THE EQUIVALENCE ABOUT THE T-STABILITIES AND THE ALMOST T-STABILITIES BETWEEN MANN AND ISHIKAWA ITERATION

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Abstract: In this paper, we prove that the equivalence about the T-stabilities and the almost T-stabilities of Mann and Ishikawa iteration.

Keywords: a T-stabilities, almost T-stabilities, Mann iteration, Ishikawa iteration.

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1. Introduction

Definition 1.1. Let $T : E \to E$ be a mapping, $x_0 \in E$ be a given point, \{\alpha_n\}, \{\beta_n\}, be two sequences in $[0,1]$, and \{u_n\}, \{v_n\}, \{r_n\}, be three bounded sequences in E.

1. the sequence $\{x_n\} \subset E$ defined by:

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n Tz_n + r_n \quad (1.1)$$

is called the Mann iteration with errors.

2. the sequence $\{x_n\} \subset E$ defined by:

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\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + u_n \]
\[ y_n = (1 - \beta_n)x_n + \beta_n Tx_n + v_n \] (1.2)

is called the Ishikawa iteration with errors.

3. the sequence \( \{x_n\} \subseteq E \) defined by:
\[ z_{n+1} = (1 - \alpha_n)z_n + \alpha_n T^n z_n + r_n \] (1.3)
is called the modified Mann iteration with errors.

4. the sequence \( \{x_n\} \subseteq E \) defined by:
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n \]
\[ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n + v_n \] (1.4)
is called the modified Ishikawa iteration with errors.

Now we give the stability definition of the sequence \( \{x_n\}_{n=1}^\infty \) defined by (1.1) and (1.2)

**Definition 1.2.** Let \( \{x_n\}_{n=1}^\infty \) be the sequence defined by (1.1), (1.2) and such that
\( x_n \to p \in F(T), F(T) \) denotes the set of fixed point of T. Let \( \{s_n\} \{q_n\} \) be any bounded sequence in E by:

(i) \[ \delta_n = \|q_{n+1} - (1 - \alpha_n)q_n - \alpha_n Tq_n - r_n\|, \quad n \geq 0 \] (1.5)

(ii) \[ \varepsilon_n = \|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ty_n - u_n\|, \quad y_n = (1 - \beta_n)s_n + \beta_n Ts_n + v_n, \quad n \geq 0 \] (1.6)

If \( \delta_n \to 0 \) implies that \( q_n \to p \), then the Mann iterative sequence \( \{x_n\} \) is said to be \( T \)-stable; \( \varepsilon_n \to 0 \) implies that \( s_n \to p \), then the Ishikawa iterative sequence \( \{x_n\} \) is said to be \( T \)-stable; if \( \sum_{n=0}^{\infty} \delta_n < +\infty \) implies that \( q_n \to p \), then the Mann sequence \( \{x_n\} \) is said to be almost \( T \)-stable; \( \sum_{n=0}^{\infty} \varepsilon_n < +\infty \) implies that \( s_n \to p \), then the Ishikawa sequence \( \{x_n\} \) is said to be almost \( T \)-stable.

The equivalence between the Ishikawa iteration and Mann iteration for several
classes of maps has been studied in [1-5], and proved the equivalence between T-stable of (1.1) and (1.2). In this paper, we shall prove the equivalence between T-stable and almost T-stable of (1.3) and (1.4). Throughout this paper, we shall assume that X is a normed space. T is a map on X with a bounded range and assume that both Mann and Ishikawa iterations with errors converge to a fixed point of T.

2. The equivalence between T-stabilities

Theorem 2.1. Let X be a normed space and T: X → X be a map with a bounded range. For all \( \alpha_n, \beta_n \in (0,1) \), \( u_n \), \( r_n \), satisfy

\[
\lim_{n \to \infty} \alpha_n = 0, \lim_{n \to \infty} \beta_n = 0, \|u_n\| \to 0, \|r_n\| \to 0, \quad \text{as} \quad n \to \infty
\]

The following are equivalent:

(i) the Ishikawa iteration sequence with errors (1.2) is T-stable.

(ii) the Mann iteration sequence with errors (1.1) is T-stable.

Proof:

We first prove (ii) ⇒ (i) In (1.6), let \( s_n = q_n \), we have:

\[
\|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Tq_n - r_n\| \\
\leq \|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Ty_n - u_n + \alpha_n Ty_n + u_n - \alpha_n Tq_n - r_n\| \\
\leq \|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Ty_n - u_n\| + \|\alpha_n Ty_n - \alpha_n Tq_n\| + \|u_n - r_n\| \\
\leq \|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Ty_n - u_n\| + \alpha_n \|Ty_n - Tq_n\| + \|u_n\| + \|r_n\|
\]

from the definition, we can know \( \forall \{s_n\} \subset X \) are bounded sequence such that \( \{Ts_n\} \) is bounded so we can let

\[
M := \max\left\{\sup_{n \in \mathbb{N}}\|T(y_n)\|, \sup_{n \in \mathbb{N}}\|T(s_n)\|, \sup_{n \in \mathbb{N}}\|T(p_n)\|\right\}
\]

then \( M < +\infty \). Then we obtain:

\[
\|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Tq_n - r_n\| \\
\leq \|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Ty_n - u_n\| + 2\alpha_n M + \|u_n\| + \|r_n\| \to 0 \quad (n \to \infty)
\]

We have \( \lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \|q_{n+1} - (1- \alpha_n)q_n - \alpha_n Tq_n - r_n\| = 0 \Rightarrow \lim_{n \to \infty} q_n = p \) by the condition (ii), thus, for \( \forall \{q_n\} \subset X \) satisfying
\[ \lim_{n \to \infty} \varepsilon_n = \lim_{n \to \infty} \|q_{n+1} - (1 - \alpha_n)q_n - \alpha_n Ty_n - u_n\| = 0, \]
we have \( \lim_{n \to \infty} q_n = p \) i.e. (ii) \( \Rightarrow \) (i).

Then we prove (i) \( \Rightarrow \) (ii). In (1.5), Let \( q_n = s_n \), we have:

\[
\|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ts_n - u_n\|
\leq\|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ts_n - r_n\| + \|\alpha_n Ty_n - \alpha_n Ts_n\| + \|u_n - r_n\|
\leq\|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ts_n - r_n\| + \alpha_n\|Ty_n - Ts_n\| + \|u_n\| + \|r_n\|
\leq\|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ts_n - r_n\| + 2\alpha_n M + \|u_n\| + \|r_n\| \to 0 \quad (n \to \infty)
\]

We have \( \lim_{n \to \infty} \varepsilon_n = \lim_{n \to \infty} \|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ts_n - u_n\| = 0 \Rightarrow \lim_{n \to \infty} s_n = p \) by the condition (i), thus, for \( \forall \{s_n\} \subset X \) satisfying \( \lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \|s_{n+1} - (1 - \alpha_n)s_n - \alpha_n Ts_n - r_n\| = 0 \), we have \( \lim_{n \to \infty} s_n = p \) i.e.(i) \( \Rightarrow \) (ii).

3. Main results

**Theorem 3.1.** Let \( X \) be a normed space and \( T : X \to X \) be a map with a bounded range. For all \( \{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1) \{u_n\}, \{r_n\} \), satisfy

\[
\sum_{n=1}^{\infty} \alpha_n < +\infty, \sum_{n=1}^{\infty} \beta_n < +\infty, \sum_{n=1}^{\infty} u_n < +\infty, \sum_{n=1}^{\infty} r_n < +\infty,
\]

The following are equivalent:

(i) the Ishikawa iteration with errors (1.2)is almost T-stable.

(ii) the Mann iteration with errors (1.1) is almost T-stable.

**Proof:** in (2.1), we have

\[
\|q_{n+1} - (1 - \alpha_n)q_n - \alpha_n Tq_n - r_n\|
\leq\|q_{n+1} - (1 - \alpha_n)q_n - \alpha_n Tq_n - u_n\| + 2\alpha_n M + \|u_n\| + \|r_n\|
\leq\varepsilon_n + 2\alpha_n M + \|u_n\| + \|r_n\| \tag{3.1}
\]

Because \( \sum_{n=1}^{\infty} \alpha_n < +\infty, M < +\infty, \sum_{n=1}^{\infty} u_n < +\infty, \sum_{n=1}^{\infty} r_n < +\infty \) and we have

\[
\sum_{n=0}^{\infty} \delta_n = \sum_{n=0}^{\infty} \|q_{n+1} - (1 - \alpha_n)q_n - \alpha_n Tq_n - r_n\| < +\infty \Rightarrow \lim_{n \to \infty} q_n = p \] by the
condition (ii), thus, for \( \forall \{q_n\} \subset X \) satisfying

\[
\sum_{n=0}^{\infty} e_n = \sum_{n=0}^{\infty} \| q_{n+1} - (1 - \alpha_n) q_n - \alpha_n T y_n - u_n \| < +\infty, \text{ we have } \lim_{n \to \infty} q_n = p \quad \text{i.e. (ii) } \Rightarrow (i).
\]

Conversely, In (2.2) we have:

\[
\| s_{n+1} - (1 - \alpha_n) s_n - \alpha_n T y_n - u_n \| \\
\leq \| s_{n+1} - (1 - \alpha_n) s_n - \alpha_n T s_n - r_n \| + 2 \alpha_n M + \| u_n \| + \| r_n \| 
\]

(3.2)

Because \( \sum_{n=0}^{\infty} \alpha_n < +\infty, M < +\infty, \sum_{n=0}^{\infty} u_n < +\infty, \sum_{n=0}^{\infty} r_n < +\infty \) and we have

We have

\[
\sum_{n=0}^{\infty} e_n = \sum_{n=0}^{\infty} \| s_{n+1} - (1 - \alpha_n) s_n - \alpha_n T y_n - u_n \| < +\infty \Rightarrow \lim_{n \to \infty} s_n = p \quad \text{by the}
\]

condition(i),thus, for \( \forall \{s_n\} \subset X \) satisfying

\[
\sum_{n=0}^{\infty} \delta_n = \sum_{n=0}^{\infty} \| s_{n+1} - (1 - \alpha_n) s_n - \alpha_n T s_n - r_n \| < +\infty, 
\]

We have \( \lim_{n \to \infty} s_n = p \quad \text{i.e. (i) } \Rightarrow (\text{ii}). \)

In (1.1) and (1.2), set \( T := T^n \), we can obtain the modified Mann and Modified Ishikawa iteration with errors (1.3) (1.4). We also suppose that the modified Mann and Modified Ishikawa iteration with errors converge to a fixed point of \( T \). Note that Definition1.2 and Theorem 2.1 and Theorem 3.1 hold in this case too.

**Theorem 3.2.** Let \( X \) be a normed space and \( T : X \to X \) be a map with a bounded range. For all \( \{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1), \{u_n\}, \{v_n\}, \) satisfy

\[
\lim_{n \to \infty} \alpha_n = 0, \lim_{n \to \infty} \beta_n = 0, \| u_n \| \to 0, \| r_n \| \to 0 \quad \text{as} \quad n \to \infty \]

The following are equivalent:

(i) the modified Ishikawa iteration with errors is \( T \)-stable.

(ii) the modified Mann iteration with errors is \( T \)-stable.

**Theorem 3.3.** Let \( X \) be a normed space and \( T : X \to X \) be a map with a bounded range. For all \( \{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1), \{r_n\}, \{u_n\}, \) satisfy
\[ \sum_{n=1}^{\infty} \alpha_n < +\infty, \sum_{n=1}^{\infty} \beta_n < +\infty, \sum_{n=1}^{\infty} u_n < +\infty, \sum_{n=1}^{\infty} r_n < +\infty \]

The following are equivalent:

(i) the modified Ishikawa iteration with errors is almost T-stable.

(ii) the modified Mann iteration with errors is almost T-stable.

REFERENCES


