SOME FIXED POINT THEOREMS IN S-METRIC SPACES WITH NEW CONTRACTIVE MAPPINGS

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Abstract: The purpose of this paper is to introduce new contractive mapping in S-metric space using new class of function. We establish some fixed point theorems in context of these new contractive mapping in S-metric spaces.

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1. INTRODUCTION

In mathematics, Banach [1] in 1922, introduced a well-known theorem which is called Banach fixed point theorem (or Banach contraction Principle). It is a beautiful mixture of analysis, topology and geometry. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to find those fixed points.

Let $X$ be a complete metric space. A mapping $T:X \to X$ is called a contraction mapping on $X$ if there exists $q \in [0, 1)$ such that $d(Tx, Ty) \leq q \cdot d(x, y)$, for all $x, y \in X$. Then $T$ has a unique fixed point. Because of its simplicity and usefulness, it has become very powerful tool in solving
various problem such as nonlinear analysis, integral and differential equations, inclusions, dynamical system theory and mathematical economics. Later, in 1968, Kannan [9], proved the following fixed point theorem which is independent of the Banach contraction principle. Let \((X,d)\) be a metric space and \(T\) be a self-map on \(X\), if there exists \(c \in (0, \frac{1}{2}]\) such that 
\[
d(Tx,Ty) \leq c \ d(x,Tx) + c \ d(y,Ty),
\]
for all \(x,y \in X\). Then \(T\) has a unique fixed point.


Later on, this theorem was generalized by many authors in different metric spaces and also in generalized metric spaces. To study these generalizations, we can refer to ([3], [4], [5], [6], [7], [8], [10], [12]). Sedghi et al. [15] introduce the concept of S-metric space and proved that this concept is the generalization of G-metric space [10] and D* metric space [17]. Further in [15] authors proved that the notion of S-metric space is not the generalization of G–metric space and both metric spaces are independent of each other. More details regarding this space can be found in ([12], [14], [16]).

In this paper, we define a new class of function. Furthermore, we define some new contractive mappings which combine with the terms 
\[
d(x,x,y), d(x,x,Tx), d(y,y,Ty), d(x,x,Ty), d(y,y,Tx) \text{ and } d(Tx,Tx,y)
\]
by means of the member of newly defined class and prove some fixed point theorem using this new class of functions. We begin by recalling some basic definitions and results for S-metric spaces that will be needed in the sequel.

2. PRELIMINARIES

Definition 2.1 [15]: Let \(X\) be a non-empty set. An S-metric on \(X\) is a mapping \(S : X^3 \to \mathbb{R}_+\) that satisfies the following condition:

\[
(S_1) \ S(x,y,z) = 0 \text{ if and only if } x = y = z = 0;
\]
\( (S_2) \) \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \) for all \( x, y, z, a \in X \)

The pair \( (X, S) \) is called an S-metric space.

**Example 2.2:** Let \( X = \mathbb{R} \). \( S(x, y, z) = |x - z| + |y - z| \). Then \( S(x, y, z) \) is an S-metric on \( \mathbb{R} \), which is known as usual S-metric space on \( X \).

**Lemma 2.3**[15]: If \( S \) is an S-metric on a non-empty set \( X \), then \( S(x, x, y) = S(y, y, x) \), for all \( x, y \in X \).

**Definition 2.4**[13]: Let \( (X, S) \) be a S-metric space. For \( r > 0 \) and \( x \in X \), we define the open ball with center in \( x \) and radius \( r \), the set

\[ B_s(x, r) = \{ y \in X : S(x, y) < r \} \]

The topology induced by the S-metric is the topology determined by the base of all open balls in \( X \).

**Definition 2.5**[12]: A sequence \( \{x_n\} \) in \( (X, S) \) is said to be convergent to \( x \), denoted by

\[ \lim_{n \to \infty} x_n = x \text{ if } x_n \to x, \text{ or } S(x_n, x, x) \to 0 \text{ as } n \to \infty. \]

**Definition 2.6**[12]: A sequence \( \{x_n\} \) in \( (X, S) \) is said to be Cauchy sequence if

\[ S(x_n, x_n, x_m) \to 0 \text{ as } n, m \to \infty. \]

**Definition 2.7**[12]: An S-metric space \( (X, S) \) is said to be complete if every Cauchy sequence in \( X \) is convergent.

**Example 2.8:** Let \( (X, S) \) be S-metric space, then the usual metric space on \( X \) defined in Example 2.2 is complete.

**Lemma 2.9**([13, 14]): Let \( (X, S) \) be S-metric space. If \( x_n \to x \) and \( y_n \to y \) then \( S(x_n, x_n, y_n) \to S(x, x, y) \).

**Lemma 2.10**[16]: Let \( (X, S) \) be an S-metric space and \( x_n \to x \). Then \( \lim_{n \to \infty} x_n \) is unique.

**Lemma 2.11**([13, 15]): Every sequence \( \{x_n\} \) of elements from S-metric space \( (X, S) \), having the property that there exists \( \lambda \in [0, 1) \) such that

\[ S(x_n, x_n, x_{n+1}) \leq \lambda S(x_{n-1}, x_{n-1}, x_n) \]

for every \( n \in \mathbb{N} \), is a Cauchy.
3. MAIN RESULTS

In this section, we define some new class of functions and then by defining new contractive mapping in S-metric spaces we are going to prove some fixed point theorems which are as follows:

**Definition 3.1:** For any $m \in \mathbb{N}$, we define $\Xi_m$ to be the set of all functions $\eta: [0, +\infty)^m \to [0, +\infty)$ such that

(\eta_1) $\eta(t_1, t_2, \ldots, t_m) < \max\{t_1, t_2, \ldots, t_m\}$ if $(t_1, t_2, \ldots, t_m) \neq (0,0, \ldots, 0)$;

(\eta_2) for $n \in \mathbb{N}$, $1 \leq i \leq m$, are $m$ sequences in $[0, +\infty)$ such that $\lim_{n \to \infty} t_i^{(n)} = t_i < +\infty$ for all $i = 1, 2, \ldots, m$, then $\lim_{n \to \infty} \eta(t_1^{(n)}, t_2^{(n)}, \ldots, t_m^{(n)}) \leq \eta(t_1, t_2, \ldots, t_m)$.

**Definition 3.2:** Let $(X, S)$ be an S-metric space and $T: X \to X$ is said to be an $\eta$-contractive mapping of Type – I if there exists $\eta \in \Xi_3$ and $S(Tx, Tx, Ty) \leq q \eta(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x, x, Ty) + S(Tx, Tx, y)}{3})$ (3.1) for all $x, y \in X$ and $q \in (0, 1)$.

**Theorem 3.3:** Let $(X, S)$ be a complete S-metric space and $T: X \to X$ be an $\eta$-contractive mapping of Type – I. Then $T$ has a unique fixed point.

**Proof:** Let $x_0 \in X$. Define a sequence $\{x_n\}$ in $X$ as $x_n = Tx_{n-1}$ for all $n \geq 1$. Assume that any two consecutive terms of the sequence $\{x_n\}$ are distinct; otherwise, $T$ has a fixed point. Consider

$$S(x_n, x_n, x_{n+1}) \leq q \eta\left(S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3}\right)$$

(3.2)

$$< q \max\left\{S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3}\right\}$$

$$= q \max\left\{S(x_{n-1}, x_{n-1}, x_n)\right\}$$

$$\leq q \max\left\{S(x_{n-1}, x_{n-1}, x_n), \frac{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}{3}\right\}$$

(3.3)

Case 1: If $S(x_n, x_n, x_{n+1}) \leq q S(x_{n-1}, x_{n-1}, x_n)$ then by Lemma 2.11 in view of (3.3), $\{x_n\}$ is a Cauchy sequence for $q \in (0, 1)$.

Case 2: If $S(x_n, x_n, x_{n+1}) \leq q \left\{\frac{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}{3}\right\}$
Then,
\[ 3S(x_n, x_n, x_{n+1}) \leq q \left( 2 S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_{n+1}) \right), \]
that is,
\[ (3 - q)S(x_n, x_n, x_{n+1}) \leq 2 qS(x_{n-1}, x_{n-1}, x_n), \]
that is,
\[ S(x_n, x_n, x_{n+1}) \leq \frac{2q}{3-q} S(x_{n-1}, x_{n-1}, x_n), \]
that is,
\[ S(x_n, x_n, x_{n+1}) \leq t S(x_{n-1}, x_{n-1}, x_n), \]
where \( t = \frac{2q}{3-q} \) which lies between 0 to 1 for all \( q \).

Therefore, we get
\[ S(x_n, x_n, x_{n+1}) \leq t^n S(x_0, x_0, x_1). \]

By Lemma 2.11 and (S2) it follows that \( \{x_n\} \) is a Cauchy sequence in \( X \).

As \((X, S)\) is a complete \( S \)-metric space, it follows that sequence \( \{x_n\} \) is convergent to some \( x \in X \). Thus, \( x_n \to x \) as \( n \to \infty \).

Now, Consider
\[ S(Tx_n, Tx_n, Tx) \leq q \eta \left( S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x_n, Tx), \frac{S(x_n, x_n, Tx) + S(x_n, Tx)}{3} \right). \]
Taking limit as \( n \to \infty \), we get
\[ S(x, x, Tx) \leq q \eta \left( S(x, x, x), S(x, x, x), S(x, x, Tx), \frac{S(x, x, Tx) + S(x, x, Tx)}{3} \right) \]
\[ < q \max \left\{ 0, 0, S(x, x, Tx), \frac{2S(x, x, Tx)}{3} \right\} \]
\[ = q \ S(x, x, Tx). \]

Thus, we get, \( S(x, x, Tx) < q \ S(x, x, Tx) \), where \( q \in (0, 1) \), which is contradiction.

Therefore, \( Tx = x \).

Now let \( Ty = y \) for some \( y \in X \) and suppose that \( x \neq y \), then consider
\[ S(x, x, y) = S(Tx, Tx, Ty) \]
\[ \leq q \eta \left( S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x, x, Ty) + S(y, y, Ty)}{3} \right) \]
\[ q \max \left\{ S(x, x, y), S(x, x, x), S(y, y, y), \frac{S(x, x, y) + S(y, y, y)}{3} \right\} \]

\[ = q \max \left\{ S(x, x, y), 0, 0, \frac{2S(x, x, y)}{3} \right\} \]

\[ = qS(x, x, y), \]

a contradiction. Therefore, \( x = y \). ■

**Theorem 3.4:** Let \( (X, S) \) be a complete S-metric space and \( T : X \to X \) be an \( \eta \)-contractive mapping where \( \eta \in \Xi_4 \) given by

\[ S(Tx, Tx, Ty) \leq \eta \left( S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x, x, y) + S(Tx, Tx, Ty)}{3} \right), \quad (3.4) \]

for all \( x, y \in X \). Then \( T \) has a unique fixed point.

**Proof:** Let \( x_0 \in X \). Define a sequence \( \{x_n\} \) in \( X \) as \( x_n = Tx_{n-1} \) for all \( n \geq 1 \). Assume that any two consecutive terms of the sequence \( \{x_n\} \) are distinct; otherwise, \( T \) has a fixed point.

Consider

\[ S(x_n, x_n, x_{n+1}) \leq \eta \left( S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \right) \]

\[ < \max \left\{ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \right\} \]

\[ = \max \left\{ S(x_{n-1}, x_{n-1}, x_n), \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \right\} \]

\[ \leq \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \]

\[ \leq \frac{2S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \]

If \( \max \left\{ S(x_{n-1}, x_{n-1}, x_n), \frac{2S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \right\} \]

then \( S(x_n, x_n, x_{n+1}) \leq \frac{2S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \).

Therefore, we have

\[ S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n) \]

\[ \text{(3.7)} \]

From (3.7), we see that \( \{S(x_n, x_n, x_{n+1})\} \) is monotonically decreasing sequence and bounded below. Therefore, \( S(x_n, x_n, x_{n+1}) \to k, k \geq 0 \).

Now suppose that \( k > 0 \), taking \( \lim \inf \to +\infty \) in (3.5), we have \( k \leq \eta(k, k, k, k') \), where \( k' = \lim_{n \to +\infty} \sup \frac{S(x_{n-1}, x_{n-1}, x_{n+1})}{3} \).
\[
\leq \limsup_{n \to +\infty} \frac{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_{n+1})}{3} = \frac{2k+k}{3} = k.
\]

Therefore, \( k \leq \eta(k, k, k') < \max\{k, k, k'\} = k \), a contradiction.

Therefore, we get

\[
\lim_{n \to +\infty} S(x_n, x_n, x_{n+1}) = 0 \tag{3.8}
\]

Now suppose \( \{x_n\} \) is not a Cauchy sequence then there exist \( \varepsilon > 0 \) such that for any \( c \in N \) there exist \( m_c > n_c \geq c \) such that

\[
S(x_{m_c}, x_{m_c}, x_{n_c}) \geq \varepsilon \tag{3.9}
\]

Suppose \( m_c \) is smallest natural number greater than \( n_c \) such that (3.9) holds. Then,

\[
\varepsilon \leq S(x_{m_c}, x_{m_c}, x_{n_c})
\]

\[
\leq 2S(x_{m_c}, x_{m_c}, x_{m_c-1}) + S(x_{n_c}, x_{n_c}, x_{m_c-1})
\]

\[
< 2S(x_{m_c}, x_{m_c}, x_{m_c-1}) + \varepsilon.
\]

Taking \( \lim c \to +\infty \), we get

\[
\lim_{c \to +\infty} S(x_{m_c}, x_{m_c}, x_{n_c}) = \varepsilon \tag{3.10}
\]

Now consider,

\[
S(x_{m_c}, x_{m_c}, x_{n_c}) \leq 2S(x_{m_c}, x_{m_c}, x_{m_c+1}) + S(x_{n_c}, x_{n_c}, x_{m_c+1})
\]

\[
\leq 2S(x_{m_c}, x_{m_c}, x_{m_c+1}) + \eta \left( \frac{S(x_{m_c}, x_{m_c}, x_{n_c-1}), S(x_{m_c}, x_{m_c}, x_{m_c+1}), S(x_{n_c-1}, x_{n_c-1}, x_{n_c})}{3} \right)
\]

Taking limit as \( c \to +\infty \), on both sides and using (3.8) and (3.10) we get,

\[
\varepsilon \leq 2.0 + \eta(\varepsilon, 0, 0, \varepsilon'), \text{ where}
\]

\[
\varepsilon' = \lim_{c \to +\infty} \sup \frac{S(x_{m_c}, x_{m_c}, x_{m_c+1}) + S(x_{n_c}, x_{n_c}, x_{m_c+1})}{3}
\]

\[
\leq \lim_{c \to +\infty} \sup \frac{S(x_{m_c}, x_{m_c}, x_{m_c+1}) + 2S(x_{n_c}, x_{n_c}, x_{m_c}) + S(x_{m_c}, x_{m_c}, x_{m_c+1})}{3}
\]
\[
\frac{0+2\varepsilon+0}{3} = \frac{2\varepsilon}{3}.
\]

Therefore, \(\varepsilon \leq 0 + \eta \left(\varepsilon, 0, 0, \frac{2\varepsilon}{3}\right) < \max \left\{\varepsilon, 0, 0, \frac{2\varepsilon}{3}\right\} = \varepsilon,
\]

which is contradiction. Thus, \(\{x_n\}\) is a Cauchy sequence. As \((X, S)\) is a complete S-metric space, it follows that sequence \(\{x_n\}\) is convergent to some \(x \in X\).

Thus, \(x_n \to x\) as \(n \to \infty\).

Now, consider
\[
S(Tx_n, Tx_n, Tx) \leq \eta \left(S(x_n, x_n, x), S(x_n, x_n, x_{n+1}), S(x, x, Tx), \frac{S(x_n, x_n, Tx) + S(x, x, Tx)}{3}\right)
\]

Taking limit as \(n \to +\infty\) we get
\[
S(x, x, Tx) \leq \eta \left(S(x, x, x), S(x, x, x), S(x, x, Tx), \frac{S(x, x, Tx) + S(x, x, Tx)}{3}\right)
\]
\[
< \max \left\{0, 0, S(x, x, Tx), \frac{2S(x, x, Tx)}{3}\right\}
\]
\[
= S(x, x, Tx).
\]

Thus, we get, \(S(x, x, Tx) < S(x, x, Tx)\), which is contradiction.

Therefore, \(Tx = x\).

Now let \(Ty = y\) for some \(y \in X\) and suppose that \(x \neq y\), then consider
\[
S(x, x, y) = S(Tx, Tx, Ty)
\]
\[
\leq \eta \left(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x, x, Ty) + S(y, y, Tx)}{3}\right)
\]
\[
< \max \left\{S(x, x, y), S(x, x, x), S(y, y, y), \frac{S(x, x, y) + S(y, y, y)}{3}\right\}
\]
\[
= \max \left\{S(x, x, y), 0, 0, \frac{2S(x, x, y)}{3}\right\}
\]
\[
= S(x, x, y),
\]

which is contradiction. Therefore, \(x = y\). ■

**Definition 3.5:** Let \((X, S)\) be an S-metric space. The mapping \(T: X \to X\) is said to be an \(\eta\)-contractive mapping of Type – II if there exists \(\eta \in \Xi_5\) and
\[
S(Tx, Tx, Ty) \leq q \eta \left(S(x, x, y), S(x, x, Tx), S(y, y, Ty), \frac{S(x, x, Ty)}{3}, S(Tx, Tx, y)\right),
\]
for all \(x, y \in X\) and \(q \in (0, 1)\).
**Theorem 3.6:** Let \((X, S)\) be a complete \(S\)-metric space and \(T : X \to X\) be an \(\eta\)-contractive mapping of Type – II. Then \(T\) has a unique fixed point.

**Proof:** Let \(x_0 \in X\). Define a sequence \(\{x_n\}\) in \(X\) as \(x_n = Tx_{n-1}\) for all \(n \geq 1\). Assume that any two consecutive terms of the sequence \(\{x_n\}\) are distinct; otherwise, \(T\) has a fixed point.

Consider

\[
S(x_n, x_n, x_{n+1}) \leq q \eta \left( S(x_{n-1}, x_{n-1}, x_n)^3, S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}) \right)
\]

(3.12)

\[
< q \max \left\{ S(x_{n-1}, x_{n-1}, x_n), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}) \right\}
\]

\[
= q \max \left\{ S(x_{n-1}, x_{n-1}, x_n), \frac{S(x_{n-1}, x_{n-1}, x_n)}{3} \right\}
\]

\[
\leq q \max \left\{ S(x_{n-1}, x_{n-1}, x_n), \frac{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}{3} \right\}
\]

(3.13)

Case 1: If \(S(x_n, x_n, x_{n+1}) \leq q S(x_{n-1}, x_{n-1}, x_n)\) then by Lemma 2.11 in view of (3.13), \(\{x_n\}\) is a Cauchy sequence for all \(q \in (0, 1)\).

Case 2: If \(S(x_n, x_n, x_{n+1}) \leq q \left\{ \frac{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})}{3} \right\} \).

Then,

\[
3S(x_n, x_n, x_{n+1}) \leq q \left( 2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1}) \right), \text{ that is,}
\]

\[
(3 - q) S(x_n, x_n, x_{n+1}) \leq 2qS(x_{n-1}, x_{n-1}, x_n), \text{ that is,}
\]

\[
S(x_n, x_n, x_{n+1}) \leq \frac{2q}{3 - q} S(x_{n-1}, x_{n-1}, x_n), \text{ that is,}
\]

\[
S(x_n, x_n, x_{n+1}) \leq t S(x_{n-1}, x_{n-1}, x_n), \text{ where } t = \frac{2q}{3 - q} \text{ which lies between 0 to 1 for all } q.
\]

Therefore,

\[
S(x_n, x_n, x_{n+1}) \leq t \left( tS(x_{n-2}, x_{n-2}, x_{n-1}) \right) = t^2 S(x_{n-2}, x_{n-2}, x_{n-1})
\]

\[
\leq t^n S(x_0, x_0, x_1).
\]

By Lemma 2.11 and (S₂) it follows that \(\{x_n\}\) is a Cauchy sequence in \(X\).
As \((X, S)\) is a complete S-metric space, it follows that sequence \(\{x_n\}\) is convergent to some \(x \in X\). Thus, \(x_n \to x\) as \(n \to \infty\).

Now consider,

\[
S(Tx_n, Tx_n, Tx) \leq q \eta \left( S(x_n, x_n, x), S(x_n, x_{n+1}, x), S(x, x, Tx), S(x, x, Tx) \frac{S(x_n, x_n, Tx)}{3}, S(x, x, Tx) \right)
\]

Taking limit as \(n \to \infty\), we get

\[
S(x, x, Tx) \leq q \eta \left( S(x, x, x), S(x, x, x), S(x, x, Tx), S(x, x, Tx) \frac{S(x, x, Tx)}{3}, S(x, x, x) \right)
\]

\[
< q \max \left\{ 0, 0, S(x, x, Tx), \frac{2 S(x, x, Tx)}{3}, 0 \right\}
\]

\[
= q S(x, x, Tx).
\]

Thus, we get, \(S(x, x, Tx) < q S(x, x, Tx)\) where \(q \in (0, 1)\), which is contradiction.

Therefore, \(Tx = x\).

Now let \(Ty = y\) for some \(y \in X\) and suppose that \(x \neq y\), then consider

\[
S(x, y) = S(Tx, Tx, Ty)
\]

\[
\leq q \eta \left( S(x, y), S(x, x, Tx), S(y, y, Ty), S(x, x, Tx) \frac{S(x, x, Ty)}{3}, S(y, y, Tx) \right)
\]

\[
< q \max \left\{ S(x, y), S(x, x, x), S(y, y, y), S(x, x, y) \frac{S(x, x, y)}{3}, S(y, y, x) \right\}
\]

\[
= q \max \left\{ S(x, y), 0, 0, \frac{S(x, x, y)}{3}, S(x, x, y) \right\}
\]

\[
= q S(x, x, y),
\]

which is a contradiction. Therefore, \(x = y\). 

**Theorem 3.7:** Let \((X, S)\) be a complete S-metric space. Let \(T\) be a mapping \(T : X \to X\) such that

\[
S(Tx, Tx, Ty) \leq \alpha \max \{S(x, x, Tx), S(y, y, Ty), S(x, x, y)\} + \beta \{S(x, x, Ty) + S(y, y, Tx)\},
\]

\[
(3.14)
\]

\(\alpha, \beta > 0\) such that \(\alpha + 3\beta < 1\) for all \(x, y \in X\). Then \(T\) has a unique fixed point.

**Proof:** Let \(x_0 \in X\). Define a sequence \(\{x_n\}\) in \(X\) as

\[
x_n = Tx_{n-1} = T^n x_0,
\]

\[
(3.15)
\]
for all $n \geq 1$.

By (3.14) and (3.15) we obtain that

$$S(x_n, x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n)$$

$$\leq \alpha \max\{S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_{n-1}, Tx_n), S(x_{n-1}, x_{n-1}, x_n)\} +$$

$$\beta \{S(x_{n-1}, x_{n-1}, Tx_n) + S(x_n, x_{n-1}, Tx_{n-1})\}$$

$$= \alpha \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1}), S(x_{n-1}, x_{n-1}, x_n)\} +$$

$$\beta \{S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\}$$

$$\leq \alpha \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} +$$

$$\beta \{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\},$$

that is,

$$S(x_n, x_n, x_{n+1}) \leq \alpha A_1 + \beta \{2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\},$$

where $A_1 = \max\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\}$

(3.16)

Now two cases arise:

**Case I:** Suppose that $A_1 = S(x_{n-1}, x_{n-1}, x_n)$, then from (3.16)

$$S(x_n, x_n, x_{n+1}) \leq \alpha S(x_{n-1}, x_{n-1}, x_n) + 2\beta S(x_{n-1}, x_{n-1}, x_n) + \beta S(x_n, x_n, x_{n+1}),$$

that is,

$$(1 - \beta)S(x_n, x_n, x_{n+1}) \leq (\alpha + 2\beta)S(x_{n-1}, x_{n-1}, x_n),$$

that is,

$$S(x_n, x_n, x_{n+1}) \leq \frac{\alpha + 2\beta}{(1 - \beta)} S(x_{n-1}, x_{n-1}, x_n),$$

that is,

$$S(x_n, x_n, x_{n+1}) \leq \kappa S(x_{n-1}, x_{n-1}, x_n),$$

where $\kappa = \frac{\alpha + 2\beta}{(1 - \beta)} < 1$.

Thus, we have

$$S(x_n, x_n, x_{n+1}) \leq \kappa S(x_{n-2}, x_{n-2}, x_{n-1}).$$

Continue this process, we get

$$S(x_n, x_n, x_{n+1}) \leq \kappa^n S(x_0, x_0, x_1).$$

**Case II:** Suppose that $A_1 = S(x_n, x_n, x_{n+1})$, then by (3.16), we have

$$S(x_n, x_n, x_{n+1}) \leq \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_{n-1}, x_{n-1}, x_n) + \beta S(x_n, x_n, x_{n+1}),$$

that is,

$$(1 - \alpha - \beta)S(x_n, x_n, x_{n+1}) \leq 2\beta S(x_{n-1}, x_{n-1}, x_n),$$

or,

$$S(x_n, x_n, x_{n+1}) \leq \frac{2\beta}{1 - \alpha - \beta} S(x_{n-1}, x_{n-1}, x_n),$$

that is,

$$S(x_n, x_n, x_{n+1}) \leq \kappa S(x_{n-1}, x_{n-1}, x_n),$$

where $\kappa = \frac{2\beta}{1 - \alpha - \beta} < 1$. 
\[ S(x_n, x_n, x_{n+1}) \leq \kappa \kappa (S(x_{n-2}, x_{n-2}, x_{n-1})). \]

Continue this process, we get
\[ S(x_n, x_n, x_{n+1}) \leq \kappa^n S(x_0, x_0, x_1). \]

Thus, from both the cases and by Lemma 2.11 it follows that \( \{x_n\} \) is a Cauchy sequence in \( X \).

As \( (X, S) \) is a complete S-metric space, it follows that sequence \( \{x_n\} \) is convergent to some \( z \in X \). That is, \( x_n \to z \) as \( n \to \infty \).

Now, we show that \( z \) is the fixed point of \( T \).

\[ S(z, z, Tz) \leq 2S(z, z, x_{n+1}) + S(x_{n+1}, x_{n+1}, Tz) \]

\[ = 2S(z, z, x_{n+1}) + S(Tx_n, Tx_n, Tz), \text{ that is,} \]

\[ S(z, z, Tz) \leq 2S(z, z, x_{n+1}) \]

\[ + \alpha \max\{S(x_n, x_n, Tx_n), S(z, z, Tz), S(x_n, x_n, z)\} \]

\[ + \beta \{S(x_n, x_n, Tz) + S(z, z, x_{n+1})\} \]

\[ = 2S(z, z, x_{n+1}) + \alpha \max\{S(x_n, x_n, x_{n+1}), S(z, z, Tz), S(x_n, x_n, z)\} \]

\[ + \beta \{S(x_n, x_n, Tz) + S(z, z, x_{n+1})\} \]

\[ \leq 2S(z, z, x_{n+1}) + \alpha \max\{S(x_n, x_n, x_{n+1}), S(z, z, Tz), S(x_n, x_n, z)\} \]

\[ + \beta \{2S(x_n, x_n, z) + S(z, z, Tz) + S(z, z, x_{n+1})\}, \]

that is,

\[ (1 - \beta)S(z, z, Tz) \leq (2 + \beta)S(z, z, x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha A_2, \quad (3.17) \]

where \( A_2 = \max\{S(x_n, x_n, x_{n+1}), S(z, z, Tz), S(x_n, x_n, z)\} \)

**Case I:** Suppose that \( A_2 = S(x_n, x_n, x_{n+1}) \), then from (3.17), we get

\[ (1 - \beta)S(z, z, Tz) \leq (2 + \beta)S(z, z, x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha S(x_n, x_n, x_{n+1}) \]

\[ \leq (2 + \beta)S(z, z, x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha \{2S(x_n, x_n, z) + S(z, z, x_{n+1})\}, \]

that is,

\[ (1 - \beta)S(z, z, Tz) \leq (2 + \beta + \alpha)S(z, z, x_{n+1}) + 2(\alpha + \beta) S(x_n, x_n, z), \text{ that is,} \]

\[ S(z, z, Tz) \leq \frac{(2 + \alpha + \beta)}{(1 - \beta)} S(z, z, x_{n+1}) + \frac{2(\alpha + \beta)}{(1 - \beta)} S(x_n, x_n, z). \]

Taking limit as \( n \to \infty \), we get
\[
\lim_{n \to \infty} S(z, z, Tz) = 0 \implies Tz = z.
\]

Therefore, \( z \) is the fixed point of \( T \).

**Case II:** Suppose that \( A_2 = S(z, z, Tz) \) then from (3.17)

\[
(1 - \beta)S(z, z, Tz) \leq (2 + \beta)S(z, z, x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha S(z, z, Tz), \text{ that is,}
\]

\[
S(z, z, Tz) \leq \frac{(2+\beta)}{(1-\alpha-\beta)}(z, z, x_{n+1}) + \frac{2\beta}{((1-\alpha-\beta))} S(x_n, x_n, z).
\]

Taking limit as \( n \to \infty \), we get

\[
\lim_{n \to \infty} S(z, z, Tz) = 0 \implies Tz = z
\]

Therefore, \( z \) is the fixed point of \( T \).

**Case III:** Suppose that \( A_2 = S(x_n, x_n, z) \) then from (3.17)

\[
(1 - \beta)S(z, z, Tz) \leq (2 + \beta)S(z, z, x_{n+1}) + 2\beta S(x_n, x_n, z) + \alpha S(x_n, x_n, z), \text{ that is,}
\]

\[
S(z, z, Tz) \leq \frac{(2+\beta)}{(1-\beta)}S(z, z, x_{n+1}) + \frac{2\beta+\alpha}{(1-\beta)} S(x_n, x_n, z).
\]

Taking limit as \( n \to \infty \), we get

\[
\lim_{n \to \infty} S(z, z, Tz) = 0 \implies Tz = z.
\]

Therefore, \( z \) is the fixed point of \( T \).

**Uniqueness of the fixed point:**

We have to show that \( z \) is the unique fixed point of \( T \).

Let \( z' \) be another fixed point of \( T \) then we have \( Tz' = z' \).

Now, \( S(z, z, z') = S(Tz, Tz, Tz') \)

\[
\leq \alpha \max\{S(z, z, Tz), S(z', z', Tz'), S(z, z, Tz')\} + \beta \{S(z, z, Tz') + S(z', z', z)\}
\]

\[
\leq \alpha \max\{S(z, z, z), S(z', z', z'), S(z, z, z')\} + \beta \{S(z, z, z') + S(z', z', z)\}
\]

\[
= \alpha S(z, z, z') + 2\beta S(z, z, z')
\]

Thus, we get

\[
S(z, z, z') \leq (\alpha + 2\beta)S(z, z, z'), \text{ that is,}
\]

\[
S(z, z, z') < (1 - \beta) S(z, z, z') \text{ where } q > 0 \text{ with } \alpha + 3\beta < 1, \text{ a contradiction.}
\]
Therefore $z = z'$. Hence $z$ is the unique fixed point. This completes the proof. 

**Example 3.8:** Let $X = [0,1]$. Let us consider the usual S-metric on $X$ defined as follows

$S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Let $Tx = \frac{x}{3}$ for all $x \in [0,1]$. Then $T$ is a self-mapping on the S-metric space $[0,1]$. Here we have $S(Tx, Tx, Ty) = \frac{2}{3}|x - y|$ and

\[
\alpha \max\{S(x, x, Tx), S(y, y, Ty), S(x, x, y)\} + \beta \{S(x, x, Ty) + S(y, y, Tx)\}
\]

\[
= \alpha \max\{2|x - Tx|, 2|y - Ty|, 2|x - y|\} + \beta\{2|x - Ty| + |y - Tx|\}
\]

which for all value of $x, y \in [0,1]$, condition (3.14) is satisfied with $\alpha + 3\beta < 1$. Therefore, $T$ has a unique fixed point $x = 0$.

**Theorem 3.9:** Let $(X, S)$ be a complete S-metric space. Let $T$ be a mapping $T: X \rightarrow X$ such that

$S(Tx, Tx, Ty) \leq \alpha\{S(x, x, Tx) + S(y, y, Ty)\} + \beta\{S(x, x, Ty) + S(y, y, Tx)\}, \quad (3.18)$

where $\alpha, \beta > 0$ such that $2\alpha + 3\beta < 1$ for all $x, y \in X$. Then $T$ has a unique fixed point.

**Proof:** Let $x_0 \in X$. Define a sequence $\{x_n\}$ in $X$ as

$x_n = Tx_{n-1} = T^n x_0, \quad (3.19)$

for all $n \geq 1$

By (3.18) and (3.19), we obtain that

\[
S(x_n, x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n)
\]

\[
\leq \alpha\{S(x_{n-1}, x_{n-1}, Tx_{n-1}) + S(x_n, x_n, Tx_n)\} + \beta\{S(x_{n-1}, x_{n-1}, Tx_n) + S(x_n, x_n, Tx_n)\}
\]

\[
+ S(x_n, x_n, Tx_{n-1})
\]

\[
= \alpha\{S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\} + \beta\{S(x_{n-1}, x_{n-1}, x_{n+1}) + S(x_n, x_n, x_{n+1})\}
\]

\[
\leq \alpha S(x_{n-1}, x_{n-1}, x_n) + \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_{n-1}, x_{n-1}, x_n) + 2\beta S(x_{n-1}, x_{n-1}, x_{n+1})
\]

\[
(1 - \alpha - \beta)S(x_n, x_n, x_{n+1}) \leq (\alpha + 2\beta)S(x_{n-1}, x_{n-1}, x_n), \text{ or,}
\]

\[
S(x_n, x_n, x_{n+1}) \leq \frac{(\alpha + 2\beta)}{(1 - \alpha - \beta)} S(x_{n-1}, x_{n-1}, x_n), \text{ that is,}
\]
$S(x_n, x_n, x_{n+1}) \leq \kappa S(x_{n-1}, x_{n-1}, x_n)$, where \( \kappa = \frac{(\alpha + 2\beta)}{(1 - \alpha - \beta)} < 1 \).

Hence, by Lemma 2.11 and (S2) it follows that \( \{x_n\} \) is a Cauchy sequence in \( X \).

As \((X, S)\) is a complete S-metric space, it follows that sequence \( \{x_n\} \) is convergent to some point \( z \in X \).

That is, \( x_n \to z \) as \( n \to \infty \).

Now, we show that \( z \) is the fixed point of \( T \).

\[
S(z, z, Tz) \leq 2S(z, z, x_{n+1}) + S(x_{n+1}, x_{n+1}, Tz)
\]
\[
= 2S(z, z, x_{n+1}) + S(Tx_n, Tx_n, Tz)
\]
\[
\leq 2S(z, z, x_{n+1}) + \alpha[S(x_n, x_n, Tx_n) + S(z, z, Tz)]
\]
\[
+ \beta[S(x_n, x_n, Tz) + S(z, z, Tx_n)]
\]
\[
\leq 2S(z, z, x_{n+1}) + \alpha[S(x_n, x_n, x_{n+1}) + S(z, z, Tz)]
\]
\[
+ \beta[S(x_n, x_n, Tz) + S(z, z, x_{n+1})]
\]
\[
(1 - \alpha)S(z, z, Tz) \leq (2 + \beta)S(z, z, x_{n+1}) + \alpha S(x_n, x_n, x_{n+1}) + \beta S(x_n, x_n, Tz)
\]
\[
\leq (2 + \beta)S(z, z, x_{n+1}) + \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_n, x_n, z) + \beta S(z, z, Tz),
\]
that is,
\[
(1 - \alpha - \beta)S(z, z, Tz) \leq (2 + \beta)S(z, z, x_{n+1}) + \alpha S(x_n, x_n, x_{n+1}) + 2\beta S(x_n, x_n, z),
\]
that is,
\[
S(z, z, Tz) \leq \frac{(2 + \beta)}{(1 - \alpha - \beta)} S(z, z, x_{n+1}) + \frac{\alpha}{(1 - \alpha - \beta)} S(x_n, x_n, x_{n+1})
\]
\[
+ \frac{2\beta}{(1 - \alpha - \beta)} S(x_n, x_n, z)
\]

Now, taking \( \lim_{n \to \infty} \) we get
\[
\lim_{n \to \infty} S(z, z, Tz) = 0 \Rightarrow Tz = z
\]

Therefore, \( z \) is the fixed point of \( T \).

**Uniqueness of the fixed point:**

We have to show that \( z \) is the unique fixed point of \( T \).

Let \( z' \) be another fixed point of \( T \) then we have \( Tz' = z' \).

Now, \( S(z, z, z') = S(Tz, Tz, Tz') \)
\[
\leq \alpha[S(z, z, Tz) + S(z', z', Tz')] + \beta[S(z, z, Tz') + S(z', z', Tz)]
\]
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\[ S(z, z, z) + S(z', z', z') \leq \alpha \{ S(z, z, z) + S(z', z', z') \} + \beta \{ S(z, z, z') + S(z', z', z) \}, \]

that is, \( S(z, z, z) \leq 2\beta S(z, z, z') \) with \( \beta > 0 \) such that \( 2\alpha + 3\beta < 1 \), a contradiction.

Therefore \( z = z' \).

Hence \( z \) is the unique fixed point. This completes the proof. \( \blacksquare \)

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES


