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COUPLED FIXED POINTS OF GENERALIZED SUZUKI TYPE $\mathscr{Z}$-CONTRACTION MAPS WITH RATIONAL EXPRESSIONS IN b-METRIC SPACES

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Abstract. In this paper, we introduce generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions for a single map $\mathfrak{A} : S \times S \to S$ where S is a b-metric space and prove the existence and uniqueness of coupled fixed points. We extend it to a pair of selfmaps by defining generalized Suzuki type $\mathscr{Z}$-contraction pair of maps with rational expressions. Two corollaries are drawn from our results and we provide examples in support of our results.

Keywords: coupled fixed points; b-metric space; b-Cauchy sequence; generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions.

2020 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The study of existence and uniqueness of coincidence points of mappings satisfying certain contractive conditions has been interesting field, when Banach stated and proved his famous result Banach contraction principle and it plays an important role in solving nonlinear functional analysis. In the direction of generalization of contraction condition, Dass and Gupta [14] initiated a contraction condition involving rational expression and established the existence of

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fixed points in complete metric spaces. In 2008, Suzuki [30] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

In the direction of generalization of metric spaces, Bourbaki [11] and Bakhtin [6] initiated the idea of b-metric spaces. The concept of b-metric space or metric type space was introduced by Czerwik [12] as a generalization of metric space. Afterwards, many authors studied the existence of fixed points for a single-valued and multi-valued mappings in b-metric spaces under certain contraction conditions. For more details, we refer [2, 9, 10, 13, 15, 17, 20, 26, 27]. In 2006, Bhaskar and Lakshmikantham [7] introduced the notion of coupled fixed point and established the existence of coupled fixed points for mixed monotone mappings in ordered metric spaces. Later, Lakshmikantham and Ćirić [21] introduced the notion of coupled coincidence points of mappings in two variables. Afterwards, many authors studied coupled fixed point theorems, we refer [22, 24, 28, 29].

2. PRELIMINARIES

Definition 2.1. [12] Let S be a non-empty set. A function $l_b : S \times S \to [0,\infty)$ is said to be a b-metric if the following conditions are satisfied: for any $\mu, \xi, \nu \in S$

- (i) $0 \leq l_b(\mu,\xi)$ and $l_b(\mu,\xi) = 0$ if and only if $\mu = \xi$,
- (ii) $l_b(\mu,\xi) = l_b(\xi,\mu)$,
- (iii) there exists $\tau \geq 1$ such that $l_b(\mu,\nu) \leq \tau[l_b(\mu,\xi) + l_b(\xi,\nu)]$.

In this case, the pair (S, l_b) is called a b-metric space with coefficient τ.

Every metric space is a b-metric space with $\tau = 1$. In general, every b-metric space is not a metric space.

Definition 2.2. [10] Let (S, l_b) be a b-metric space.

- (i) A sequence $\{\mu_n\}$ in S is called b-convergent if there exists $\mu \in S$ such that $l_b(\mu_n,\mu) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} \mu_n = \mu$.
- (ii) A sequence $\{\mu_n\}$ in S is called b-Cauchy if $l_b(\mu_n,\mu_m) \to 0$ as $n,m \to \infty$.
- (iii) A b-metric space (S, l_b) is said to be a complete b-metric space if every b-Cauchy sequence in S is b-convergent in S.

(iv) A set $B \subset S$ is said to be b-closed if for any sequence $\{\mu_n\}$ in B such that $\{\mu_n\}$ is b-convergent to $\nu \in S$ then $\nu \in B$.

In general, a b-metric is not necessarily continuous.

In this paper, we denote $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{N}$ is the set of all natural numbers.

Example 2.3.[16] Let $S = \mathbb{N} \cup \{\infty\}$. We define a mapping $l_b : S \times S \to \mathbb{R}^+$ as follows:

$$l_b(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (S, l_b) is a b-metric space with coefficient $\tau = \frac{5}{2}$.

Definition 2.4. [10] Let (S, d_S) and (T, d_T) be two b-metric spaces. A function $f : S \to T$ is a b-continuous at a point $\mu \in S$, if it is b-sequentially continuous at μ. i.e., whenever $\{\mu_n\}$ is b-convergent to μ we have $f\mu_n$ is b-convergent to $f\mu$.

Definition 2.5. [7] Let S be a nonempty set and $f : S \times S \to S$ be a mapping. Then we say that an element $(\mu, \xi) \in S \times S$ is a coupled fixed point, if $f(\mu, \xi) = \mu$ and $f(\xi, \mu) = \xi$.

Definition 2.6. [21] Let S be a nonempty set. Let $F : S \times S \to S$ and $g : S \to S$ be two mappings. An element $(\mu, \xi) \in S \times S$ ia called

(i) a coupled coincidence point of the mappings F and g if $F(\mu, \xi) = g\mu$ and $F(\xi, \mu) = g\xi$;

(ii) a common coupled fixed point of mappings F and g if $F(\mu, \xi) = g\mu = \mu$ and $F(\xi, \mu) = g\xi = \xi$.

The following lemma is useful in proving our main results.

Lemma 2.7. [1] Let (S, l_b) be a b-metric space with coefficient $\tau \geq 1$. Suppose that $\{\mu_n\}$ and $\{\xi_n\}$ are b-convergent to μ and ξ respectively. Then we have

$$\frac{1}{\tau^2} l_b(\mu, \xi) \leq \liminf_{n\to\infty} l_b(\mu_n, \xi_n) \leq \limsup_{n\to\infty} l_b(\mu_n, \xi_n) \leq \tau^2 l_b(\mu, \xi).$$

In particular, if $\mu = \xi$, then we have $\lim_{n\to\infty} l_b(\mu_n, \xi_n) = 0$. Moreover for each $\nu \in S$ we have

$$\frac{1}{\tau}l_b(\mu,\nu) \le \liminf_{n\to\infty} l_b(\mu_n,\nu) \le \limsup_{n\to\infty} l_b(\mu_n,\nu) \le \tau l_b(\mu,\nu).$$

In 2015, Khojasteh, Shukla and Radenović [18] introduced simulation function and defined $\mathscr{Z}$-contraction with respect to a simulation function.

Definition 2.8. [18] A simulation function is a mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ for all $s,t > 0$;

(ζ_3) if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(t_n,s_n) < 0$.

Remark 2.9.[3] Let ζ be a simulation function. If $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(kt_n,s_n) < 0$ for any $k > 1$.
The following are examples of simulation functions.

Example 2.10. [3] Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ be defined by

(i) $\zeta(t,s) = \lambda s - t$ for all $t,s \in \mathbb{R}^+,$ where $\lambda \in [0,1)$;

(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $s,t \in \mathbb{R}^+$;

(iii) $\zeta(t,s) = s - kt$ for all $t,s \in \mathbb{R}^+,$ where $k > 1$;

(iv) $\zeta(t,s) = \frac{1}{1+s} - (1+t)$ for all $s,t \in \mathbb{R}^+$;

(v) $\zeta(t,s) = \frac{1}{k+s} - t$ for all $s,t \in \mathbb{R}^+$ where $k > 1$.

Definition 2.11. [18] Let (S,l_b) be a metric space and $f : S \to S$ be a selfmap of S. We say that f is a $\mathscr{Z}$-contraction with respect to ζ if there exists a simulation function ζ such that

$$\zeta(l_b(f\mu,f\xi),l_b(\mu,\xi)) \ge 0 \text{ for all } \mu,\xi \in S.$$

Theorem 2.12. [18] Let (S,l_b) be a complete metric space and $f : S \to S$ be a $\mathscr{Z}$-contraction with respect to a certain simulation function ζ. Then for every $\mu_0 \in S$, the Picard sequence $\{f^n\mu_0\}$ converges in S and $\lim_{n\to\infty} f^n\mu_0 = u$(say) in S and u is the unique fixed point of f in S.

Recently, Olgun, Bicer and Alyildiz [23] proved the following result in complete metric spaces.

Theorem 2.13. [23] Let (S,l_b) be a complete metric space and $f : S \to S$ be a selfmap on S. If there exists a simulation function ζ such that

$$\zeta(l_b(f\mu, f\xi), M(\mu,\xi)) \geq 0$$

for all $\mu,\xi \in S$, where $M(\mu,\xi) = \max\{l_b(\mu,\xi), l_b(\mu, f\mu), l_b(\xi, f\xi), \frac{l_b(\mu,f\xi)+l_b(\xi,f\mu)}{2}\}$,then for every $\mu_0 \in S$, the Picard sequence $\{f^n\mu_0\}$ converges in S and $\lim_{n\to\infty} f^n\mu_0 = u$(say) in S and u is the unique fixed point of f in S.

In 2018, Babu, Dula and Kumar [3] extended Theorem 1.13 [23] to pair of selfmaps in the setting of b-metric spaces as follows.

Theorem 2.14. [3] Let (S,l_b) be a complete b-metric space with coefficient $\tau \geq 1$ and $f,g : S \to S$ be a selfmaps on S. If there exists a simulation function ζ such that

$$\zeta(\tau^4 l_b(f\mu, g\xi), M(\mu,\xi)) \geq 0$$

for all $\mu,\xi \in S$, where $M(\mu,\xi) = \max\{l_b(\mu,\xi), l_b(\mu, f\mu), l_b(\xi, g\xi), \frac{l_b(\mu,g\xi)+l_b(\xi,f\mu)}{2\tau}\}$, then f and g have a unique common fixed point in S, provided either f or g is b-continuous.

The following theorem is due to Kumam, Gopal and Budhia [19].

Theorem 2.15. [19] Let (S,l_b) be a complete metric space and $f : S \to S$ be a selfmap on S. If there exists a simulation function ζ such that

$$\frac{1}{2} l_b(\mu, f\mu) < l_b(\mu,\xi) \implies \zeta(l_b(f\mu, f\xi), l_b(\mu,\xi)) \geq 0$$

for all $\mu,\xi \in S$, then for every $\mu_0 \in S$, the Picard sequence $\{\mu_n\}$, where $\mu_n = f\mu_{n-1}$ for all $n \in \mathbb{N}$ converges to the unique fixed point of f.

In 2018, Padcharoen, Kumam, Saipara and Chaipunya [25], proved the following theorem in complete metric spaces.

Theorem 2.16. [25] Let (S,l_b) be a complete metric space and $f : S \to S$ be a selfmap on S. If there exists a simulation function ζ such that

$$\frac{1}{2} l_b(\mu, f\mu) < l_b(\mu,\xi) \implies \zeta(l_b(f\mu, f\xi), M(\mu,\xi)) \geq 0$$

for all $\mu,\xi \in S$, where $M(\mu,\xi) = \max\{l_b(\mu,\xi), l_b(\mu, f\mu), l_b(\xi, f\xi), \frac{l_b(\mu,f\xi)+l_b(\xi,f\mu)}{2}\}$, then for every $\mu_0 \in S$, the Picard sequence $\{\mu_n\}$, where $\mu_n = f\mu_{n-1}$ for all $n \in \mathbb{N}$ converges to the unique fixed point of f.

Recently, Babu and Babu [4, 5] proved the following theorems in the setting of b-metric spaces.

Definition 2.17. [4] Let (S,l_b) be a b-metric space with coefficient $\tau \geq 1$ and $f : S \to S$ be a selfmap. We say that f is a Suzuki $\mathscr{Z}$-contraction type (I) map, if there exists a simulation function ζ such that

$$\frac{1}{2\tau} l_b(\mu, f\mu) < l_b(\mu,\xi) \text{ implies that } \zeta(\tau^4 l_b(f\mu, f\xi), M_1(\mu,\xi)) \geq 0$$

for all distinct $\mu,\xi \in S$, where
$M_1(\mu,\xi) = \max\{l_b(\mu,\xi), l_b(\mu, f\mu), l_b(\xi, f\xi), \frac{l_b(\mu,f\xi)+l_b(\xi,f\mu)}{2\tau}\}$.

Theorem 2.18.[4] Let (S,l_b) be a complete b-metric space with coefficient $\tau \geq 1$ and $f : S \to S$ be a Suzuki $\mathscr{Z}$-contraction type (I) map. Then f has a unique fixed point in S.

Definition 2.19. [5] Let (S,l_b) be a b-metric space with coefficient $\tau \geq 1$ and $f,g : S \to S$ be a selfmaps on S. We say that (f,g) is a Suzuki $\mathscr{Z}$-contraction type (I) maps, if there exists a simulation function ζ such that

$$\frac{1}{2\tau} \min\{l_b(\mu, f\mu), l_b(\xi, g\xi)\} \leq l_b(\mu,\xi) \text{ implies that } \zeta(\tau^4 l_b(f\mu, g\xi), M_1(\mu,\xi)) \geq 0$$

for all $\mu,\xi \in S$, where $M_1(\mu,\xi) = \max\{l_b(\mu,\xi), l_b(\mu, f\mu), l_b(\xi, g\xi), \frac{l_b(\mu,g\xi)+l_b(\xi,f\mu)}{2\tau}\}$.

Theorem 2.20.[5] Let (S,l_b) be a complete b-metric space with coefficient $\tau \geq 1$ and (f,g) be a Suzuki $\mathscr{Z}$-contraction type (I) maps. If either f (or) g is b-continous then f and g have a unique common fixed point in S.

In 2018, Bindu and Malhotra [8] proved the existence of common coupled fixed points as follows:

Theorem 2.21. Let (S,l_b) be a complete b-metric space with parameter $\tau \geq 1$ and let the mappings $F,G : S \times S \to S$ satisfy

$$\begin{aligned} l_b(F(\mu,\xi),G(u,v)) \leq{}& \alpha_1 \tfrac{l_b(\mu,u)+l_b(\xi,v)}{2} + \alpha_2 \tfrac{l_b(\mu,F(\mu,\xi))l_b(u,G(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} \\ &+ \alpha_3 \tfrac{l_b(u,F(\mu,\xi))l_b(\mu,G(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} + \alpha_4 \tfrac{l_b(F(\mu,\xi),G(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} \\ &+ \alpha_5 \tfrac{l_b(F(\mu,\xi),G(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} + \alpha_6 \tfrac{l_b(u,G(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} \\ &+ \alpha_7 \tfrac{l_b(u,F(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} + \alpha_8 \tfrac{l_b(u,F(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,F(\mu,\xi))} \\ &+ \alpha_9 \max l_b(u,F(\mu,\xi)), l_b(F(\mu,\xi),G(u,v)) \end{aligned}$$

for all $\mu,\xi,u,v \in S$ and $\alpha_i \geq 0, i = 1,2,\ldots,9$ with $\tau\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \tau\alpha_9 < 1$ and

$\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_7+\alpha_8+\alpha_9<1$. Then F and G have a unique common coupled fixed point in S.

Motivated by the works of Babu and Babu [4, 5], Bindu and Malhotra [8], in Section 3, we introduce generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions for a single map $\mathfrak{A}: S\times S\to S$ where S is a b-metric space and we extend it to a pair of maps. In Section 4, we prove the existence and uniqueness of coupled fixed points and common coupled fixed points in complete b-metric spaces. Some examples are provided in support of our results and we draw some corollaries in Section 5.

3. Generalized Suzuki Type $\mathscr{Z}$-Contraction Maps With Rational Expressions

The following we introduce generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions for a single and a pair of maps in b-metric spaces as follows:

Definition 3.1. Let (S,l_b) be a b-metric space with coefficient $\tau\geq 1$ and $\mathfrak{A}: S\times S\to S$ be a map. We say that $\mathfrak{A}$ is a generalized Suzuki type $\mathscr{Z}$-contraction map with rational expressions, if there exists a simulation function ζ such that

(3.1)
$$\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi))<l_b(\mu,\xi)\implies\zeta(\tau^3l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v)),M(\mu,\xi,u,v))\geq 0 \text{ for all } \mu,\xi,u,v\in S,$$

where

$$\begin{aligned}M(\mu,\xi,u,v)=\max\{&\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau},\tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\&\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\&\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\&\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}.\end{aligned}$$

Remark 3.2. It is clear that from definition of simulation function that $\zeta(a,b)<0$, for all $a\geq b>0$. Therefore if $\mathfrak{A}$ satisfies the inequality (3.1), then

$$\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi))<l_b(\mu,\xi)\implies\tau^3l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))<M(\mu,\xi,u,v), \text{ for all } \mu,\xi,u,v\in S.$$

Example 3.3. Let $S=[0,1]$ and let $l_b : S\times S\to\mathbb{R}^+$ defined by

$$l_b(\mu,\xi)=\begin{cases} 0 & \text{if } \mu=\xi \\ (\mu+\xi)^2 & \text{if } \mu\neq\xi. \end{cases}$$

Then clearly (S,l_b) is a b-metric space with coefficient $\tau=2$.

We define $\mathfrak{A}: S\times S\to S$ by $\mathfrak{A}(\mu,\xi)=\frac{\mu^2e^{\xi}}{16}$ for all $\mu\in[0,1]$ and

$\zeta:\mathbb{R}^+\times\mathbb{R}^+\to(-\infty,\infty)$ by $\zeta(t,s)=\frac{1}{2}s-t, t\geq 0, s\geq 0$.

Since $\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi))=\frac{1}{4}(\mu+\frac{\mu^2e^{\xi}}{16})^2<(\mu+\xi)^2=l_b(\mu,\xi)$.

From the inequality (3.1), we have

$$\begin{aligned}\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v)) &= 8[\tfrac{\mu^2e^{\xi}}{16}+\tfrac{u^2e^{v}}{16}]^2\\ &\leq \tfrac{1}{8}[(\mu+u)^2+(\xi+v)^2]\\ &= \tfrac{1}{2}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau})\\ &\leq \tfrac{1}{2}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\qquad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\qquad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\qquad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\qquad \max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}).\end{aligned}$$

Therefore $\mathfrak{A}$ is a generalized Suzuki type $\mathscr{Z}$-contraction map with rational expressions.

Definition 3.4. Let (S,l_b) be a b-metric space with coefficient $\tau\geq 1$ and $\mathfrak{A},\mathfrak{B}: S\times S\to S$ be two maps. We say that the pair $(\mathfrak{A},\mathfrak{B})$ is a generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions, if there exists a simulation function ζ such that

(3.2)

$$\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)),l_b(u,\mathfrak{B}(u,v))\}\leq\max\{l_b(\mu,u),l_b(\xi,v)\}$$
$$\implies \zeta(\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)),M(\mu,\xi,u,v))\geq 0,$$

for all $\mu,\xi,u,v\in S$, where

$$\begin{aligned}M(\mu,\xi,u,v)=\max\{&\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\}.\end{aligned}$$

Remark 3.5. It is clear that from definition of simulation function that $\zeta(a,b) < 0$, for all $a \geq b > 0$. Therefore if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the inequality (3.2), then

$$\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)), l_b(u,\mathfrak{B}(u,v))\} \leq \max\{l_b(\mu,u), l_b(\xi,v)\}$$

$$\implies \tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)) < M(\mu,\xi,u,v),$$

for all $\mu,\xi,u,v \in S$.

Example 3.6. Let $S = [0,1]$ and let $l_b : S\times S \to \mathbb{R}^+$ defined by

$$l_b(\mu,\xi) = \begin{cases} 0 & \text{if } \mu = \xi \\ (\mu+\xi)^2 & \text{if } \mu \neq \xi. \end{cases}$$

Then clearly (S,l_b) is a b-metric space with coefficient $\tau = 2$.

We define $\mathfrak{A},\mathfrak{B} : S\times S \to S$ by

$\mathfrak{A}(\mu,\xi) = \frac{\mu^2 e^{\xi}}{16}$ and $\mathfrak{B}(\mu,\xi) = \frac{\log(1+\mu^2+\xi^2)}{8}$

$\zeta : \mathbb{R}^+\times\mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{99}{100}s - t, t \geq 0, s \geq 0$.

Since $\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)), l_b(u,\mathfrak{B}(u,v))\} = \frac{1}{4}\min\{(\mu+\frac{\mu^2e^{\xi}}{16})^2, (u+\frac{\log(1+\mu^2+\xi^2)}{8})^2\}$

$$\leq \max\{(\mu+u)^2,(\xi+v)^2\} = \max\{l_b(\mu,u), l_b(\xi,v)\}.$$

From the inequality (3.2), we have

$$\begin{aligned}
\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)) &= 8[\tfrac{\mu^2e^{\xi}}{16} + \tfrac{\log(1+\mu^2+\xi^2)}{8}]^2 \\
&\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2] \\
&= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}) \\
&\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \max\{l_b(u,\mathfrak{A}(\mu,\xi)), l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\}).
\end{aligned}$$

Therefore the pair $(\mathfrak{A},\mathfrak{B})$ is a generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions.

4. Main Results

Theorem 4.1. Let (S,l_b) be a complete b-metric space with coefficient $\tau \geq 1$ and $\mathfrak{A} : S\times S \to S$ be a generalized Suzuki type $\mathscr{Z}$-contraction map with rational expressions. Then $\mathfrak{A}$ has a unique coupled fixed point in S.

Proof. Let μ_0 and ξ_0 be arbitrary points in S.

We define

$\mu_{i+1} = \mathfrak{A}(\mu_i, \xi_i)$ and $\xi_{i+1} = \mathfrak{A}(\xi_i, \mu_i)$ for $i = 0, 1, 2, \ldots$.

Since $\frac{1}{2\tau} l_b(\mu_n, \mathfrak{A}(\mu_n, \xi_n)) < l_b(\mu_n, \mu_{n+1}))$, from the inequality (3.1), we have

$$\zeta(\tau^3 l_b(\mu_{n+1}, \mu_{n+2}), M(\mu_n, \xi_n, \mu_{n+1}, \xi_{n+1})) = \zeta(\tau^3 l_b(\mathfrak{A}(\mu_n, \xi_n), \mathfrak{A}(\mu_{n+1}, \xi_{n+1})), M(\mu_n, \xi_n, \mu_{n+1}, \xi_{n+1})) \geq 0, \tag{4.1}$$

where

$$\begin{aligned} M(\mu_n, \xi_n, \mu_{n+1}, \xi_{n+1}) &= \max\{\tfrac{l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})}{2\tau}, \tfrac{l_b(\mu_n,\mathfrak{A}(\mu_n,\xi_n))l_b(\mu_{n+1},\mathfrak{A}(\mu_{n+1},\xi_{n+1}))}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \\ &\tfrac{l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))l_b(\mu_n,\mathfrak{A}(\mu_{n+1},\xi_{n+1}))}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \tfrac{l_b(\mathfrak{A}(\mu_n,\xi_n),\mathfrak{A}(\mu_{n+1},\xi_{n+1}))l_b(\mu_n,\mu_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \\ &\tfrac{l_b(\mathfrak{A}(\mu_n,\xi_n),\mathfrak{A}(\mu_{n+1},\xi_{n+1}))l_b(\xi_n,\xi_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \tfrac{l_b(\mu_{n+1},\mathfrak{A}(\mu_{n+1},\xi_{n+1}))l_b(\xi_n,\xi_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \\ &\tfrac{l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))l_b(\mu_n,\mu_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \tfrac{l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))l_b(\xi_n,\xi_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mathfrak{A}(\mu_n,\xi_n))}, \\ &\max\{l_b(\mu_{n+1}, \mathfrak{A}(\mu_n, \xi_n)), l_b(\mathfrak{A}(\mu_n, \xi_n), \mathfrak{A}(\mu_{n+1}, \xi_{n+1}))\}\} \\ &= \max\{\tfrac{l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})}{2\tau}, \tfrac{l_b(\mu_n,\mu_{n+1})l_b(\mu_{n+1},\mu_{n+2})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \\ &\tfrac{l_b(\mu_{n+1},\mu_{n+1})l_b(\mu_n,\mu_{n+2})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \tfrac{l_b(\mu_{n+1},\mu_{n+2})l_b(\mu_n,\mu_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \\ &\tfrac{l_b(\mu_{n+1},\mu_{n+2})l_b(\xi_n,\xi_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \tfrac{l_b(\mu_{n+1},\mu_{n+2})l_b(\xi_n,\xi_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \\ &\tfrac{l_b(\mu_{n+1},\mu_{n+1})l_b(\mu_n,\mu_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \tfrac{l_b(\mu_{n+1},\mu_{n+1})l_b(\xi_n,\xi_{n+1})}{1+l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})+l_b(\mu_{n+1},\mu_{n+1})}, \\ &\max\{l_b(\mu_{n+1}, \mu_{n+1}), l_b(\mu_{n+1}, \mu_{n+2})\}\} \\ &\leq \max\{\tfrac{l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1})}{2\tau}, l_b(\mu_{n+1}, \mu_{n+2})\} \end{aligned}$$

If $M(\mu_n, \xi_n, \mu_{n+1}, \xi_{n+1}) = l_b(\mu_{n+1}, \mu_{n+2})$ then from (4.1), we have

$$\begin{aligned} 0 \leq \zeta(\tau^3 l_b(\mu_{n+1}, \mu_{n+2}), M(\mu_n, \xi_n, \mu_{n+1}, \xi_{n+1})) &= \zeta(\tau^3 l_b(\mu_{n+1}, \mu_{n+2}), l_b(\mu_{n+1}, \mu_{n+2})) \\ &< l_b(\mu_{n+1}, \mu_{n+2}) - \tau^3 l_b(\mu_{n+1}, \mu_{n+2}), \end{aligned}$$

which is a contradiction.

Therefore

$$l_b(\mu_{n+1}, \mu_{n+2}) \leq \frac{l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1})}{2\tau} \tag{4.2}$$

for all $n = 0, 1, 2, \ldots$.

Similarly we can prove that

$$l_b(\xi_{n+1}, \xi_{n+2}) \leq \frac{l_b(\xi_n, \xi_{n+1}) + l_b(\mu_n, \mu_{n+1})}{2\tau} \tag{4.3}$$

for all $n = 0, 1, 2, \dots$.

Adding the inequalities (4.2) and (4.3), we have

$l_b(\mu_{n+1}, \mu_{n+2}) + l_b(\xi_{n+1}, \xi_{n+2}) \leq h[l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1})]$, where $h = \frac{1}{2\tau} < 1$.

Also, it is easy to see that $l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1}) \leq h[l_b(\mu_{n-1}, \mu_n) + l_b(\xi_{n-1}, \xi_n)]$.

Therefore $l_b(\mu_{n+1}, \mu_{n+2}) + l_b(\xi_{n+1}, \xi_{n+2}) \leq h^2[l_b(\mu_{n-1}, \mu_n) + l_b(\xi_{n-1}, \xi_n)]$.

Continuing in the same way, we get that

$l_b(\mu_{n+1}, \mu_{n+2}) + l_b(\xi_{n+1}, \xi_{n+2}) \leq h^n[l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)]$.

For $m > n, m, n \in \mathbb{N}$, we have

$$\begin{aligned} l_b(\mu_n, \mu_m) + l_b(\xi_n, \xi_m) &\leq \tau[l_b(\mu_n, \mu_{n+1}) + l_b(\mu_{n+1}), \mu_m)] + \tau[l_b(\xi_n, \xi_{n+1}) + l_b(\xi_{n+1}, \xi_m)] \\ &\leq \tau[l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1})] + \tau^2[l_b(\mu_{n+1}, \mu_{n+2}) + l_b(\mu_{n+2}, \mu_m)] \\ &\qquad + \tau^2[l_b(\xi_{n+1}, \xi_{n+2}) + l_b(\xi_{n+2}, \xi_m)] \\ &= \tau[l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1})] + \tau^2[l_b(\mu_{n+1}, \mu_{n+2}) + l_b(\xi_{n+1}, \xi_{n+2})] \\ &\quad + \tau^2[l_b(\mu_{n+2}, \mu_m) + l_b(\xi_{n+2}, \xi_m)] \dots + \tau^m[l_b(\mu_{m-1}, \mu_m) \\ &\quad + l_b(\xi_{m-1}, \xi_m)] \\ &\leq [\tau h^n + \tau^2 h^{n+1} + \dots + \tau^{m-1} h^{m-1}][l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)] \\ &\leq \tau h^n[1 + \tau h + (\tau h)^2 \dots + (\tau h)^{m-1} + \dots][l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)] \\ &= \tau h^n(\tfrac{1}{1-\tau h})[l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\{\mu_n\}$ and $\{\xi_n\}$ are b-Cauchy sequences in S.

Since S is b-complete, there exist $\mu, \xi \in S$ such that $\mu_n \to \mu$ and $\xi_n \to \xi$ as $n \to \infty$.

We now prove that $\mu = \mathfrak{A}(\mu, \xi)$ and $\xi = \mathfrak{A}(\xi, \mu)$.

On the contrary suppose that $\mu \neq \mathfrak{A}(\mu, \xi)$ and $\xi \neq \mathfrak{A}(\xi, \mu)$. We now show that

$$\begin{aligned} &\text{either (a): } \tfrac{1}{2\tau} l_b(\mu_n, \mu_{n+1})) \leq l_b(\mu_n, \mu), \tfrac{1}{2\tau} l_b(\xi_n, \xi_{n+1}) \leq l_b(\xi_n, \xi) \text{ or} \\ &\qquad \text{(b): } \tfrac{1}{2\tau} l_b(\mu_{n+1}, \mu_{n+2}) \leq l_b(\mu_{n+1}, \mu), \tfrac{1}{2\tau} l_b(\xi_{n+1}, \xi_{n+2}) \leq l_b(\xi_{n+1}, \xi) \end{aligned} \tag{4.4}$$

hold.

On the contrary, suppose that

$\frac{1}{2\tau} l_b(\mu_n, \mu_{n+1}) > l_b(\mu_n, \mu), \frac{1}{2\tau} l_b(\xi_n, \xi_{n+1}) > l_b(\xi_n, \xi)$ and

$\frac{1}{2\tau} l_b(\mu_{n+1}, \mu_{n+2}) > l_b(\mu_{n+1}, \mu), \frac{1}{2\tau} l_b(\xi_{n+1}, \xi_{n+2}) > l_b(\xi_{n+1}, \xi)$.

By b-triangular property, we have

$l_b(\mu_n, \mu_{n+1}) \leq \tau[l_b(\mu_n, \mu) + l_b(\mu, \mu_{n+1})]$

$$< \tau \tfrac{1}{2\tau}[l_b(\mu_n,\mu_{n+1})+l_b(\mu_{n+1},\mu_{n+2})]$$
$$= \tfrac{1}{2}[l_b(\mu_n,\mu_{n+1})+l_b(\mu_{n+1},\mu_{n+2})].$$

Similarly we can prove that

$$l_b(\xi_n,\xi_{n+1}) < \tfrac{1}{2}[l_b(\xi_n,\xi_{n+1})+l_b(\xi_{n+1},\xi_{n+2})].$$

Adding the above inequalities, we get

$$\begin{aligned} l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1}) &< \tfrac{1}{2}[l_b(\mu_n,\mu_{n+1}))+l_b(\mu_{n+1},\mu_{n+2})+l_b(\xi_n,\xi_{n+1})+l_b(\xi_{n+1},\xi_{n+2})] \\ &\leq l_b(\mu_n,\mu_{n+1})+l_b(\xi_n,\xi_{n+1}), \end{aligned}$$

which is a contradiction.

Therefore (4.4) holds.

Since $\frac{1}{2\tau}l_b(\mu_n,\mathfrak{A}(\mu_n,\xi_n)) \leq l_b(\mu_n,\mu)$, from the inequality (3.1), we have

(4.5)
$$\zeta(\tau^3 l_b(\mu_{n+1},\mathfrak{A}(\mu,\xi)),M(\mu_n,\xi_n,\mu,\xi)) = \zeta(\tau^3 l_b(\mathfrak{A}(\mu_n,\xi_n),\mathfrak{A}(\mu,\xi)),M(\mu_n,\xi_n,\mu,\xi)) \geq 0,$$

where

$$\begin{aligned} M(\mu_n,\xi_n,\mu,\xi) = \max\{ & \tfrac{l_b(\mu_n,\mu)+l_b(\xi_n,\xi)}{2\tau}, \tfrac{l_b(\mu_n,\mathfrak{A}(\mu_n,\xi_n))l_b(\mu,\mathfrak{A}(\mu,\xi))}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \\ & \tfrac{l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))l_b(\mu_n,\mathfrak{A}(\mu,\xi))}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \tfrac{l_b(\mathfrak{A}(\mu_n,\xi_n),\mathfrak{A}(\mu,\xi))l_b(\mu_n,\mu)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \\ & \tfrac{l_b(\mathfrak{A}(\mu_n,\xi_n),\mathfrak{A}(\mu,\xi))l_b(\xi_n,\xi)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(\xi_n,\xi)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \\ & \tfrac{l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))l_b(\mu_n,\mu)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \tfrac{l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))l_b(\xi_n,\xi)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mathfrak{A}(\mu_n,\xi_n))}, \\ & \max\{l_b(\mu,\mathfrak{A}(\mu_n,\xi_n)),l_b(\mathfrak{A}(\mu_n,\xi_n),\mathfrak{A}(\mu,\xi))\}\} \\ = \max\{ & \tfrac{l_b(\mu_n,\mu)+l_b(\xi_n,\xi)}{2\tau}, \tfrac{l_b(\mu_n,\mu_{n+1})l_b(\mu,\mathfrak{A}(\mu,\xi))}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \\ & \tfrac{l_b(\mu,\mu_{n+1}))l_b(\mu_n,\mathfrak{A}(\mu,\xi))}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \tfrac{l_b(\mu_{n+1}),\mathfrak{A}(\mu,\xi))l_b(\mu_n,\mu)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \\ & \tfrac{l_b(\mu_{n+1}),\mathfrak{A}(\mu,\xi))l_b(\xi_n,\xi)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(\xi_n,\xi)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \\ & \tfrac{l_b(\mu,\mu_{n+1}))l_b(\mu_n,\mu)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \tfrac{l_b(\mu,\mu_{n+1}))l_b(\xi_n,\xi)}{1+l_b(\mu_n,\mu)+l_b(\xi_n,\xi)+l_b(\mu,\mu_{n+1})}, \\ & \max\{l_b(\mu_n,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mu_{n+1}))\}\}. \end{aligned}$$

On taking limit superior as $n \to \infty$ in $M(\mu,\xi,\mu_n,\xi_n)$, we have

$$\limsup_{n\to\infty} M(\mu,\xi,\mu_n,\xi_n) \leq \tau l_b(\mu,\mathfrak{A}(\mu,\xi)).$$

On letting limit superior as $n \to \infty$ in (4.5) and using the Lemma 2.7, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mu_{n+1}),M(\mu,\xi,\mu_n,\xi_n)) \\ &= \limsup_{n\to\infty} M(\mu,\xi,\mu_n,\xi_n) - \liminf_{n\to\infty} \tau^3 l_b(\mathfrak{A}(\mu,\xi),\mu_{n+1})) \\ &\leq l_b(\mu,\mathfrak{A}(\mu,\xi)) - \tau^3 \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))}{\tau}, \end{aligned}$$

a contradiction. Therefore $\mu = \mathfrak{A}(\mu,\xi)$.

Similarly we can prove that $\xi = \mathfrak{A}(\xi,\mu)$.

Therefore (μ,ξ) is a coupled fixed point of $\mathfrak{A}$.

Let $(\mu',\xi') \in S\times S$ be another coupled fixed point of $\mathfrak{A}$ with $(\mu'.\xi') \neq (\mu,\xi)$.

Since $\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi)) < l_b(\mu,\mu')$, from the inequality (3.1), we have

$$\zeta(\tau^3 l_b(\mu,\mu'),M(\mu,\xi,\mu',\xi')) = \zeta(\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(\mu',\xi')),M(\mu,\xi,\mu',\xi')) \geq 0,$$

where

$$\begin{aligned} M(\mu,\xi,\mu',\xi') = \max\{&\tfrac{l_b(\mu,\mu')+l_b(\xi,\xi')}{2\tau}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(\mu',\mathfrak{A}(\mu',\xi'))}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mu',\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(\mu',\xi'))}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \\ &\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(\mu',\xi'))l_b(\mu,\mu')}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(\mu',\xi'))l_b(\xi,\xi')}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \\ &\tfrac{l_b(\mu',\mathfrak{A}(\mu',\xi'))l_b(\xi,\xi')}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mu',\mathfrak{A}(\mu,\xi))l_b(\mu,\mu')}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \\ &\tfrac{l_b(\mu',\mathfrak{A}(\mu,\xi))l_b(\xi,\xi')}{1+l_b(\mu,\mu')+l_b(\xi,\xi')+l_b(\mu',\mathfrak{A}(\mu,\xi))}, \max\{l_b(\mu',\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(\mu',\xi'))\}\} \\ &\leq \max\{\tfrac{l_b(\mu,\mu')+l_b(\xi,\xi')}{2\tau}, l_b(\mu,\mu')\}. \end{aligned}$$

If $M(\mu,\xi,\mu',\xi') = l_b(\mu,\mu')$ then we have

$$\zeta(\tau^3 l_b(\mu,\mu'),M(\mu,\xi,\mu',\xi')) = l_b(\mu,\mu') - \tau^3 l_b(\mu,\mu') \geq 0,$$

which is a contradiction.

Therefore

$$l_b(\mu,\mu') \leq \frac{l_b(\mu,\mu')+l_b(\xi,\xi')}{2\tau}. \tag{4.6}$$

Similarly, we can prove that

$$l_b(\xi,\xi') \leq \frac{l_b(\mu,\mu')+l_b(\xi,\xi')}{2\tau}.. \tag{4.7}$$

Adding the inequalities (4.6) and (4.7), we get that

$l_b(\mu,\mu')+l_b(\xi,\xi') \leq \frac{l_b(\mu,\mu')+l_b(\xi,\xi')}{2\tau} < l_b(\mu,\mu')+l_b(\xi,\xi')$,

it is a contradiction.

Therefore $(\mu,\xi) = (\mu',\xi')$ is the unique coupled fixed point of $\mathfrak{A}$ in S. □

Proposition 4.2. Let (S,l_b) be a b-metric space with coefficient $s \geq 1$ and $\mathfrak{A},\mathfrak{B} : S\times S \to S$ be two selfmaps. Assume that the pair $(\mathfrak{A},\mathfrak{B})$ is generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions. Then (u,v) is a coupled fixed point of $\mathfrak{A}$ if and only if (u,v) is a coupled fixed point of $\mathfrak{B}$. Moreover, (u,v) is unique in this case.

Proof. Let (u,v) be a coupled fixed point of $\mathfrak{A}$. Then $u=\mathfrak{A}(u,v)$ and $v=\mathfrak{A}(v,u)$.
Suppose that $u\neq\mathfrak{B}(u,v)$.
Since $\frac{1}{2\tau}\min\{l_b(u,\mathfrak{A}(u,v)),l_b(u,\mathfrak{B}(u,v))\}\leq\max\{l_b(u,u),l_b(v,v)\}$ from the inequality (3.2), we have

$$\zeta(\tau^3 l_b(u,\mathfrak{B}(u,v)),M(u,v,u,v))=\zeta(\tau^3 l_b(\mathfrak{A}(u,v),\mathfrak{B}(u,v)),M(u,v,u,v))\geq 0, \tag{4.8}$$

where

$$\begin{aligned}M(u,v,u,v)=\max\{&\tfrac{l_b(u,u)+l_b(v,v)}{2\tau},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(u,\mathfrak{B}(u,v))}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(u,\mathfrak{B}(u,v))}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\\&\tfrac{l_b(\mathfrak{A}(u,v),\mathfrak{B}(u,v))l_b(u,u)}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\tfrac{l_b(\mathfrak{A}(u,v),\mathfrak{B}(u,v))l_b(v,v)}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\tfrac{l_b(u,\mathfrak{B}(u,v))l_b(v,v)}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\\&\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(u,u)}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(v,v)}{1+l_b(u,u)+l_b(v,v)+l_b(u,\mathfrak{A}(u,u))},\\&\max\{l_b(u,\mathfrak{A}(u,v)),l_b(\mathfrak{A}(u,v),\mathfrak{B}(u,v))\}\}=l_b(u,\mathfrak{B}(u,v)).\end{aligned}$$

From the inequality (4.8), we have
$0\leq\zeta(\tau^3 l_b(u,\mathfrak{B}(u,v)),M(u,v,u,v))=l_b(u,\mathfrak{B}(u,v))-\tau^3 l_b(u,\mathfrak{B}(u,v))$,
which is a contradiction.
Therefore $u=\mathfrak{B}(u,v)$. Similarly, we can prove that $v=\mathfrak{B}(v,u)$.
Hence, (u,v) is a coupled fixed point of $\mathfrak{B}$.

In the similar lines as above, it is easy to see that (u,v) is a coupled fixed point of $\mathfrak{A}$ whenever (u,v) is a coupled fixed point of $\mathfrak{B}$.
Let $(u,v),(u',v')\in S\times S$ be two coupled fixed points of $\mathfrak{A}$ and $\mathfrak{B}$ with $(u.v)\neq(u',v')$.
Since $\frac{1}{2\tau}\min\{l_b(u,\mathfrak{A}(u,v)),l_b(u',\mathfrak{B}(u',v'))\}\leq\max\{l_b(u,u'),l_b(v,v')\}$ from the inequality (3.2), we have

$$\zeta(\tau^3 l_b(u,u'),M(u,v,u',v'))=\zeta(\tau^3 l_b(\mathfrak{A}(u,v),\mathfrak{B}(u',v')),M(u,v,u',v'))\geq 0,$$

where

$$\begin{aligned}M(u,v,u',v')=\max\{&\tfrac{l_b(u,u')+l_b(v,v')}{2\tau},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(u',\mathfrak{B}(u',v'))}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\tfrac{l_b(u',\mathfrak{A}(u,v))l_b(u,\mathfrak{B}(u',v'))}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\\&\tfrac{l_b(\mathfrak{A}(u,v),\mathfrak{B}(u',v'))l_b(u,u')}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\tfrac{l_b(\mathfrak{A}(u,v),\mathfrak{B}(u',v'))l_b(v,v')}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\\&\tfrac{l_b(u',\mathfrak{B}(u',v'))l_b(v,v')}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\tfrac{l_b(u',\mathfrak{A}(u,v))l_b(u,u')}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\\&\tfrac{l_b(u',\mathfrak{A}(u,v))l_b(v,v')}{1+l_b(u,u')+l_b(v,v')+l_b(u',\mathfrak{A}(u,v))},\max\{l_b(u',\mathfrak{A}(u,v)),l_b(\mathfrak{A}(u,v),\mathfrak{B}(u',v'))\}\}\end{aligned}$$

$\leq \max\{\frac{l_b(u,u')+l_b(v,v')}{2\tau}, l_b(u,u')\}.$

If $M(u,v,u',v') = l_b(u,u')$ then we have

$$\zeta(\tau^3 l_b(u,u'), M(u,v,u',v')) = l_b(u,u') - \tau^3 l_b(u,u') \geq 0,$$

which is a contradiction.

Therefore

$$l_b(u,u') \leq \frac{l_b(u,u')+l_b(v,v')}{2\tau}. \tag{4.9}$$

Similarly, we can prove that

$$l_b(v,v') \leq \frac{l_b(u,u')+l_b(v,v')}{2\tau}.. \tag{4.10}$$

Adding the inequalities (4.9) and (4.10), we get that

$l_b(u,u') + l_b(v,v') \leq \frac{l_b(u,u')+l_b(v,v')}{2\tau} < l_b(u,u') + l_b(v,v')$,

it is a contradiction.

Therefore $(u,v) = (u',v')$ is the unique coupled fixed point of $\mathfrak{A}$ and $\mathfrak{B}$ in S. □

Theorem 4.3. Let (S,l_b) be a complete b-metric space with coefficient $\tau \geq 1$ and the pair $(\mathfrak{A},\mathfrak{B})$ be a generalized Suzuki type $\mathscr{Z}$-contraction maps with rational expressions. Then $\mathfrak{A}$ and $\mathfrak{B}$ have a unique coupled fixed point in S.

Proof. Let μ_0 and ξ_0 be arbitrary points in S.

We define $\mu_{2i+1} = \mathfrak{A}(\mu_{2i},\xi_{2i}), \xi_{2i+1} = \mathfrak{A}(\xi_{2i},\mu_{2i})$ and

$\mu_{2i+2} = \mathfrak{B}(\mu_{2i+1},\xi_{2i+1}), \xi_{2i+2} = \mathfrak{B}(\xi_{2i+1},\mu_{2i+1})$ for $i = 0,1,2,\ldots$.

Since $\frac{1}{2\tau}\min\{l_b(\mu_{2n},\mathfrak{A}(\mu_{2n},\xi_{2n})), l_b(\mu_{2n+1},\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))\} \leq \max\{l_b(\mu_{2n},\mu_{2n+1}), l_b(\xi_{2n},\xi_{2n+1})\}$

from the inequality (3.2), we have

(4.11)

$$\zeta(\tau^3 l_b(\mu_{2n+1},\mu_{2n+2}), M(\mu_{2n},\xi_{2n},\mu_{2n+1},\xi_{2n+1})) = \zeta(\tau^3 l_b(\mathfrak{A}(\mu_{2n},\xi_{2n}),\mathfrak{B}(\mu_{2n+1},\xi_{2n+1})), M(\mu_{2n},\xi_{2n},\mu_{2n+1},\xi_{2n+1})) \geq 0,$$

where

$M(\mu_{2n},\xi_{2n},\mu_{2n+1},\xi_{2n+1}) = \max\{\frac{l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})}{2\tau},$

$\frac{l_b(\mu_{2n},\mathfrak{A}(\mu_{2n},\xi_{2n}))l_b(\mu_{2n+1},\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$

$$\frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))l_b(\mu_{2n},\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$$
$$\frac{l_b(\mathfrak{A}(\mu_{2n},\xi_{2n}),\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))l_b(\mu_{2n},\mu_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$$
$$\frac{l_b(\mathfrak{A}(\mu_{2n},\xi_{2n}),\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))l_b(\xi_{2n},\xi_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$$
$$\frac{l_b(\mu_{2n+1},\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))l_b(\xi_{2n},\xi_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$$
$$\frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))l_b(\mu_{2n},\mu_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$$
$$\frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))l_b(\xi_{2n},\xi_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n}))},$$
$$\max\{l_b(\mu_{2n+1},\mathfrak{A}(\mu_{2n},\xi_{2n})),l_b(\mathfrak{A}(\mu_{2n},\xi_{2n}),\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))\}\}$$

$$= \max\{\frac{l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})}{2\tau},$$
$$\frac{l_b(\mu_{2n},\mu_{2n+1})l_b(\mu_{2n+1},\mu_{2n+2})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\frac{l_b(\mu_{2n+1},\mu_{2n+1})l_b(\mu_{2n},\mu_{2n+2})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\frac{l_b(\mu_{2n+1},\mu_{2n+2})l_b(\mu_{2n},\mu_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\frac{l_b(\mu_{2n+1},\mu_{2n+2})l_b(\xi_{2n},\xi_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\frac{l_b(\mu_{2n+1},\mu_{2n+2})l_b(\xi_{2n},\xi_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\frac{l_b(\mu_{2n+1},\mu_{2n+1})l_b(\mu_{2n},\mu_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\frac{l_b(\mu_{2n+1},\mu_{2n+1})l_b(\xi_{2n},\xi_{2n+1})}{1+l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n+1},\mu_{2n+1})},$$
$$\max\{l_b(\mu_{2n+1},\mu_{2n+1}),l_b(\mu_{2n+1},\mu_{2n+2})\}\}$$
$$\leq \max\{\frac{l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})}{2\tau},l_b(\mu_{2n+1},\mu_{2n+2})\}.$$

If $M(\mu_{2n},\xi_{2n},\mu_{2n+1},\xi_{2n+1})=l_b(\mu_{2n+1},\mu_{2n+2})$ then from (4.11), we have

$$0\leq\zeta(\tau^3 l_b(\mu_{2n+1},\mu_{2n+2}),M(\mu_{2n},\xi_{2n},\mu_{2n+1},\xi_{2n+1}))=\zeta(\tau^3 l_b(\mu_{2n+1},\mu_{2n+2}),l_b(\mu_{2n+1},\mu_{2n+2}))$$
$$< l_b(\mu_{2n+1},\mu_{2n+2})-\tau^3 l_b(\mu_{2n+1},\mu_{2n+2}),$$

which is a contradiction.

Therefore

$$l_b(\mu_{2n+1},\mu_{2n+2})\leq\frac{l_b(\mu_{2n},\mu_{2n+1})+l_b(\xi_{2n},\xi_{2n+1})}{2\tau} \tag{4.12}$$

for all $n=0,1,2,\ldots$.

Similarly, we can prove that

$$l_b(\xi_{2n+1},\xi_{2n+2})\leq\frac{l_b(\xi_{2n},\xi_{2n+1})+l_b(\mu_{2n},\mu_{2n+1})}{2\tau} \tag{4.13}$$

for all $n = 0, 1, 2, \ldots$.

Adding the inequalities (4.12) and (4.13), we have

$l_b(\mu_{2n+1}, \mu_{2n+2}) + l_b(\xi_{2n+1}, \xi_{2n+2}) \leq h[l_b(\mu_{2n}, \mu_{2n+1}) + l_b(\xi_{2n}, \xi_{2n+1})]$, where $h = \frac{1}{2\tau} < 1$.

Also, it is easy to see that $l_b(\mu_{2n+2}, \mu_{2n+3}) + l_b(\xi_{2n+2}, \xi_{2n+3}) \leq h[l_b(\mu_{2n+1}, \mu_{2n+2}) + l_b(\xi_{2n+1}, \xi_{2n+2})]$.

Therefore $l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1}) \leq h[l_b(\mu_{n-1}, \mu_n) + l_b(\xi_{n-1}, \xi_n)]$ for all $n = 1, 2, 3, \ldots$.

Continuing in the same way, we get that

$l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1}) \leq h^n[l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)]$.

For $m > n, m, n \in \mathbb{N}$, we have

$$\begin{aligned} l_b(\mu_n, \mu_m) + l_b(\xi_n, \xi_m) &\leq \tau[l_b(\mu_n, \mu_{n+1}) + l_b(\mu_{n+1}), \mu_m)] + \tau[l_b(\xi_n, \xi_{n+1}) + l_b(\xi_{n+1}, \xi_m)] \\ &\leq \tau[l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1})] + \tau^2[l_b(\mu_{n+1}, \mu_{n+2}) + l_b(x_{n+2}, x_m)] \\ &\qquad + \tau^2[l_b(\xi_{n+1}, \xi_{n+2}) + l_b(\xi_{n+2}, \xi_m)] \\ &= \tau[l_b(\mu_n, \mu_{n+1}) + l_b(\xi_n, \xi_{n+1})] + \tau^2[l_b(\mu_{n+1}, \mu_{n+2}) + l_b(\xi_{n+1}, \xi_{n+2})] \\ &\quad + \tau^2[l_b(\mu_{n+2}, \mu_m) + l_b(\xi_{n+2}, \xi_m)] \ldots + \tau^m[l_b(\mu_{m-1}, \mu_m) + l_b(\xi_{m-1}, \xi_m)] \\ &\leq [\tau h^n + \tau^2 h^{n+1} + \ldots + \tau^{m-1} h^{m-1}][l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)] \\ &\leq \tau h^n[1 + \tau h + (\tau h)^2 \ldots + (\tau h)^{m-1} + \ldots][l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)] \\ &= \tau h^n(\tfrac{1}{1-\tau h})[l_b(\mu_0, \mu_1) + l_b(\xi_0, \xi_1)] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\{\mu_n\}$ and $\{\xi_n\}$ are b-Cauchy sequences in S.

Since S is b-complete, there exist $\mu, \xi \in S$ such that $\mu_n \to \mu$ and $\xi_n \to \xi$ as $n \to \infty$.

We now prove that $\mu = \mathfrak{A}(\mu, \xi)$ and $\xi = \mathfrak{A}(\xi, \mu)$.

On the contrary suppose that $\mu \neq \mathfrak{A}(\mu, \xi)$ and $\xi \neq \mathfrak{A}(\xi, \mu)$.

Since $\frac{1}{2\tau}\min\{l_b(\mu, \mathfrak{A}(\mu, \xi)), l_b(\mu_{2n+1}, \mathfrak{B}(\mu_{2n+1}, \xi_{2n+1}))\} \leq \max\{l_b(\mu, \mu_{2n+1}), l_b(\xi, \xi_{2n+1})\}$

from the inequality (3.2), we have

(4.14)
$$\begin{aligned} \zeta(\tau^3 l_b(\mathfrak{A}(\mu, \xi), \mu_{2n+2}), M(\mu, \xi, \mu_{2n+1}, \xi_{2n+1})) = \zeta(\tau^3 l_b(\mathfrak{A}(\mu, \xi), \mathfrak{B}(\mu_{2n+1}, \xi_{2n+1})), \\ M(\mu, \xi, \mu_{2n+1}, \xi_{2n+1})) \geq 0, \end{aligned}$$

where

$$\begin{aligned} M(\mu, \xi, \mu_{2n+1}, \xi_{2n+1}) = \max\{&\tfrac{l_b(\mu, \mu_{2n+1}) + l_b(\xi, \xi_{2n+1})}{2\tau}, \tfrac{l_b(\mu, \mathfrak{A}(\mu, \xi)) l_b(\mu_{2n+1}, \mathfrak{B}(\mu_{2n+1}, \xi_{2n+1}))}{1 + l_b(\mu, \mu_{2n+1}) + l_b(\xi, \xi_{2n+1}) + l_b(\mu_{2n+1}, \mathfrak{A}(\mu, \xi))}, \\ &\tfrac{l_b(\mu_{2n+1}, \mathfrak{A}(\mu, \xi)) l_b(\mu, \mathfrak{B}(\mu_{2n+1}, \xi_{2n+1}))}{1 + l_b(\mu, \mu_{2n+1}) + l_b(\xi, \xi_{2n+1}) + l_b(\mu_{2n+1}, \mathfrak{A}(\mu, \xi))}, \tfrac{l_b(\mathfrak{A}(\mu, \xi), \mathfrak{B}(\mu_{2n+1}, \xi_{2n+1})) l_b(\mu, \mu_{2n+1})}{1 + l_b(\mu, \mu_{2n+1}) + l_b(\xi, \xi_{2n+1}) + l_b(\mu_{2n+1}, \mathfrak{A}(\mu, \xi))}, \end{aligned}$$

$$\frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))l_b(\xi,\xi_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mu_{2n+1},\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))l_b(\xi,\xi_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))l_b(\mu,\mu_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))l_b(\xi,\xi_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))},$$
$$\max\{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi)), l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(\mu_{2n+1},\xi_{2n+1}))\}\}$$
$$= \max\{\frac{l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})}{2\tau}, \frac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(\mu_{2n+1},\mu_{2n+2})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))l_b(\mu,\mu_{2n+2})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mathfrak{A}(\mu,\xi),\mu_{2n+2})l_b(\mu,\mu_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(\mathfrak{A}(\mu,\xi),\mu_{2n+2})l_b(\xi,\xi_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mu_{2n+1},\mu_{2n+2})l_b(\xi,\xi_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))l_b(\mu,\mu_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))l_b(\xi,\xi_{2n+1})}{1+l_b(\mu,\mu_{2n+1})+l_b(\xi,\xi_{2n+1})+l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi))},$$
$$\max\{l_b(\mu_{2n+1},\mathfrak{A}(\mu,\xi)), l_b(\mathfrak{A}(\mu,\xi),\mu_{2n+2})\}\}.$$

On taking limit superior as $n \to \infty$ in $M(\mu,\xi,\mu_n,\xi_n)$ and using Lemma 2.7, we have

$\limsup_{n\to\infty} M(\mu,\xi,\mu_n,\xi_n) \leq \tau l_b(\mu,\mathfrak{A}(\mu,\xi))$.

On letting limit superior as $n \to \infty$ in (4.14) and using the Lemma 2.7, we have

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \zeta(\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mu_{2n+2}), M(\mu,\xi,\mu_{2n+1},\xi_{2n+1}))\\
&= \limsup_{n\to\infty} M(\mu,\xi,\mu_{2n+1},\xi_{2n+1}) - \liminf_{n\to\infty} \tau^3 l_b(\mathfrak{A}(\mu,\xi),\mu_{2n+2})\\
&\leq \tau l_b(\mu,\mathfrak{A}(\mu,\xi)) - \tau^3 \frac{l_b(\mu,\mathfrak{A}(\mu,\xi))}{\tau},
\end{aligned}$$

a contradiction.

Therefore $\mu = \mathfrak{A}(\mu,\xi)$.

Similarly we can prove that $\xi = \mathfrak{A}(\xi,\mu)$.

Therefore (μ,ξ) is a coupled fixed point of $\mathfrak{A}$.

By Proposition 4.2, we have (μ,ξ) is a unique common coupled fixed point of $\mathfrak{A}$ and $\mathfrak{B}$ in S. □

5. Corollaries and Examples

Corollary 5.1. Let (S,l_b) be a complete b-metric space with coefficient $\tau \geq 1$. $\mathfrak{A}: S\times S \to S$ be a selfmap. Assume that there exist two continuous functions $\varphi,\psi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t \leq \psi(t)$ for all $t > 0$ and $\varphi(t) = \psi(t) = 0$ if and onlt if $t = 0$ such that

$$\frac{1}{2\tau} l_b(\mu,\mathfrak{A}(\mu,\xi)) < l_b(\mu,\xi) \implies \psi(\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))) \leq \varphi(M(\mu,\xi,u,v))$$

where

$M(\mu,\xi,u,v) = \max\{\frac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \frac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$

$$\frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}, \text{ for all } \mu,\xi,u,v\in S.$$

Then $\mathfrak{A}$ has a unique common coupled fixed point in S.

Proof. Follows from Theorem 4.1 by choosing $\zeta(s,t)=\varphi(t)-\psi(s)$ for all $t,s\in[0,\infty)$. □

Corollary 5.2. Let (S,l_b) be a complete b-metric space with coefficient $\tau\geq 1$. $\mathfrak{A},\mathfrak{B}: S\times S\to S$ be two selfmaps. Assume that there exist two continuous functions $\varphi,\psi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t\leq\psi(t)$ for all $t>0$ and $\varphi(t)=\psi(t)=0$ if and onlt if $t=0$ such that

$$\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)),l_b(u,\mathfrak{B}(u,v))\}\leq\max\{l_b(\mu,u),l_b(\xi,v)\}$$
$$\implies \psi(\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)))\leq\varphi(M(\mu,\xi,u,v))$$

where

$$M(\mu,\xi,u,v)=\max\{\frac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \frac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\}, \text{ for all } \mu,\xi,u,v\in S.$$

Then $\mathfrak{A}$ and $\mathfrak{B}$ have a unique common coupled fixed point in S.

Proof. Follows by taking $\zeta(s,t)=\varphi(t)-\psi(s)$ in Theorem 4.3. □

The following is an example in support of Theorem 4.1.

Example 5.3. Let $S=[0,1]$ and let $l_b: S\times S\to\mathbb{R}^+$ defined by

$$l_b(\mu,\xi)=\begin{cases} 0 & \text{if } \mu=\xi \\ (\mu+\xi)^2 & \text{if } \mu\neq\xi. \end{cases}$$

Then clearly (S,l_b) is a b-metric space with coefficient $\tau=2$.

We define $\mathfrak{A}: S\times S\to S$ by

$$\mathfrak{A}(\mu,\xi)=\begin{cases} \frac{\mu^3+\xi^3}{256} & \text{if } \mu,\xi\in[0,\frac{1}{2}) \\ \frac{1}{32} & \text{if } \mu,\xi\in[\frac{1}{2},1] \\ 0 & \text{otherwise} \end{cases}$$

$\zeta:\mathbb{R}^+\times\mathbb{R}^+\to(-\infty,\infty)$ by $\zeta(t,s)=\frac{99}{100}s-t, t\geq 0, s\geq 0$.

Case (i). $\mu,\xi,u,v\in[0,\frac{1}{2})$.

Since $\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi))=\frac{1}{4}[\mu+\frac{\mu^3+\xi^3}{256}]^2<(\mu+\xi)^2=l_b(\mu,\xi)$.

From the inequality (3.1), we have

$$\begin{aligned}\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v)) &= 8[\tfrac{\mu^2+\xi^2}{16}+\tfrac{u^2+v^2}{16}]^2\\ &\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2]\\ &= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau},\tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}).\end{aligned}$$

Case (ii). $\mu,\xi,u,v\in[\frac{1}{2},1]$.

Since $\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi))=\frac{1}{4}[\mu+\frac{1}{32}]^2<(\mu+\xi)^2=l_b(\mu,\xi)$.

From the inequality (3.1), we have

$$\begin{aligned}\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v)) &= 8[\tfrac{1}{32}+\tfrac{1}{32}]^2\\ &\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2]\\ &= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau},\tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}).\end{aligned}$$

Case (iii). $\mu,\xi\in[\frac{1}{2},1],u,v\in[0,\frac{1}{2})$.

Since $\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi))=\frac{1}{4}[\mu+\frac{1}{32}]^2<(\mu+\xi)^2=l_b(\mu,\xi)$.

From the inequality (3.1), we have

$$\begin{aligned}\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v)) &= 8[\tfrac{1}{32}+\tfrac{u^3+v^3}{256}]^2\\ &\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2]\\ &= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau},\tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\ &\quad \max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}).\end{aligned}$$

Case (iv). $\mu,\xi\in[0,\frac{1}{2}),u,v\in[\frac{1}{2},1]$.

Since $\frac{1}{2\tau}l_b(\mu,\mathfrak{A}(\mu,\xi)) = \frac{1}{4}[\mu+\frac{\mu^3+\xi^3}{256}]^2 < (\mu+\xi)^2 = l_b(\mu,\xi)$.

From the inequality (3.1), we have

$$\begin{aligned}
\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v)) &= 8[\tfrac{\mu^3+\xi^3}{256}+\tfrac{1}{32}]^2 \\
&\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2] \\
&= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}) \\
&\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{A}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \max\{l_b(u,\mathfrak{A}(\mu,\xi)), l_b(\mathfrak{A}(\mu,\xi),\mathfrak{A}(u,v))\}\}).
\end{aligned}$$

Therefore $\mathfrak{A}$ satisfies all the hypotheses of Theorem 4.1 and $(0,0)$ is a unique coupled fixed point of $\mathfrak{A}$.

The following is an example in support of Theorem 4.3.

Example 4.4. Let $S=[0,1]$ and let $l_b : S\times S \to \mathbb{R}^+$ defined by

$$l_b(\mu,\xi) = \begin{cases} 0 & \text{if } \mu=\xi \\ (\mu+\xi)^2 & \text{if } \mu\neq\xi. \end{cases}$$

Then clearly (S,l_b) is a b-metric space with coefficient $\tau=2$.

We define $\mathfrak{A},\mathfrak{B} : S\times S\to S$ by

$$\mathfrak{A}(\mu,\xi) = \begin{cases} \frac{\log(1+\mu^2+\xi^2)}{16} & \text{if } \mu,\xi\in[0,\frac{1}{2}) \\ \frac{1}{32} & \text{if } \mu,\xi\in[\frac{1}{2},1] \end{cases} \text{ and } \mathfrak{B}(\mu,\xi) = \begin{cases} \frac{\mu^2\xi^2e^{\mu\xi}}{8} & \text{if } \mu,\xi\in[0,\frac{1}{2}) \\ \log(\mu+\xi) & \text{if } \mu,\xi\in[\frac{1}{2},1]. \end{cases}$$

$\zeta : \mathbb{R}^+\times\mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{99}{100}s - t, t\geq 0, s\geq 0$.

Case (i). $\mu,\xi,u,v\in[0,\frac{1}{2})$.

$$\begin{aligned}
\text{Since } \tfrac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)), l_b(u,\mathfrak{B}(u,v))\} &= \tfrac{1}{4}\min\{[\mu+\tfrac{\log(1+\mu^2+\xi^2)}{16}]^2, [u+\tfrac{u^2v^2e^{uv}}{8}]^2\} \\
&\leq \max\{(\mu+u)^2,(\xi+v)^2\} = \max\{l_b(\mu,u), l_b(\xi,v)\}.
\end{aligned}$$

From the inequality (3.2), we have

$$\begin{aligned}
\tau^3 l_b(f(\mu,\xi),\mathfrak{B}(u,v)) &= 8[\tfrac{\log(1+\mu^2+\xi^2)}{16}+\tfrac{u^2v^2e^{uv}}{8}]^2 \\
&\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2] \\
&= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}) \\
&\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \\
&\quad \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},
\end{aligned}$$

$\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\}).$

Case (ii). $\mu,\xi,u,v\in[\frac{1}{2},1]$.

Since $\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)),l_b(u,\mathfrak{B}(u,v))\}=\frac{1}{4}\min\{[\mu+\frac{1}{32}]^2,[u+\log(u+v)]^2\}$
$\leq\max\{(\mu+u)^2,(\xi+v)^2\}=\max\{l_b(\mu,u),l_b(\xi,v)\}.$

From the inequality (3.2), we have

$$\begin{aligned}
\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)) &= 8[\tfrac{1}{32}+\log(u+v)]^2\\
&\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2]\\
&= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau})\\
&\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau},\tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\
&\quad\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\
&\quad\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\
&\quad\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\})
\end{aligned}$$

Case (iii). $\mu,\xi\in[\frac{1}{2},1],u,v\in[0,\frac{1}{2})$.

Since $\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)),l_b(u,\mathfrak{B}(u,v))\}=\frac{1}{4}\min\{[\mu+\frac{1}{32}]^2,[u+\frac{u^2v^2e^{uv}}{8}]^2\}$
$\leq\max\{(\mu+u)^2,(\xi+v)^2\}=\max\{l_b(\mu,u),l_b(\xi,v)\}.$

From the inequality (3.2), we have

$$\begin{aligned}
\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)) &= 8[\tfrac{1}{16}+\tfrac{u^2v^2e^{uv}}{8}]^2\\
&\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2]\\
&= \tfrac{99}{100}(\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau})\\
&\leq \tfrac{99}{100}(\max\{\tfrac{l_b(\mu,u)+l_b(\xi,v)}{2\tau},\tfrac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\
&\quad\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\
&\quad\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\tfrac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},\\
&\quad\max\{l_b(u,\mathfrak{A}(\mu,\xi)),l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\})
\end{aligned}$$

Case (iv). $\mu,\xi\in[0,\frac{1}{2}),u,v\in[\frac{1}{2},1]$.

Since $\frac{1}{2\tau}\min\{l_b(\mu,\mathfrak{A}(\mu,\xi)),l_b(u,\mathfrak{B}(u,v))\}=\frac{1}{4}\min\{[\mu+\frac{\log(1+\mu^2+\xi^2)}{16}]^2,[u+\log(u+v)]^2\}$
$\leq\max\{(\mu+u)^2,(\xi+v)^2\}=\max\{l_b(\mu,u),l_b(\xi,v)\}.$

From the inequality (3.2), we have

$$\begin{aligned}
\tau^3 l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v)) &= 8[\tfrac{\log(1+\mu^2+\xi^2)}{8}+\log(\mu+\xi)]^2\\
&\leq \tfrac{99}{400}[(\mu+u)^2+(\xi+v)^2]
\end{aligned}$$

$$= \frac{99}{100}\left(\frac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}\right)$$
$$\leq \frac{99}{100}\Big(\max\Big\{\frac{l_b(\mu,u)+l_b(\xi,v)}{2\tau}, \frac{l_b(\mu,\mathfrak{A}(\mu,\xi))l_b(u,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,\mathfrak{B}(u,v))}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{B}(u,v))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\mu,u)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))}, \frac{l_b(u,\mathfrak{A}(\mu,\xi))l_b(\xi,v)}{1+l_b(\mu,u)+l_b(\xi,v)+l_b(u,\mathfrak{A}(\mu,\xi))},$$
$$\max\{l_b(u,\mathfrak{A}(\mu,\xi)), l_b(\mathfrak{A}(\mu,\xi),\mathfrak{B}(u,v))\}\Big\}\Big)$$

Therefore the pair $(\mathfrak{A},\mathfrak{B})$ satisfies all the hypotheses of Theorem 4.3 and $(0,0)$ is a unique common coupled fixed point of $\mathfrak{A}$ and $\mathfrak{B}$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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