(z) \right|.
\]
If (3.30) is not true, then there exist a point $z_0$ on the unit circle that
\[
\left| z_0 D_\alpha H(z_0) + n\beta \frac{\alpha - A}{1 + A} H(z_0) \right| < \left| z_0 D_\alpha G(z_0) + n\beta \frac{\alpha - A}{1 + A} G(z_0) \right|.
\]

If we choose
\[
\lambda = \frac{z_0 D_\alpha H(z_0) + n\beta \frac{\alpha - A}{1 + A} H(z_0)}{z_0 D_\alpha G(z_0) + n\beta \frac{\alpha - A}{1 + A} G(z_0)},
\]
then $|\lambda| < 1$ and hence (3.29) gives $T(z_0) = 0$ for $|z_0| = 1$. This is a contradiction to (3.28). Hence, (3.30) must hold for $\beta \in \mathbb{C}$ with $|\beta| < 1$. For $|\beta| = 1$, (3.30) holds by continuity. This completes the proof of Lemma 2.7.

\[\Box\]

4. PROOF OF THE THEOREM

We now prove Theorem 2.1.

Proof of Theorem 2.1. Since $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, does not vanish in $|z| < k$, and if $m = \min_{|z|=k} |p(z)|$, then $m \leq |p(z)|$ for $|z| = k$. Now for real or complex $\lambda$ with $|\lambda| < 1$, we have $|\lambda m| < m \leq |p(z)|$ for $|z| = k$. Therefore, it follows by Rouche’s theorem that the polynomial $G(z) = p(z) - \lambda m$ has no zero in $|z| < k$. Therefore, the polynomial

\begin{equation}
H(z) = \left( \frac{z}{k} \right)^n G \left( \frac{k^2}{z} \right) = Q(z) - \overline{\lambda} m \left( \frac{z}{k} \right)^n,
\end{equation}

where $Q(z) = \left( \frac{z}{k} \right)^n p \left( \frac{k^2}{z} \right)$, will have all its zeros in $|z| \leq k$, $k \leq 1$. Also, $|G(z)| = |H(z)|$ for $|z| = k$. Applying Lemma 3.7 for the polynomial $H(z)$ and $G(z)$, we have

\begin{equation}
\left| z D_\alpha G(z) + n\beta \frac{\alpha - A}{1 + A} G(z) \right| \leq \left| z D_\alpha H(z) + n\beta \frac{\alpha - A}{1 + A} \right|,
\end{equation}

where $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$.

Substituting for $G(z)$ and $H(z)$ in (4.2), we conclude that for every $\alpha$, $\beta$ with $|\alpha| \geq A$, $|\beta| \leq 1$ and $|z| = 1$
Combining the right hand sides of (4.3) and (4.5), we can rewrite inequality (4.5) as

\[ \left| zD_\alpha p(z) - \lambda mn z + n\beta \frac{\alpha - A}{1 + A} (p(z) - \lambda m) \right| \leq \left| zD_\alpha Q(z) - \overline{\lambda} \alpha mn \left( \frac{z}{k} \right)^n \right| + n\beta \frac{\alpha - A}{1 + A} \left\{ Q(z) - \overline{\lambda} m (\frac{z}{k})^n \right\}, \]

which implies that

\[ \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) - \lambda mn \left( z + \beta \frac{\alpha - A}{1 + A} \right) \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) - \overline{\lambda} mn \left( \frac{z}{k} \right)^n \left( \alpha + \beta \frac{\alpha - A}{1 + A} \right) \right|. \]  

(4.3)

Since all the zeros of \( Q(z) \) lie in \(|z| \leq k\) and \(|p(z)| = |Q(z)|\) for \(|z| = k\), therefore, by applying Lemma 2.4 to \( Q(z) \), we get

\[ \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) \right| \leq nk^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| \min_{|z|=k} |Q(z)| \]

(4.4)

\[ = nk^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| \min_{|z|=k} |p(z)|. \]

Then, for an appropriate choice of the argument of \( \lambda \), we have

\[ \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) - \lambda mn z + \beta \frac{\alpha - A}{1 + A} \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) - \lambda \overline{mnk}^{-n} \alpha + \beta \frac{\alpha - A}{1 + A} \right|, \text{ on } |z| = 1. \]

(4.5)

Combining the right hand sides of (4.3) and (4.5), we can rewrite inequality (4.5) as

\[ \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) \right| - |\lambda| mn \left| z + \beta \frac{\alpha - A}{1 + A} \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) \right| - |\lambda| \overline{mnk}^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| \text{ for } |z| = 1. \]

Equivalently,

\[ \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) \right| - |\lambda| mn \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| - \left| z + \beta \frac{\alpha - A}{1 + A} \right| \right\}. \]
As \(|\lambda| \to 1\), we have
\[
\left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) \right| \leq \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) \right|
- mn \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| + \left| z + \beta \frac{\alpha - A}{1 + A} \right| \right\}.
\]
It implies for every real or complex number \(\beta\) with \(|\beta| \leq 1\) and \(|z| = 1\),
\[
2 \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) \right| \leq \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) \right|
+ \left| zD_\alpha Q(z) + n\beta \frac{\alpha - A}{1 + A} Q(z) \right|
- mn \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| - \left| z + \beta \frac{\alpha - A}{1 + A} \right| \right\}.
\]
Inequality (4.6) in conjunction with Lemma 3.6 gives for \(|\beta| \leq 1\) and \(|z| = 1\),
\[
2 \left| zD_\alpha p(z) + n\beta \frac{\alpha - A}{1 + A} p(z) \right| \leq \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| + \left| z + \beta \frac{\alpha - A}{1 + A} \right| \right\} \max_{|z|=1} \left| p(z) \right|
- \left\{ k^{-n} \left| \alpha + \beta \frac{\alpha - A}{1 + A} \right| + \left| z + \beta \frac{\alpha - A}{1 + A} \right| \right\} \min_{|z|=k} \left| p(z) \right|,
\]
from which (2.1) follows.

\(\square\)

**CONFLICT OF INTERESTS**

The authors declare that there is no conflict of interests.

**REFERENCES**


