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# ON $(p,q)$-ANALOGUES OF SOME GENERALIZED OPIAL'S INTEGRAL INEQUALITIES

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**Abstract.** In this work, we obtain $(p,q)$-analogues of generalized Opial's integral inequalities. We also present some further extensions of the new analogues. The fundamental theorem of $(p,q)$-calculus and the $(p,q)$-Hölder's integral inequality were employed to establish the results.

**Keywords:** generalized opial integral inequality; $(p,q)$-Hölder's integral inequality; $(p,q)$-analogue; $(p,q)$-calculus.

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## 1. INTRODUCTION

Opial established an inequality involving integral of a function and its derivative in [13] as

$$\int_0^h |f(t)f'(t)|dt \leq \frac{h}{4}\int_0^h (f'(t))^2 dt, \tag{1}$$

where $f \in C^1[0,h]$, such that $f(0) = f(h) = 0$, $f'(t) > 0$ and $t \in [0,h]$. The coefficient $h/4$ is the best constant possible.

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This inequality, due to its significance, experienced a lot of extensions and generalizations over time in the classical field. See [3], [4], [5] and [16], among others.

In [16], generalizations of the classical Opial's inequality were established as

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{(b-a)}{2}\int_a^b \left|f'(x)\right|^2 dx \tag{2}$$

and

$$\int_a^b \left|f(x)f'(x)\right| dx \leq \frac{(b-a)}{4}\int_a^b \left|f'(x)\right|^2 dx, \tag{3}$$

where the coefficients $(b-a)/2$ and $(b-a)/4$ are their respective best constants possible.

$(p,q)$-Calculus is a generalization of $q$-calculus. There has been a lot of development in the study of $(p,q)$-calculus. Recently, Sadjang [15] investigated on fundamental concepts of $(p,q)$-calculus. In [8], $(p,q)$-derivatives and $(p,q)$-integrals and their properties are also presented.

In [7], the authors established a $(p,q)$-analogue of a generalized Opial type inequality as

$$\int_0^b |\omega(px)||D_{p,q}\omega(x)|d_{p,q}x \leq \frac{b}{4}\int_0^b |(D_{p,q}\omega(x)|^2 d_{p,q}x. \tag{4}$$

where $\omega \in C[0,b]$ with $\omega(0) = \omega(b) = 0$ and $0 < q < p \leq 1$.

See also [1], [2], [7] and [11] for more analogues of the Opial's type inequalities.

The Opial inequality plays essential role in establishing the existence and uniqueness of initial and boundary values problems for both ordinary and partial differential equations [2] and [7].

The objective of this paper is to establish $(p,q)$-analogues of the generalized Opial integral inequalities (2) and (3).

## 2. PRELIMINARIES

The basic concepts and terminologies of $(p,q)$-calculus which will be used to prove our results are presented in this section. The definitions provided can also be seen in [8], [9] [11], [14], [12] and [15].

**Definition 2.1.** [8] For any arbitrary function $f$ in the real-line, the $(p,q)$-derivative is defined as

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0. \tag{5}$$

**Definition 2.2.** [8] For any positive real $\alpha$, the twin basic number or the $(p,q)$-Number $\alpha$ is defined as

$$[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p-q} = p^{\alpha-1} + p^{\alpha-2}q + \cdots + pq^{\alpha-2} + q^{\alpha-1}, \tag{6}$$

$0 < q < p \leq 1, \quad \alpha \in \mathbf{R}^+.$

The $(p,q)$-Derivative of sum or difference of $f$ and $g$ is defined as

$$D_{p,q}(\alpha f(x) \pm \beta g(x)) = \alpha D_{p,q} f(x) \pm \beta D_{p,q} g(x). \tag{7}$$

The $(p,q)$-Derivative of product of $f$ and $g$ is defined as

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) \\ &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x). \end{aligned} \tag{8}$$

The $(p,q)$-Derivative of a quotient of $f$ and $g$ is defined as

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)} \\ &= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \quad g(px)g(qx) \neq 0. \end{aligned} \tag{9}$$

**Definition 2.3.** [6] (Composite Rule) Let $f$ be a function of a power function $g$, the $(p,q)$-derivative of $f(g(x))$ is defined as

$$D_{p,q}(f(g(x))) = D_{p^k,q^k} f(g(x)) D_{p,q} g(x), \tag{10}$$

where $k$ is real and index of $g$.

**Lemma 2.1.** *Let $\alpha \in \mathbf{R}^+$, then*

$$D_{p,q}(x-a)^\alpha = [\alpha]_{p,q}(x-a)^{\alpha-1}. \tag{11}$$

*Proof.*

$$\begin{aligned} D_{p,q}(x-a)^\alpha &= \frac{p(x-a)^\alpha - ((x-a)q)^\alpha}{(p-q)(x-a)} \\ &= \frac{(p-q^\alpha)}{(p-q)}(x-a)^{\alpha-1} \\ &= [\alpha]_{p,q}(x-a)^{\alpha-1}. \end{aligned}$$

This completes the proof.

**Definition 2.4.** [15] Let $f : [0,b] \to \mathbf{R}$ be a continuous function and $0 < q < p \leq 1$. The definite $(p,q)$-integral of $f$ on $[0,b]$ is defined as

$$\int_0^b f(x)d_{p,q}x = (p-q)b\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}b\right). \tag{12}$$

If $a \in (0,b)$, the definite $(p,q)$-integral of $f$ on $[a,b]$ is defined as

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x. \tag{13}$$

**Remark 2.1.** Taking $p = 1$, equation (12) reduces to the well known Jackson $q$-integral [10]

$$\int_0^b f(x)d_qx = (1-q)b\sum_{j=0}^{\infty}q^j f(bq^j). \tag{14}$$

**Definition 2.5.** [12]The function $f$ defined on $[a,b]$ is called $(p,q)$-increasing or $(p,q)$-decreasing on $[a,b]$, if $f(qx) \leq f(px)(f(qx) \geq f(px))$, for $qx$, $px \in [a,b]$.

It is easily observed that if the function $f$ is increasing (decreasing), then it is also $(p,q)$-increasing ($(p,q)$-decreasing).

**Definition 2.6.** [15](Fundamental Theorem of $(p,q)$-Calculus) If $f \in C[a,b]$, $F$ is an antiderivative of $f$ on $x \in [a,b]$, then

$$F(x) = \int_a^x f(t)d_{p,q}t. \tag{15}$$

**Lemma 2.2.** [17] *($(p,q)-$Hölder's Inequality ) Let $\alpha, \beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. If $f$ and $g$ are continuous real-valued functions on $[a,b]$, then*

$$\int_a^b |f(x)g(x)|d_{p,q}x \leq \left(\int_a^b |f(x)|^{\alpha}d_{p,q}x\right)^{\frac{1}{\alpha}}\left(\int_a^b |g(x)|^{\beta}d_{p,q}x\right)^{\frac{1}{\beta}}, \tag{16}$$

*holds. With equality when $|g(x)| = c|f(x)|^{\alpha-1}$. If $\alpha = \beta = 2$, the inequality becomes $(p,q)$-Cauchy-Bunyakovsky-Schwartz's Integral Inequality.*

## 3. Main Results

**Lemma 3.1.** *Let $h:[a,b]\to\mathbf{R}$ be an absolutely continuous and a differentiable function, such that $D_{p,q}h\in L_\beta[a,b]$, $1\le\beta<\infty$ and $0<q<p\le 1$. Then*

$$\left(\int_a^b |D_{p,q}h(x)|d_{p,q}x\right)^\beta \le (b-a)^{\beta-1}\int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x \tag{17}$$

*holds.*

*Proof.* Applying $(p,q)$-Hölder's inequality we have

$$\begin{aligned}\left(\int_a^b |D_{p,q}h(x)|d_{p,q}x\right)^\beta &= \left(\int_a^b t^{\frac{1}{\beta}}|D_{p,q}h(x)|t^{-\frac{1}{\beta}}d_{p,q}x\right)^\beta\\ &\le \left[\left(\int_a^b t|D_{p,q}h(x)|^\beta d_{p,q}x\right)^{\frac{1}{\beta}}\left(\int_a^b (t^{-\frac{1}{\beta}})^{\frac{\beta}{\beta-1}}d_{p,q}x\right)^{\frac{\beta-1}{\beta}}\right]^\beta\\ &= \int_a^b t|D_{p,q}h(x)|^\beta d_{p,q}x\left(\int_a^b t^{-\frac{1}{\beta-1}}d_{p,q}x\right)^{\beta-1}\\ &= (b-a)^{\beta-1}\int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x.\end{aligned}$$

This completes the proof.

**Theorem 3.1.** *Let $h:[a,b]\to\mathbf{R}$ be an absolutely continuous function, such that $D_{p,q}h\in L_\beta[a,b]$, $h(a)=0,$ (or $\quad h(b)=0$), $1\le\beta<\infty$ and $0<q<p\le 1$. Then*

$$\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \le \frac{(b-a)^{\beta-1}}{[\beta]_{p,q}}\int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x \tag{18}$$

*holds.*

*Proof.* Let $\phi$ be a convex function on $[0,\infty)$ with $\phi(0)=0$, $x\in[a,b]$, $h(a)=0$ and

$$y(x)=\int_a^x |D_{p,q}h(t)|d_{p,q}t.$$

Then

$$J(x)=\phi(y(x))=\phi\left(\int_a^x |D_{p,q}h(t)|d_{p,q}t\right). \tag{19}$$

Since $D_{p,q}y(x)=|D_{p,q}h(x)| \quad$ and $\quad |h(x)|\le y(x)$, then we have

$$D_{p,q}J(x)=D_{p,q}\phi(y(x))|D_{p,q}h(x)|\ge D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|. \tag{20}$$

Thus

$$\int_a^b D_{p,q}J(x)d_{p,q}x = \phi\left(y(b)\right) - \phi(y(a)) \geq \int_a^b D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x. \tag{21}$$

Since $\phi(0) = 0$, (21) becomes

$$\int_a^b D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x \leq \phi\left(\int_a^b |D_{p,q}h(x)|d_{p,q}x\right). \tag{22}$$

Letting $\phi(x) = \frac{x^\beta}{\beta}$ for $1 \leq \beta < \infty$ in (22) we obtain

$$\frac{[\beta]_{p,q}}{\beta}\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \leq \frac{1}{\beta}\left(\int_a^b |D_{p,q}h(x)|d_{p,q}x\right)^\beta. \tag{23}$$

Applying Lemma 3.1 to (23) yields

$$\frac{[\beta]_{p,q}}{\beta}\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \leq \frac{(b-a)^{\beta-1}}{\beta}\int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x, \tag{24}$$

which implies

$$\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \leq \frac{(b-a)^{\beta-1}}{[\beta]_{p,q}}\int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x.$$

This completes the proof.

**Remark 3.1.** Letting $\beta = 2$, $p = 1$ and taking limit of (26) as $q \to 1$ yields (2).

**Remark 3.2.** Putting $\beta = 3$ into (18) yields

$$\int_a^b |D_{p,q}h(x)||h(px)|^2 d_{p,q}x \leq \frac{(b-a)^2}{[3]_{p,q}}\int_a^b |D_{p,q}h(x)|^3 d_{p,q}x. \tag{25}$$

This simplifies to

$$\begin{aligned}\int_a^b |D_{p,q}h(x)||h(px)|^2 d_{p,q}x &\leq \frac{(p-q)(b-a)^2}{(p^3-q^3)}\int_a^b |D_{p,q}h(x)|^3 d_{p,q}x \\ &= \frac{(b-a)^2}{p^2+pq+q^2}\int_a^b |D_{p,q}h(x)|^3 d_{p,q}x. \end{aligned} \tag{26}$$

which is the $(p,q)$-extension of (2).

**Theorem 3.2.** *Let $h \in C^n[a,b]$ be a differentiable function, such that $h(a) = 0$, for $1 \leq i \leq n-1$, $1 \leq \beta < \infty$ and $0 < q < p \leq 1$. Then*

$$\int_a^b (x-a)^{n-1}|D^n_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \leq \frac{(b-a)^{\beta n-1}}{[\beta]_{p,q}}\int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x \tag{27}$$

*holds.*

*Proof.* Let $\phi$ be a convex function on $[0,\infty)$ with $\phi(0)=0$, $x\in[a,b]$, $h(a)=0$ and

$$y(x)=\int_a^x\int_a^{x_{n-1}}\cdots\int_a^{x_1}|D_{p,q}^{(n)}h(s)|d_{p,q}sd_{p,q}x_1\dots d_{p,q}x_{n-1}.$$

So that

$D_{p,q}^{(n)}y(x)=|D_{p,q}^{(n)}h(x)|$, $y(x)\geq|h(x)|$ and $D_{p,q}^{(i)}y(x)\geq 0$

By [15], it follows that

$$D_{p,q}^{(i)}y(x)\leq(x-a)D_{p,q}^{(i+1)}y(x),\quad x\in[a,b],\quad 0\leq i\leq n-2. \tag{28}$$

It implies that

$$|h(x)|\leq y(x)\leq(x-a)D_{p,q}y(x)\leq\cdots\leq(x-a)^{(n-1)}D_{p,q}^{(n-1)}y(x). \tag{29}$$

Consider

$$W(x)=\phi((x-a)^{(n-1)}D_{p,q}^{(n-1)}y(x)). \tag{30}$$

Applying Lemma 2.1, then

$$\begin{aligned}D_{p,q}W(x)&=D_{p,q}\phi((x-a)^{(n-1)}D_{p,q}^{(n-1)}y(x))D_{p,q}[(x-a)^{(n-1)}D_{p,q}^{(n-1)}y(x)]\\&=D_{p,q}\phi((x-a)^{(n-1)}D_{p,q}^{(n-1)}y(x))D_{p,q}^{(n-1)}y(x)\frac{(p(x-a))^{(n-1)}-((x-a)q)^{(n-1)}}{(p-q)(x-a)}+\\&(x-a)^{(n-1)}D_{p,q}D_{p,q}^{(n-1)}y(x)\\&=D_{p,q}\phi((x-a)^{(n-1)}D_{p,q}^{(n-1)}y(x))[[n-1]_{p,q}(x-a)^{(n-2)}D_{p,q}^{(n-1)}y(x)\\&+(x-a)^{(n-1)}D_{p,q}^{(n)}y(x)].\end{aligned} \tag{31}$$

From (31) we have

$$\begin{aligned}D_{p,q}W(x)&\geq D_{p,q}\phi(|h(x)|)(x-a)^{(n-1)}D_{p,q}^{(n)}y(x)\\&=D_{p,q}\phi(|h(x)|)(x-a)^{(n-1)}|D_{p,q}^{(n)}h(x)|.\end{aligned} \tag{32}$$

Thus

$$\int_a^b D_{p,q}W(x)d_{p,q}x = \phi((b-a)^{n-1}D_{p,q}^{(n-1)}y(b)) - \phi(0)$$

$$\geq \int_a^b D_{p,q}\phi(|h(x)|)(x-a)^{n-1}|D_{p,q}^{(n)}h(x)|d_{p,q}x. \tag{33}$$

Since $\phi(0)=0$, (33) becomes

$$\int_a^b D_{p,q}\phi(|h(x)|)(x-a)^{n-1}|D_{p,q}^{(n)}h(x)|d_{p,q}x \leq \phi\left((b-a)^{n-1}\int_a^b D_{p,q}^{(n)}y(x)d_{p,q}x\right), \tag{34}$$

which results

$$\int_a^b D_{p,q}\phi(|h(x)|)(x-a)^{n-1}|D_{p,q}^{(n)}h(x)|d_{p,q}x$$

$$\leq \phi\left((b-a)^{n-1}\int_a^b |D_{p,q}^{(n)}h(x)|d_{p,q}x\right). \tag{35}$$

Considering $\phi(x)=\frac{x^\beta}{\beta}$ for $1\leq\beta<\infty$ in (35) we obtain

$$\frac{[\beta]_{p,q}}{\beta}\int_a^b (x-a)^{n-1}|D_{p,q}^{(n)}h(x)||h(px)|^{\beta-1}d_{p,q}x$$

$$\leq \frac{1}{\beta}\left((b-a)^{n-1}\int_a^b |D_{p,q}^{(n)}h(x)|d_{p,q}x\right)^\beta. \tag{36}$$

This simplifies to

$$\frac{[\beta]_{p,q}}{\beta}\int_a^b (x-a)^{n-1}|D_{p,q}^{(n)}h(x)||h(px)|^{\beta-1}d_{p,q}x$$

$$\leq \frac{(b-a)^{\beta(n-1)}}{\beta}\left(\int_a^b |D_{p,q}^{(n)}h(x)|d_{p,q}x\right)^\beta. \tag{37}$$

Applying Lemma 3.1 to the right-hand side of (37) yields

$$\frac{[\beta]_{p,q}}{\beta}\int_a^b (x-a)^{n-1}|D_{p,q}^{(n)}h(x)||h(px)|^{\beta-1}d_{p,q}x$$

$$\leq \frac{(b-a)^{\beta(n-1)}(b-a)^{\beta-1}}{\beta}\int_a^b |D_{p,q}^{(n)}h(x)|^\beta d_{p,q}x, \tag{38}$$

which implies

$$\int_a^b (x-a)^{n-1}|D^{(n)}p,qh(x)||h(px)|^{\beta-1}d_{p,q}x \leq \frac{(b-a)^{\beta n-1}}{[\beta]_{p,q}}\int_a^b |D_{p,q}^{(n)}h(x)|^\beta d_{p,q}x. \tag{39}$$

This completes the proof.

**Remark 3.3.** By letting $\beta=2$, $n=1$ $p=1$ and taking limit of (41) as $q\to 1$ yields (2).

**Remark 3.4.** Putting $\beta = 4$ in (27) yields

$$\int_a^b (x-a)^{n-1}|D_{p,q}^{(n)}h(x)||h(px)|^3 d_{p,q}x \le \frac{(b-a)^{4n-1}}{[4]_{p,q}} \int_a^b |D_{p,q}^{(n)}h(x)|^4 d_{p,q}x. \tag{40}$$

This simplifies to

$$\begin{aligned}\int_a^b (x-a)^{n-1}|D_{p,q}^{(n)}h(x)||h(px)|^3 d_{p,q}x &\le \frac{(p-q)(b-a)^{4n-1}}{(p^4-q^4)} \int_a^b |D_{p,q}^{(n)}h(x)|^4 d_{p,q}x \\ &= \frac{(b-a)^{4n-1}}{p^3+p^2q+pq^2+q^3} \int_a^b |D_{p,q}^{(n)}h(x)|^4 d_{p,q}x,\end{aligned} \tag{41}$$

for $n \ge 1$.

which is the $(p,q)$-extension of (2).

**Theorem 3.3.** *Let $h : [a,b] \to \mathbf{R}$ be an absolutely continuous function, such that $D_{p,q}h \in L_\beta[a,b]$, $h(a) = h(b) = 0$, $1 \le \beta \le \infty$ and $0 < q < p \le 1$. Then*

$$\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1} d_{p,q}x \le \frac{(b-a)^{\beta-1}}{2^{\beta-1}[\beta]_{p,q}} \int_a^b |D_{p,q}h(x)|^\beta d_{p,q}x \tag{42}$$

*holds.*

*Proof.* Let $\phi$ be a convex function on $[0,\infty)$ with $\phi(0) = 0$, $x \in [a,b]$, $h(a) = 0$ and

$$y(x) = \int_a^x |D_{p,q}h(t)| d_{p,q}t.$$

Then

$$J(x) = \phi(y(x)) = \phi\left(\int_a^x |D_{p,q}h(t)| d_{p,q}t\right). \tag{43}$$

Since $D_{p,q}y(x) = |D_{p,q}h(x)|$ and $|h(x)| \le y(x)$, then

$$D_{p,q}J(x) = D_{p,q}\phi(y(x))|D_{p,q}h(x)| \ge D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|. \tag{44}$$

Also, let

$$z(x) = \int_x^b |D_{p,q}h(t)| d_{p,q}t \tag{45}$$

for $h(b) = 0$, then

$$T(x) = -\phi(z(x)) = -\phi\left(\int_x^b |D_{p,q}h(t)| d_{p,q}t\right). \tag{46}$$

Since $D_{p,q}z(x) = -|D_{p,q}h(x)|$ and $|h(x)| \leq z(x)$, then

$$D_{p,q}T(x) = D_{p,q}\phi(z(x))|D_{p,q}h(x)| \geq D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|. \tag{47}$$

Let $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ be subintervals of $[a,b]$.
By (44) we obtain

$$\begin{aligned} \int_a^{\frac{a+b}{2}} D_{p,q}J(x)d_{p,q}x &= \phi\left(y\left(\frac{a+b}{2}\right)\right) - \phi(y(a)) \\ &\geq \int_a^{\frac{a+b}{2}} D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x. \end{aligned} \tag{48}$$

Since $\phi(0) = 0$, (48) becomes

$$\phi\left(\int_a^{\frac{a+b}{2}} |D_{p,q}h(x)|d_{p,q}x\right) \geq \int_a^{\frac{a+b}{2}} D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x. \tag{49}$$

Also, by (47) we obtain

$$\begin{aligned} \int_{\frac{a+b}{2}}^b D_{p,q}T(x)d_{p,q}x &= \phi(z(b)) - \phi\left(z\left(\frac{a+b}{2}\right)\right) \\ &\geq \int_{\frac{a+b}{2}}^b D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x. \end{aligned} \tag{50}$$

Since $\phi(0) = 0$, (50) becomes

$$\phi\left(\int_{\frac{a+b}{2}}^b |D_{p,q}h(x)|d_{p,q}x\right) \geq \int_{\frac{a+b}{2}}^b D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x. \tag{51}$$

Adding inequalities (49) and (51) we obtain

$$\begin{aligned} \int_a^b D_{p,q}\phi(|h(x)|)|D_{p,q}h(x)|d_{p,q}x &\leq \phi\left(\int_a^{\frac{a+b}{2}} |D_{p,q}h(x)|d_{p,q}x\right) \\ &\quad + \phi\left(\int_{\frac{a+b}{2}}^b |D_{p,q}h(x)|d_{p,q}x\right). \end{aligned} \tag{52}$$

Now, for $\phi(x) = \frac{x^\beta}{\beta}$, $1 \leq \beta < \infty$ in (52) we have

$$\begin{aligned} \frac{[\beta]_{p,q}}{\beta} \int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x &\leq \frac{1}{\beta}\left(\int_a^{\frac{a+b}{2}} |D_{p,q}h(x)|d_{p,q}x\right)^\beta \\ &\quad + \frac{1}{\beta}\left(\int_{\frac{a+b}{2}}^b |D_{p,q}h(x)|d_{p,q}x\right)^\beta. \end{aligned} \tag{53}$$

Applying Lemma 3.1 to (53) yields

$$\begin{aligned}\frac{[\beta]_{p,q}}{\beta}\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \leq & \frac{(b-a)^{\beta-1}}{2^{\beta-1}\beta}\int_a^{\frac{a+b}{2}} |D_{p,q}h(x)|^{\beta}d_{p,q}x \\ &+\frac{(b-a)^{\beta-1}}{2^{\beta-1}\beta}\int_{\frac{a+b}{2}}^b |D_{p,q}h(x)|^{\beta}d_{p,q}x, \end{aligned} \tag{54}$$

which simplifies to

$$\int_a^b |D_{p,q}h(x)||h(px)|^{\beta-1}d_{p,q}x \leq \frac{(b-a)^{\beta-1}}{2^{\beta-1}[\beta]_{p,q}}\int_a^b |D_{p,q}h(x)|^{\beta}d_{p,q}x. \tag{55}$$

This completes the proof.

**Remark 3.5.** The inequality 55 is sharper than the inequality 39 .

**Remark 3.6.** By letting $\beta=2$, $p=1$ and taking limit of (55) as $q\to 1$ we obtain (3).

**Remark 3.7.** Putting $\beta=3$ into (42) yields

$$\int_a^b |D_{p,q}h(x)||h(px)|^{2}d_{p,q}x \leq \frac{(b-a)^{2}}{4[3]_{p,q}}\int_a^b |D_{p,q}h(x)|^{3}d_{p,q}x. \tag{56}$$

This simplifies to

$$\begin{aligned}\int_a^b |D_{p,q}h(x)||h(px)|^{2}d_{p,q}x \leq & \frac{(p-q)(b-a)^{2}}{4(p^3-q^3)}\int_a^b |D_{p,q}h(x)|^{3}d_{p,q}x \\ =& \frac{(b-a)^{2}}{4(p^2+pq+q^2)}\int_a^b |D_{p,q}h(x)|^{3}d_{p,q}x, \end{aligned} \tag{57}$$

which is the $(p,q)$-extension of (3).

**Theorem 3.4.** *Let $h:[a,b]\to \mathbf{R}$ be an absolutely continuous function, such that $D_{p,q}h\in L_{\beta}[a,b]$, $h(a)=0$, $1\leq\beta\leq\infty$ and $0<q<p\leq 1$.*

$$\int_a^b \left|h^{\beta}(x)D_{p,q}h(x)\right| d_{p,q}x \leq \frac{(b-a)^{\beta}}{\beta+1}\int_a^b \left|D_{p,q}h(x)\right|^{\beta+1}d_{p,q}x \tag{58}$$

*holds.*

*Proof.* Let $x\in[a,b]$, $0<q<p\leq 1$ and by [15] we have

$$y(x)=\int_a^x |D_{p,q}h(t)|d_{p,q}t. \tag{59}$$

So that

$D_{p,q}y(x) = |D_{p,q}h(x)|$ and $|h(x)| \leq y(x)$.

It follows that

$$\begin{aligned}\int_a^b |h^\beta(x)D_{p,q}h(x)|d_{p,q}x &\leq \int_a^b y^\beta(x)D_{p,q}y(x)d_{p,q}x \\ &= \frac{1}{\beta+1}y^{(\beta+1)}(b).\end{aligned}$$

But

$$y^{(\beta+1)}(b) = \left(\int_a^b |D_{p,q}h(x)|d_{p,q}x\right)^{\beta+1}. \tag{60}$$

Implying that

$$\int_a^b |h^\beta(x)D_{p,q}h(x)|d_{p,q}x \leq \frac{1}{\beta+1}\left(\int_a^b |D_{p,q}h(x)|d_{p,q}x\right)^{\beta+1}. \tag{61}$$

Applying Lemma 3.1 to (61) we obtain

$$\int_a^b |h^\beta(x)D_{p,q}h(x)|d_{p,q}x \leq \frac{(b-a)^\beta}{\beta+1}\int_a^b |D_{p,q}h(x)|^{\beta+1}d_{p,q}x. \tag{62}$$

This completes the proof.

**Remark 3.8.** By letting $\beta = 1$, $p = 1$ and taking limit of (62) as $q \to 1$ we obtain (2).

## CONCLUSION

In this work, $(p,q)$-analogues of generalized Opial's integral inequalities and their further extensions were established. The basic definitions of $(p,q)$-calculus, the fundamental theorem of $(p,q)$-calculus and convexity properties of functions were employed to obtain the results. The $(p,q)$-Hölder's integral inequality was also applied in proving the theorems. It is hoped that these results will be very useful to the mathematics community.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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