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A NEW ITERATIVE ALGORITHM OF COMMON SOLUTIONS TO QUASI-VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS★

ZI-MING WANG*, AND WANDONG LOU

Department of Foundation, Shandong Yingcai University, Jinan 250104, P.R. China

Abstract. In this paper, quasi-variational inclusions and fixed point problems are considered. A general iterative algorithm is introduced for finding a common element in the zero set of the sum of two monotone operators and the fixed point set of a nonexpansive mapping. Furthermore, strong convergence results for common elements in two sets mentioned above are established in real Hilbert space.

Keywords: nonexpansive mapping; maximal monotone operator; inverse strongly monotone mapping; equilibrium problem; fixed point.

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1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C is a nonempty closed convex subset of H. Let $A : C \to H$ be a single-valued nonlinear mapping and let $B : H \to 2^H$ be a multi-valued mapping. The "so-called" quasi-variational inclusion problem [1-3] is to find an $u \in H$ such that

$$0 \in Au + Bu. \tag{1.1}$$

*Corresponding author

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The set of solution to quasi-variational inclusion problem is denoted by $(A+B)^{-1}0$. It is known that (1.1) provides a convenient framework for the unified study of optimal solutions in many optimization-related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, and so on. see, for instance,[4-6].

The problem (1.1) includes many problems as special cases:

(a) If $B = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the subdifferential of ϕ, then the variational inclusion problem (1.1) is equivalent to find $u \in H$ such that

$$\langle Au, y-u\rangle + \phi(y) - \phi(u) \geq 0, \quad \forall\, y \in H, \tag{1.2}$$

which is called the mixed quasi-variational inequality; see Noor [7].

(b) If $B = \partial\delta_C$, where C is nonempty closed convex subset of H and $\delta_C : H \to [0,\infty]$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{1.3}$$

Then the variational inclusion problem (1.1) is equivalent to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \forall\, v \in C. \tag{1.4}$$

This problem is called Hartman-Stampacchia variational inequality; see [8].

Recently, Takahashi et al. [5] introduced a new iterative algorithm for finding a common element of the set of solutions to the inclusion problem (1.1) with set-valued maximal monotone mapping and inverse strongly monotone mappings, and the set of fixed points of a nonexpansive mapping in Hilbert spaces. Then, they prove a strong convergence theorem using their iterative algorithm. Further, they give some interesting applications. For some more related works, see [1-3,7-9] and the references therein.

In this paper, inspired and motivated by Takahashi et al. [5] and Liou [6], we introduce a new iterative scheme for finding a common element of the set of solution to the inclusion problem (1.1) and the set of fixed points of a nonexpansive mapping. The results presented in this paper improve and extend the related results announced by S. Takahashi et al. [5] and Liou [6] and others.

2. Preliminaries

Let C be a nonempty closed convex subset of H. The nearest point projection of H onto C is denoted by P_C, that is,

$$\|x - P_C x\| \leq \|x - y\|$$

for all $x \in H$ and $y \in C$. The operator P_C is called the metric projection of H onto C. It is known that the metric projection P_C is firmly nonexpansive, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x,\ y \in H$. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0 \tag{2.1}$$

for all $y \in C$; see [10]. Next, recall the following definitions:

(1) A mapping $S : C \to C$ is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall\ x,\ y \in C.$$

(2) A mapping $A : C \to H$ is said to be $\alpha-$inverse strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall\ x,\ y \in C.$$

It is known that if A is an $\alpha-$inverse strongly monotone mapping, then

$$\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|, \quad \forall\ x,\ y \in C.$$

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is,

$$dom(B) = \{x \in H : Bx \neq \emptyset\}.$$

(3) A multi-valued mapping B is said to be a monotone operator on H iff

$$\langle x - y, u - v \rangle \geq 0$$

for all $x,\ y \in dom(B)$, $u \in Bx$ and $v \in By$.

(4) A monotone operator B on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H.

Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. For $\lambda > 0$, we may define a single-valued operator:

$$J_\lambda^B = (I + \lambda B)^{-1} : H \to dom(B),$$

which is called the resolvent of B for λ. It is well known that the resolvent J_λ^B is firmly nonexpansive and $B^{-1}0 = F(J_\lambda^B)$ for all λ. It is also known that

$$J_\lambda^B x = J_\mu^B \left(\frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_\lambda^B x\right) \tag{2.2}$$

holds for all $\lambda,\ \mu > 0$ and $x \in H$.

In order to prove our main results, we need the following lemmas:

Lemma 2.1 (see [11]) *Let B be a uniformly convex Banach space, C be a nonempty closed convex subset of B and $S : C \to B$ be a nonexpansive mapping with a fixed point, then $I - T$ is demi-closed in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

Lemma 2.2 (see [12]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be $\alpha-$inverse strongly monotone and let $\lambda > 0$ be a constant. Then, the following inequality*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2$$

holds for all $x,\ y \in C$. In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.3 (see [13]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* B *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$$

for all $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.4 (see [14]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n,$$

where $\{b_n\}$ *is a sequence in (0,1) and* $\{c_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} b_n = \infty$;

(2) $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n c_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Now, we will give our main result in this paper.

Theorem 3.1. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* A *be an* $\alpha-$*inverse strongly monotone mapping of* C *into* H *and let* B *be a maximal monotone operator on* H*, such that the domain of* B *is included in* C*. Let* $J_\lambda^B = (I+\lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$ *and let* S *be a nonexpansive mapping of* C *into itself, such that* $F = F(S) \cap (A+B)^{-1}0 \neq \emptyset$*. For* $u \in C$ *and given* $x_0 \in C$*, let* $\{x_n\} \subset C$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S J_{\lambda_n}^B (\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)) \tag{3.1}$$

for all $n \geq 0$*, where* $\{\lambda_n\} \subset (0, 2\alpha)$*,* $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy* *(i)* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \liminf_{n\to\infty}\beta_n < 1$;

(iii) $a \le \lambda_n \le b$ *where* $[a,b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_{\mathcal{F}}(u)$*, where* $\Pi_{\mathcal{F}}$ *is the generalized projection from* C *onto* $\mathcal{F}$.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. We choose any $z \in (A+B)^{-1}0 \cap F(S)$. Note that

$$z = J_{\lambda_n}^B(z - \lambda_n(1-\alpha_n)Az) = J_{\lambda_n}^B(\alpha_n z + (1-\alpha_n)(z - \lambda_n Az)) \tag{3.2}$$

for all $n \ge 0$. Since J_λ^B is nonexpansive for all $\lambda > 0$, we have

$$\begin{aligned}
&\|J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)) - z\|^2 \\
&\quad = \|J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)) - J_{\lambda_n}^B(\alpha_n z + (1-\alpha_n)(z - \lambda_n Az))\|^2 \\
&\quad \le \|(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)) - (\alpha_n z + (1-\alpha_n)(z - \lambda_n Az))\|^2 \\
&\quad = \|(1-\alpha_n)((x_n - \lambda_n Ax_n) - (z - \lambda_n Az)) + \alpha_n(u-z)\|^2.
\end{aligned} \tag{3.3}$$

And since A is $\alpha-$inverse strongly monotone, we get

$$\begin{aligned}
&\|(1-\alpha_n)((x_n - \lambda_n Ax_n) - (z - \lambda_n Az)) + \alpha_n(u-z)\|^2 \\
&\quad \le (1-\alpha_n)\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 + \alpha_n\|u-z)\|^2 \\
&\quad = (1-\alpha_n)\|(x_n - z) - \lambda_n(Ax_n - Az)\|^2 + \alpha_n\|u-z)\|^2 \\
&\quad = (1-\alpha_n)(\|x_n - z\|^2 - 2\lambda_n\langle Ax_n - Az, x_n - z\rangle + \lambda_n^2\|Ax_n - Az\|^2) + \alpha_n\|u-z)\|^2 \\
&\quad \le (1-\alpha_n)(\|x_n - z\|^2 - 2\alpha\lambda_n\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2) + \alpha_n\|u-z)\|^2 \\
&\quad = (1-\alpha_n)(\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2) + \alpha_n\|u-z)\|^2.
\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}
&\|J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)) - z\|^2 \\
&\quad \le (1-\alpha_n)(\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2) + \alpha_n\|u-z)\|^2 \\
&\quad \le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|u-z)\|^2.
\end{aligned} \tag{3.5}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(SJ^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z)\|^2 \\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|SJ^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-Sz\|^2 \\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|J^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z\|^2 \\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)((1-\alpha_n)\|x_n-z\|^2+\alpha_n\|u-z)\|^2) \\
&= [1-(1-\beta_n)\alpha_n]\|x_n-z\|^2+(1-\beta_n)\alpha_n\|u-z\|^2 \\
&\le \max\{\|x_n-z\|^2,\|u-z\|^2\}.
\end{aligned} \tag{3.6}$$

By mathematical induction, we have

$$\|x_{n+1}-z\| \le \max\{\|x_0-z\|,\|u-z\|\}. \tag{3.7}$$

Therefore, the sequence $\{x_n\}$ is bounded. We deduce immediately that $\{Ax_n\}$ is also bounded. Set $u_n=\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n)$ and $v_n=J^B_{\lambda_n}u_n$ for all $n\ge 0$. Then $\{u_n\}$ and $\{v_n\}$ are also bounded.

In the other hand, we compute that

$$\begin{aligned}
\|Sv_{n+1}-Sv_n\| \le \|v_{n+1}-v_n\| &= \|J^B_{\lambda_{n+1}}u_{n+1}-J_{\lambda_n}u_n\| \\
&\le \|J^B_{\lambda_{n+1}}(\alpha_{n+1}u+(1-\alpha_{n+1})(x_{n+1}-\lambda_{n+1}Ax_{n+1})) \\
&\quad -J^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))\| \\
&\le \|J^B_{\lambda_{n+1}}(\alpha_{n+1}u+(1-\alpha_{n+1})(x_{n+1}-\lambda_{n+1}Ax_{n+1})) \\
&\quad -J^B_{\lambda_{n+1}}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n)) \\
&\quad +\|J^B_{\lambda_{n+1}}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n)) \\
&\quad -J^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))\| \\
&\le \|(\alpha_{n+1}u+(1-\alpha_{n+1})(x_{n+1}-\lambda_{n+1}Ax_{n+1}))-(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n)) \\
&\quad +\|J^B_{\lambda_{n+1}}u_n-J^B_{\lambda_n}u_n\|
\end{aligned}$$

$$= \|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_{n+1}A)x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\|$$
$$+ \alpha_{n+1}(\|u\| + \|x_{n+1}\| + \lambda_{n+1}\|Ax_{n+1}\|) + \alpha_n(\|u\| + \|x_n\| + \lambda_n\|Ax_n\|) \tag{3.8}$$
$$+ \|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\|.$$

Since $I-\lambda_{n+1}A$ is nonexpansive for $\lambda_{n+1} \in (0, 2\alpha)$, we have

$$\|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_{n+1}A)x_n\| \le \|x_{n+1}-x_n\| \tag{3.9}$$

By the resolvent identity (2.2), we have

$$J^B_{\lambda_{n+1}}u_n = J^B_{\lambda_n}(\frac{\lambda_n}{\lambda_{n+1}}u_n + (1-\frac{\lambda_n}{\lambda_{n+1}})J^B_{\lambda_{n+1}}u_n) \tag{3.10}$$

It follows from (3.10) that

$$\begin{aligned}\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\| &= \|J^B_{\lambda_n}(\frac{\lambda_n}{\lambda_{n+1}}u_n + (1-\frac{\lambda_n}{\lambda_{n+1}})J^B_{\lambda_{n+1}}u_n) - J^B_{\lambda_n}u_n\| \\ &\le \|(\frac{\lambda_n}{\lambda_{n+1}}u_n + (1-\frac{\lambda_n}{\lambda_{n+1}})J^B_{\lambda_{n+1}}u_n) - u_n\| \\ &\le \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\|u_n - J^B_{\lambda_{n+1}}u_n\|\end{aligned} \tag{3.11}$$

Therefore, from (3.8), (3.9) and (3.11), we have

$$\begin{aligned}\|Sv_{n+1}-Sv_n\| &\le \|v_{n+1}-v_n\| \\ &\le \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ &\quad + \alpha_{n+1}(\|u\| + \|x_{n+1}\| + \lambda_{n+1}\|Ax_{n+1}\|) + \alpha_n(\|u\| + \|x_n\| + \lambda_n\|Ax_n\|) \\ &\quad + \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\|u_n - J^B_{\lambda_{n+1}}u_n\|.\end{aligned} \tag{3.12}$$

Thus,

$$\limsup_{n\to\infty}(\|Sv_{n+1}-Sv_n\| - \|x_{n+1}-x_n\|) \le 0 \tag{3.13}$$

and

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\|) \le 0. \tag{3.14}$$

From Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|Sv_n - x_n\| = 0 \quad and \quad \lim_{n\to\infty}\|v_n - x_n\| = 0. \tag{3.15}$$

Then, from (3.1), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|Sv_n - x_n\| = 0. \tag{3.16}$$

And from (3.15), we also learn that

$$\begin{aligned}\lim_{n\to\infty}\|Sx_n - x_n\| &\le \lim_{n\to\infty}(\|Sx_n - Sv_n\| + \|Sv_n - x_n\|)\\ &\le \lim_{n\to\infty}(\|x_n - v_n\| + \|Sv_n - x_n\|)\\ &\le \lim_{n\to\infty}\|x_n - v_n\| + \lim_{n\to\infty}\|Sv_n - x_n\| = 0\end{aligned} \tag{3.17}$$

Since $\{x_n\}$ is bounded. we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some point $\bar{x}$. By virtue of Lemma 1.2, it follows that $\bar{x} \in F(S)$. Further we show that $\bar{x} \in (A+B)^{-1}0$. In fact, notice that

$$v_n = J^B_{\lambda_n}(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)),$$

we have that

$$\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n) \in v_n + \lambda_n B v_n.$$

Let $\xi \in B\eta$. Since B is monotone, we get

$$\langle \frac{\alpha_n u + (1-\alpha_n)x_n - v_n}{\lambda_n} - (1-\alpha_n)Ax_n - \xi, v_n - \eta\rangle \ge 0.$$

In view of (i), (iii) and (3.15), we obtain

$$\langle -A\bar{x} - \xi, \bar{x} - \eta\rangle \ge 0.$$

It follows that $-A\bar{x} \in B\bar{x}$, that is, $\bar{x} \in (A+B)^{-1}0$.

On the other hand, from (3.5) and (3.6), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|SJ^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z\|^2\\ &\le (1-\beta_n)\|J^B_{\lambda_n}(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z\|^2+\beta_n\|x_n-z\|^2\\ &\le (1-\beta_n)\{(1-\alpha_n)(\|x_n-z\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2)+\alpha_n\|u-z\|^2\}\\ &\quad+\beta_n\|x_n-z\|^2\\ &=[1-(1-\beta_n)\alpha_n]\|x_n-z\|^2+(1-\beta_n)\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2\\ &\quad+(1-\beta_n)\alpha_n\|u-z\|^2\\ &\le \|x_n-z\|^2+(1-\beta_n)\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2+(1-\beta_n)\alpha_n\|u-z\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Az\|^2 &\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\beta_n)\alpha_n\|u-z\|^2\\ &\le (\|x_n-z\|-\|x_{n+1}-z\|)\|x_{n+1}-x_n\|+(1-\beta_n)\alpha_n\|u-z\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\liminf_{n\to\infty}(1-\beta_n)\lambda_n(2\alpha-\lambda_n)>0$ and (3.16), we have

$$\lim_{n\to\infty}\|Ax_n-Az\|=0. \tag{3.18}$$

Put $p=P_F u$. Set $y_n=x_n-\lambda_n(Ax_n-Ap)$ for all $n\ge 0$. Next, we show that

$$\limsup_{n\to\infty}\langle u-p, y_n-p\rangle\le 0.$$

In fact, take $z=p$ in (3.18) to get $\|Ax_n-Ap\|\to 0$. We easily see from y_n that $\|x_n-y_n\|\to 0$, as $n\to\infty$. Therefore, there exists a subsequence $\{y_{n_i}\}\subset\{y_n\}$ which converges weakly to $\bar{x}\in F$, such that

$$\limsup_{n\to\infty}\langle u-p, y_n-p\rangle=\lim_{i\to\infty}\langle u-p, y_{n_i}-p\rangle=\langle u-p,\bar{x}-p\rangle\le 0.$$

Finally, we prove that $x_n \to p$, as $n \to \infty$. From (3.1), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|SJ^B_{\lambda_n}u_n-p\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|J^B_{\lambda_n}u_n-p\|^2 \\
&= \beta_n\|x_n-p\|^2+(1-\beta_n)\|J^B_{\lambda_n}u_n-J^B_{\lambda_n}(p-(1-\alpha_n)\lambda_nAp)\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-(p-(1-\alpha_n)\lambda_nAp)\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nu+(1-\alpha_n)(x_n-\lambda_nAx_n)-(p-(1-\alpha_n)\lambda_nAp)\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|(1-\alpha_n)(x_n-\lambda_nAx_n)-(p-\lambda_nAp)+\alpha_n(u-p)\|^2 \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\times\{(1-\alpha_n)^2\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle u-p,(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\rangle+\alpha_n^2\|u-p\|^2\} \\
&\le \beta_n\|x_n-p\|^2+(1-\beta_n)\times\{(1-\alpha_n)^2\|x_n-p\|^2 \\
&\quad +2\alpha_n(1-\alpha_n)\langle u-p,x_n-\lambda_n(Ax_n-Ap)-p\rangle+\alpha_n^2\|u-p\|^2\} \\
&\le [1-(1-\beta_n)\alpha_n]\|x_n-p\|^2+(1-\beta_n)\alpha_n\{2(1-\alpha_n)\langle u-p,y_n-p\rangle+\alpha_n\|u-p\|^2\}.
\end{aligned}
$$

Notice that $\sum_{n=0}^{\infty}(1-\beta_n)\alpha_n=\infty$ and $\limsup_{n\to\infty}(2(1-\alpha_n)\langle u-p,y_n-p\rangle+\alpha_n\|u-p\|^2)\le 0$. It follows from Lemma 2.4 that $x_n \to p$, as $n\to\infty$. This completes the proof. □

Remark 3.2. The iterative algorithm (3.1) is different from the one in Theorem 3.1 in [5], but the two algorithms deal with the same problem in different angle.

When $S\equiv I$ in (3.1), we can get the following corollary by using Theorem 3.1:

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be an $\alpha-$inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J^B_\lambda=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$ such that $(A+B)^{-1}0\neq\emptyset$. For $u\in C$ and given $x_0\in C$, let $\{x_n\}\subset C$ be a sequence generated by*

$$x_{n+1}=\beta_nx_n+(1-\beta_n)J^B_{\lambda_n}(\alpha_nu+(1-\alpha_n)(x_n-\lambda_nAx_n)) \tag{3.1}$$

for all $n\ge 0$, where $\{\lambda_n\}\subset(0,2\alpha)$, $\{\alpha_n\}\subset(0,1)$ and $\{\beta_n\}\subset(0,1)$ satisfy

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1$;

(iii) $a \leq \lambda_n \leq b$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_{(A+B)^{-1}0}(u)$*, where* $\Pi_{(A+B)^{-1}0}$ *is the generalized projection from* C *onto* $(A + B)^{-1}0$.

Remark 3.4. Corollary 3.3 is just the main result in Liou [6].

4. Applications

Let H be a Hilbert space and $f : H \to (-\infty, +\infty]$ be a proper convex lower semi−continuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y\rangle, \quad \forall\ z \in H\}, \quad \forall\ x \in H.$$

From Rockafellar [15, 16], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ iff $f(x) = \min_{y\in H} f(y)$. Let δ_C be the indicator function of C, i.e.,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{4.1}$$

Since δ_C is a proper lower semi-continuous convex function on H, we see that the subdifferential $\partial\delta_C$ of δ_C is a maximal monotone operator.

The following result is introduced by Takahashi et al [5]:

Lemma 4.1 (see [5]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*,* P_C *be the metric projection from* H *onto* C*,* $\partial\delta_C$ *be the subdifferential of* δ_C *and* J_λ *be the resolvent of* $\partial\delta_C$ *for* $\lambda > 0$ *where* δ_C *is as defined in (4.1) and* $J_\lambda = (I + \lambda\partial\delta_C)^{-1}$*. Then*

$$y = J_\lambda x \Leftrightarrow y = P_C x, \quad \forall\ x \in H,\ y \in C.$$

Now, we introduce an iterative scheme for approximating a common element of the set of solutions to variation inequality (1.4) and the set of fixed points of a nonexpansive mapping:

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let A be an $\alpha-$inverse strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $\digamma = F(S) \cap VI(C, A) \neq \emptyset$. For $u \in C$ and given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) SP_C(\alpha_n u + (1 - \alpha_n)(x_n - \lambda_n Ax_n))$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1$;

(iii) $a \leq \lambda_n \leq b$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\digamma(u)$, where $\Pi_\digamma$ is the generalized projection from C onto $\digamma$.

Proof. Put $B = \partial\delta_C$. Next, we show that $VI(C, A) = (A + \partial\delta_C)^{-1}0$. Notice that

$$\begin{aligned} x \in (A + \partial\delta_C)^{-1}(0) &\Longleftrightarrow 0 \in Ax + \partial\delta_C x \\ &\Longleftrightarrow -Ax \in \partial\delta_C x \\ &\Longleftrightarrow \langle Ax, y - x\rangle \geq 0, \quad (\forall\ y \in C) \\ &\Longleftrightarrow x \in VI(C, A). \end{aligned}$$

From Lemma 4.1, we know that $J_{\lambda_n} = P_C$ for all λ_n with $0 < a \leq \lambda_n \leq b < 2\alpha$. So, we can obtain that the desired result by Theorem 3.1. this completes the proof. □

As another application of Theorem 3.1, we consider the problem for finding a common element of the set of solutions to equilibrium problems and the set of fixed points of a nonexpansive mapping.

Let $F : C \times C \to R$ be a bifunction satisfying the following conditions:

(A_1) $F(x, x) = 0$ for all $x \in C$;

(A_2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x,\ y \in C$;

(A_3) for all $x,\ y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A_4) for all $x \in C$, $F(x, \cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$F(\hat{x}, y) \geq 0 \tag{4.2}$$

for all $y \in C$. The set of solutions to equilibrium problem is denoted by $EP(F)$. The following lemma was introduced by Blum and Oettli [17]:

Lemma 4.3 (see [17]) *Let C be a nonempty closed convex subset of a real Hilbert space H, F be a bifunction of $C \times C$ into R satisfying $(A_1)-(A_4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad y \in C.$$

The following lemma was given by Combettes and Hirstoaga [18]:

Lemma 4.4 (see [18]) *Assume that $F : C \times C \to R$ satisfying $(A_1) - (A_4)$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ y \in C\} \tag{4.3}$$

for all $x \in H$. Then, the following holds:

(B_1) T_r is single valued;

(B_2) T_r is a firmly nonexpansive mapping, that is, for all $x,\ y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(B_3) $F(T_r) = EP(F)$;

(B_4) $EP(F)$ is closed and convex.

The following lemma appears in Takahashi et al.[5]:

Lemma 4.5 (see [5]) *Let C be a nonempty closed convex subset of a real Hilbert space H, F be a bifunction of $C \times C$ into R satisfying $(A_1) - (A_4)$. And A_F be a set-valued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle,\ \forall\ y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then A_F *is a maximal monotone operator with the domain* $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$ *and*

$$T_r x = (I + rA_F)^{-1} x, \quad \forall\, x \in H,\ r > 0,$$

where T_r *is defined as in (4.3).*

Applying Lemma 4.5 and Theorem 3.1, we can obtain the following result immediately.

Theorem 4.6. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C \to R$ *satisfying* $(A_1) - (A_4)$ *and let* T_r *be the resolvent of* F *for* $r > 0$*. Suppose that* $\mathcal{F} = F(S) \cap EP(F) \neq \emptyset$*. For* $u \in C$ *and given* $x_0 \in C$*, let* $\{x_n\} \subset C$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S T_{r_n}(\alpha_n u + (1 - \alpha_n) x_n)$$

for all $n \geq 0$*, where* $\{r_n\} \subset (0, 2\alpha)$*,* $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1$;

(iii) $a \leq r_n \leq b$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_{\mathcal{F}}(u)$*, where* $\Pi_{\mathcal{F}}$ *is the generalized projection from* C *onto* $\mathcal{F}$*.*

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