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J. Math. Comput. Sci. 11 (2021), No. 6, 6980-6998

<https://doi.org/10.28919/jmcs/6453>

ISSN: 1927-5307

ON FI-COTORSION MODULES AND DIMENSIONS

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Abstract. In this paper we study the class of FI-cotorsion modules and we introduce there dimensions of modules and rings. An R -module M is called FI-cotorsion if $Ext_R^1(F, M) = 0$ for any FI-flat R -module F . Also, we investigate some properties of FI-cotorsion modules and FI-cotorsion envelopes and we give a characterization of IF-ring. Then, we study the FI-flat and the FI-cotorsion envelope and we show when they existed. Furthermore, we present the notation of FI-cotorsion dimension of modules and rings.

Keywords: FI-cotorsion module; FI-cotorsion dimension; cotorsion theory; semisimple ring; coherent ring.

2010 AMS Subject Classification: 13D07, 13D05.

1. INTRODUCTION

Throughout this paper all rings are commutative with identity element and all modules are unital. For an R -module M , we denote by $pd_R(M)$, $wd_R(M)$, $id_R(M)$, $FP - id_R(M)$ and $cd_R(M)$, the usual projective, weak, injective, FP-injective and cotorsion dimension of M , respectively. Also $E(M)$ is stand for the injective envelope of M and $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called the character module of M . Let R be a ring, we denote by $CD(R)$, $gldim(R)$ and $wdim(R)$ the cotorsion, global and weak dimension of R , respectively.

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Received July 02, 2021

We first recall some known notions and facts that we'll need later. Now, we recall the notion of (pre-)cover and (pre-)envelope of modules.

Definition 1. [4, 11]

Let \mathcal{M} be any class of modules.

- (1) for any R -module M , the homomorphism $\alpha : N \rightarrow M$ is called an \mathcal{M} -precover of M if $N \in \mathcal{M}$ and $\alpha^* = \text{Hom}(N', \alpha) : \text{Hom}(N', N) \rightarrow \text{Hom}(N', M)$ is surjective for every $N' \in \mathcal{M}$.
- (2) An \mathcal{M} -precover α is called an \mathcal{M} -cover of M if every endomorphism $\beta : N \rightarrow N$ such that $\alpha \circ \beta = \alpha$ implies β is an automorphism.
- (3) A homomorphism $\alpha : M \rightarrow N$ is called an \mathcal{M} -preenvelope of M if $N \in \mathcal{M}$ and $\alpha^* = \text{Hom}(\alpha, N') : \text{Hom}(N, N') \rightarrow \text{Hom}(M, N')$ is surjective for every $N' \in \mathcal{M}$.
- (4) An \mathcal{M} -preenvelope α is called an \mathcal{M} -envelope of M if every homomorphism $\beta : N \rightarrow N$ such that $\alpha = \beta \circ \alpha$ implies β is an automorphism.
- (5) An \mathcal{M} -preenvelope of M , $\alpha : M \rightarrow E$ is called special if α is injective and $\text{Coker}(\alpha) \in {}^\perp \mathcal{M}$, where ${}^\perp \mathcal{M}$ denote the left orthogonal class of \mathcal{M} .

Also, we need to recall the definition of cotorsion theory, complete and hereditary pair of class of modules.

Definition 2. [4]

Let \mathcal{M} and \mathcal{N} be two classes of modules.

- (1) A pair $(\mathcal{M}, \mathcal{N})$ is called a cotorsion theory if $\mathcal{M} = {}^\perp \mathcal{N}$ and $\mathcal{N} = \mathcal{M}^\perp$, where \mathcal{M}^\perp the right orthogonal class of \mathcal{M} .
- (2) \mathcal{M} is called a special preenvelope (resp., precover) class if every R -module has a special preenvelope (resp., precover).
- (3) \mathcal{M} is called a cover (resp., envelope) class if every R -module has a \mathcal{M} -cover (resp., \mathcal{M} -envelope).
- (4) A cotorsion theory $(\mathcal{M}, \mathcal{N})$ is called complete if every module has a special \mathcal{M} -precover and has a special \mathcal{N} -preenvelope.

(5) A cotorsion theory $(\mathcal{M}, \mathcal{N})$ is called perfect if every module has a \mathcal{M} -cover and a \mathcal{N} -envelope.

(6) We called a cotorsion theory $(\mathcal{M}, \mathcal{N})$ is an hereditary if for any short exact sequence $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$ with $M', M'' \in \mathcal{M}$, then $M \in \mathcal{M}$.

The concept of cotorsion modules is introduced by Enochs (1984) [3] defined by:

Definition 3. An R -module M is said to be a cotorsion module if $\text{Ext}_R^1(F, M) = 0$ for any flat module F .

After in (2005) Ding and Mao [9] introduced the cotorsion dimensions of modules and rings. On the other hand, Mao and Ding (2007) [8] introduced and studied the concept of FI-flat module defined by:

Definition 4. An R -module T is said to be FI-flat if $\text{Tor}_1^R(T, H) = 0$ for any FP-injective module H .

Later, Selvaraj et al. (2017) [2] give the definition of FI-cotorsion modules (Definition5). Motivated by [1, 9], we study the FI-cotorsion modules and we introduce the FI-cotorsion dimensions of modules and of rings. This paper is organized as follows:

In Section 2, we give a characterization of FI-cotorsion modules and present several properties. We show that this class behave in short exact sequence and stable under direct product. Also, we give sufficient conditions such that every cotorsion module is FI-cotorsion and such that every FI-cotorsion module is injective. Finally, we prove, for a module M , when the character module M^+ is FI-cotorsion.

In Section 3, We show that $(\mathcal{FF}, \mathcal{FC})$ is a cotorsion theory and also hereditary cotorsion theory, where \mathcal{FF} (resp., \mathcal{FC}) denote the class of FI-flat (resp., FI-cotorsion) modules. Also, we discuss when a module has a FI-flat cover and FI-cotorsion envelope and we prove that the FI-flat envelope of a module M is FI-cotorsion if and only if M is also FI-cotorsion.

In Section 4, we introduce the concept FI-cotorsion dimension of modules and rings. We give characterizations of these dimensions and we study the FI-cotorsion dimension of modules

under short exact sequences and direct product of modules. Finally, We prove that, the FI-cotorsion of a ring R is ≤ 1 if and only if every FI-flat R -module has a projective dimension at most 1.

2. FI-COTORSION MODULES

In this section we study the class of FI-cotorsion modules and we give a large number of properties.

Definition 5. A module M is called FI-cotorsion if $\text{Ext}_R^1(F, M) = 0$ for any FI-flat R -module F . The class of all FI-cotorsion module is denoted by \mathcal{FC} .

Remark 1. From the definition above, it's easy to see that we have the following inclusions between classes of modules:

$$\{\text{injective}\} \subseteq \{\text{FI-cotorsion}\} \subseteq \{\text{cotorsion}\}.$$

The following proposition gives a characterization of FI-cotorsion modules.

Proposition 1. For an R -module M , the following statements are equivalent:

- (1) M is FI-cotorsion.
- (2) $\text{Ext}_R^n(F, M) = 0$ for every $n \geq 1$ and for every FI-flat R -module F .
- (3) For every exact sequence of R -modules $0 \rightarrow A \rightarrow B \rightarrow F \rightarrow 0$ with F is FI-flat, the functor $\text{Hom}(\cdot, M)$ preserves the exactness.
- (4) Every exact sequence of R -modules $0 \rightarrow M \rightarrow B \rightarrow F \rightarrow 0$ with F is FI-flat splits.
- (5) For every exact sequence $0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$ where F is FI-flat, we have $F \rightarrow N$ is an FI-flat precover of N .
- (6) M is a kernel of an FI-flat precover $f : A \rightarrow B$ of a module B with A projective.

Proof. 1) \Rightarrow 2) We prove it by induction on n . If $n = 1$, then its true by definition. Suppose that it true for $n - 1$ and we prove it for n . Let F be FI-flat module and let the exact sequence of R -modules $0 \rightarrow K \rightarrow F_0 \rightarrow F \rightarrow 0$, where F_0 is free and it's easy to see that K is also FI-flat. Applying the long exact sequence of the functor $\text{Hom}(\cdot, M)$ we get the exact sequence:

$$0 = \text{Ext}_R^{n-1}(F_0, M) \rightarrow \text{Ext}_R^{n-1}(K, M) \rightarrow \text{Ext}_R^n(F, M) \rightarrow \text{Ext}_R^n(F_0, M) = 0,$$

where both ends vanish since F_0 is free. Now since K is FI-flat and by induction we have $Ext_R^n(F, M) \cong Ext_R^{n-1}(K, M) = 0$.

2) \Leftrightarrow 3) Follows from the long exact sequence $0 \rightarrow Hom(F, M) \rightarrow Hom(B, M) \rightarrow Hom(A, M) \rightarrow 0$, since $Ext(F, M) = 0$.

3) \Leftrightarrow 1) Let F be a FI-flat R -module. Consider the short exact sequence of R -modules $0 \rightarrow K \rightarrow F_0 \rightarrow F \rightarrow 0$ where M is R -module, $K = ker(F_0 \rightarrow F)$ and F_0 is free. Applying the long exact sequence of the exact functor $Hom_R(., M)$, we get:

$$0 \rightarrow Hom(F, M) \rightarrow Hom(F_0, M) \rightarrow Hom(K, M) \rightarrow 0$$

and so $Ext(A, M) = 0$.

1) \Leftrightarrow 4) Follows from [10, Theorem 7.31].

1) \Rightarrow 5) Suppose that M is a FI-cotorsion R -module and consider the short exact of R -modules $0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$ where F is FI-flat. Applying the long exact sequence of the functor $Hom_R(F', .)$ with F' is FI-flat, we get: $Hom_R(F', F) \rightarrow Hom_R(F', N) \rightarrow Ext_R^1(F', M) = 0$. Therefore $F \rightarrow N$ is an FI-flat precover of N since the class of FI-flat is closed under direct summands and isomorphic images.

5) \Rightarrow 6) From [11, Example 1.6], there exists an exact sequence $0 \rightarrow M \rightarrow P \rightarrow P/M \rightarrow 0$ where P is projective preenvelope of M . By hypothesis $P \rightarrow P/M$ is an FI-flat precover of P/M since P is projective and so FI-flat.

6) \Rightarrow 1) Suppose that K is the kernel of a FI-flat cover $f : F(M) \rightarrow M$ of M . As the class of FI-flat modules contains the class of projective modules, f is surjective by [11, Lemma 1.9]. So we have the exact sequence $0 \rightarrow K \rightarrow F(M) \rightarrow M \rightarrow 0$. Let F be a FI-flat module, applying the long exact sequence of the functor $Hom_R(F, .)$, we get the exact sequence:

$$Hom_R(F, A) \rightarrow Hom_R(F, B) \rightarrow Ext_R^1(F, M) \rightarrow \dots$$

By hypothesis $Hom_R(F, A) \rightarrow Hom_R(F, B) \rightarrow 0$ is exact, then $Ext_R^1(F, M) = 0$ and hence M is FI-cotorsion. \square

Proposition 2. *If the class of FI-cotorsion modules is closed under direct sums, then the following statement are equivalent:*

- (1) M is FI-cotorsion,
- (2) $P \otimes M$ is FI-cotorsion for every projective R -module P .

Proof. 1) \Rightarrow 2) Let F be a FI-flat R -module and P a projective R -module. As P projective, there exists a projective module P' such that $R^I \cong P \oplus P'$ for some index set I . Now we have $Ext_R^1(F, M) = 0$ so $Ext_R^1(F, R \otimes M) = 0$. Hence

$$\begin{aligned} \bigoplus_I Ext_R^1(F, R \otimes M) &\cong Ext_R^1(F, R^I \otimes M) \\ &\cong Ext_R^1(F, (P \oplus P') \otimes M) \\ &\cong Ext_R^1(F, (P \otimes M) \oplus (P' \otimes M)) \\ &\cong Ext_R^1(F, P \otimes M) \oplus Ext_R^1(F, P' \otimes M) = 0. \end{aligned}$$

That is $Ext_R^1(F, P \otimes M) = 0$ and $P \otimes M$ is FI-cotorsion.

2) \Rightarrow 1) Let F be a FI-flat R -module so $P \otimes M$ is FI-cotorsion for every projective R -module P . Then $R \otimes M \cong M$ is FI-cotorsion. □

Next proposition gives a characterization of FI-cotorsion modules over a coherent ring.

Proposition 3. *For an R -module M over a coherent ring, the following are equivalent:*

- 1) M is FI-cotorsion,
- 2) $Hom_R(F, M)$ is FI-cotorsion for every flat R -module F ,
- 3) $Hom_R(P, M)$ is FI-cotorsion for every projective R -module P .

Proof. 1) \Rightarrow 2) For any FI-flat R -module N there exists an exact sequence of R -modules $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$, where P is projective. Applying the long exact sequence of the functor $\cdot \otimes_R F$ where F is a flat R -module, we get the exact sequence: $0 \rightarrow K \otimes_R F \rightarrow P \otimes_R F \rightarrow N \otimes_R F \rightarrow 0$.

Now, applying the long exact sequence of the functor $(Hom_R(\cdot, M))$, we get :

$$Hom_R(P \otimes_R F, M) \rightarrow Hom_R(K \otimes_R F, M) \rightarrow Ext_R^1(N \otimes_R F, M).$$

By [7, Proposition 2.11] $N \otimes F$ is FI-flat since N is FI-flat and R is coherent, so $Ext_R^1(N \otimes_R F, M) = 0$. On the other hand, applying the long exact sequence of the functor $Hom_R(\cdot, Hom_R(F, M))$ we get: $Hom_R(P, Hom_R(F, M)) \rightarrow Hom_R(K, Hom_R(F, M)) \rightarrow Ext_R^1(N, Hom_R(F, M)) \rightarrow Ext_R^1(P, Hom_R(F, M)) = 0$, where the right Ext vanish since P is

projective. From [10, Theorem 2.75], $\text{Hom}_R(P \otimes_R F, M) \cong \text{Hom}_R(P, \text{Hom}_R(F, M))$ and $\text{Hom}_R(K, \text{Hom}_R(F, D)) \cong \text{Hom}_R(K, \text{Hom}_R(F, M))$, then $\text{Ext}_R^1(N, \text{Hom}_R(F, M)) = 0$ and so $\text{Hom}_R(F, M)$ is FI-cotorsion.

2) \Rightarrow 3) Obvious.

3) \Rightarrow 1) Suppose that $\text{Hom}_R(P, M)$ is FI-cotorsion for every projective R -module P , then for $P = R$, we have $\text{Hom}_R(R, M) \cong M$. So M is FI-cotorsion. \square

In the following proposition, we show that FI-cotorsion modules are stable under direct product and summand.

Proposition 4. *For any family of modules $(M_j)_{j \in J}$ where J is an index set, we have $\prod_{j \in J} M_j$ is FI-cotorsion if and only if every M_j is FI-cotorsion for any $j \in J$.*

Proof. Follows from [10, Theorem 7.22], since $\text{Ext}_R^1(F, \prod_{j \in J} M_j) \cong \prod_{j \in J} \text{Ext}_R^1(F, M_j)$. \square

In the following proposition, we show how FI-cotorsion behave in a short exact sequence.

Proposition 5. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of R -modules with A is FI-cotorsion, then B is FI-cotorsion if and only if C is FI-cotorsion.*

Proof. Applying the long exact sequence of the functor $\text{Hom}_R(F, \cdot)$, where F is FI-flat, we get: $0 = \text{Ext}_R^1(F, A) \rightarrow \text{Ext}_R^1(F, B) \rightarrow \text{Ext}_R^1(F, C) \rightarrow \text{Ext}_R^2(F, A) = 0$, so $\text{Ext}_R^1(F, B) \cong \text{Ext}_R^1(F, C)$ and the result holds. \square

In the following theorem we see when the character module M^+ of a module M is FI-cotorsion.

Theorem 2.1. *Let M be R -module with $\text{FP-id}(M) < \infty$, then M^+ is FI-cotorsion.*

Proof. Suppose that $\text{FP-id}(M) = m$ for some non-negative integer m and consider the FP-injective resolution of M , $0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_m \rightarrow 0$. Then we obtain the following short exact sequences:

$$0 \rightarrow M \rightarrow E_0 \rightarrow C_1 \rightarrow 0,$$

$$0 \rightarrow C_1 \rightarrow E_1 \rightarrow C_2 \rightarrow 0,$$

⋮

$$0 \rightarrow C_{m-2} \rightarrow E_{m-2} \rightarrow C_{m-1} \rightarrow 0,$$

$$0 \rightarrow C_{m-1} \rightarrow E_{m-1} \rightarrow E_m \rightarrow 0.$$

Applying the long exact sequence of the functor $(\cdot \otimes_R F)$ where F is a FI-flat module, we get:

$$0 = \text{Tor}_{n+1}^R(E_0, F) \rightarrow \text{Tor}_{n+1}^R(C_1, F) \rightarrow \text{Tor}_n^R(M, F) \rightarrow \text{Tor}_n^R(E_0, F) = 0,$$

$$0 = \text{Tor}_{n+2}^R(E_1, F) \rightarrow \text{Tor}_{n+2}^R(C_2, F) \rightarrow \text{Tor}_{n+1}^R(C_1, F) \rightarrow \text{Tor}_{n+1}^R(E_1, F) = 0,$$

⋮

$$0 = \text{Tor}_{n+m-1}^R(E_{m-1}, F) \rightarrow \text{Tor}_{n+m-1}^R(C_{m-1}, F) \rightarrow$$

$$\text{Tor}_{n+m-2}^R(C_{m-2}, F) \rightarrow \text{Tor}_{n+m-2}^R(E_{m-2}, F) = 0,$$

$$0 = \text{Tor}_{n+m}^R(E_m, F) \rightarrow \text{Tor}_{n+m-1}^R(C_{m-1}, F) \rightarrow \text{Tor}_{n+m-1}^R(E_{m-1}, F) = 0$$

where every ends vanish since every E_i is FP-injective for any $0 \leq i \leq m$ and F is a FI-flat. From the last sequence we have $\text{Tor}_{n+m-1}^R(C_{m-1}, F) = 0$. Then $\text{Tor}_n^R(M, F) \cong \text{Tor}_{n+1}^R(C_1, F) \cong \text{Tor}_{n+2}^R(C_2, F) \cong \dots \cong \text{Tor}_{n+m-1}^R(C_{m-1}, F) = 0$. There is the following standard isomorphism: $\text{Ext}^1(F, M^+) \cong (\text{Tor}_1(F, M))^+$ and hence $\text{Ext}^1(F, M^+) = 0$. Thus M^+ is FI-cotorsion. \square

Recall that R is called a IF ring if every injective R -module is flat. Next proposition gives a characterization of IF-ring using FI-cotorsion modules.

Proposition 6. *A ring R is an IF-ring if and only if every FI-cotorsion R -module is injective.*

Proof. The necessity is follows from [8, Proposition 2.9] since every module is FI-flat over any IF-rings. For the sufficiency, let E be injective module, then by Theorem 2.1 E^+ is FI-cotorsion and so injective by hypothesis. hence E is flat and R is IF-ring. \square

Example 1. *Let R be ring which is not an IF-ring (\mathbb{Z} For example). Then from Proposition 6 there is an FI-cotorsion R -module which is not injective.*

In Remark 1 we see that every FI-cotorsion module is cotorsion, in the following propositions we discus when the converse holds.

Proposition 7. *Let R be a Prüfer domain, then every cotorsion R -module is FI-cotorsion.*

Proof. Follows from [8, Proposition 2.3] since a FI-flat R -module with weak dimension ≤ 1 is flat over a coherent ring. □

Proposition 8. *Let R be a Noetherian ring, then every cotorsion R -module of finite weak dimension is FI-cotorsion.*

Proof. We prove that every cotorsion flat R -module is FI-cotorsion and we deduce by induction the case of finite weak dimension. From [4, Lemma 5.3.27] and for any flat cotorsion R -module F there exists injective modules E and I such that $\text{Hom}_R(E, I) = F \oplus M$ for some module M . Let T be a FI-flat R -module, we have

$$\text{Ext}_R^1(T, F \oplus M) = \text{Ext}_R^1(T, \text{Hom}_R(E, I)) \cong \text{Hom}_R(\text{Tor}_1^R(T, E), I) = 0$$

since I and E are injective. Then $\text{Ext}_R^1(T, F \oplus M) = 0$ and so $\text{Ext}_R^1(T, F) = 0$, thus F is FI-cotorsion. □

On the next theorem we see the ring over which every R -modules is FI-cotorsion.

Theorem 2.2. *Let R be a ring, then the following conditions are equivalent.*

- (1) *Every R -module is FI-cotorsion;*
- (2) *Every FI-flat R -module is projective.*

Moreover, if R satisfy one of the previous condition, then R is a perfect ring.

Proof. 1) \Rightarrow 2) Let F be a FI-flat R -module, so by hypothesis for any R -module M we have $\text{Ext}^1(F, M) = 0$. Then F is projective.

2) \Rightarrow 1) Obvious.

Finally, for any R -module M we have M is FI-cotorsion and hence cotorsion. [12, Proposition 3.3.1] complete the proof. □

Recall that every semisimple ring is von Neumann regular and the converse is not true in general. Next corollary shows the converse is hold when every R -module is FI-cotorsion.

Corollary 1. *Let R be a ring, the following are equivalent:*

- (1) R is a semisimple ring;
- (2) R is a von Neumann regular ring every FI-flat module is projective;
- (3) R is a von Neumann regular ring and every R -module is FI-cotorsion.

Proof. 1) \Rightarrow 2) It is obvious by [10, Proposition 4.13].

2) \Rightarrow 1) As R is a von Neumann regular ring, every module is flat and by Theorem 2.2, R is perfect and every flat module is projective. Then R is semisimple.

2) \Leftrightarrow 3) by proof Theorem 2.2. □

3. FI-FLAT COVER AND FI-COTORSION ENVELOPE

We recall the following notations, \mathcal{FC} is class of FI-cotorsion modules, \mathcal{FF} is the class of FI-flat modules and ${}^{\perp}\mathcal{FC} = \{MR\text{-module} / \text{Ext}_R(M, F) = 0, \text{ for any FI-cotorsion module } F\}$. The main aim of this section is to show that $(\mathcal{FF}, \mathcal{FC})$ is a hereditary cotorsion theory over any ring and perfect over an FI-ring. Also we study the relation between FI-cotorsion envelope and the class of FI-flat modules and the relation between FI-flat cover and the class of FI-cotorsion modules.

Next lemma we show that $(\mathcal{FF}, \mathcal{FC})$ is a cotorsion theory.

Lemma 1. ${}^{\perp}\mathcal{FC} = \mathcal{FF}$, moreover, $(\mathcal{FF}, \mathcal{FC})$ is a cotorsion theory.

Proof. It's obvious that $\mathcal{FF} \subseteq {}^{\perp}\mathcal{FC}$. For the converse, let $M \in {}^{\perp}\mathcal{FC}$, then $\text{Ext}_R^1(M, N) = 0$ for every FI-cotorsion module N . Let G be an FP-injective module, by Theorem 2.1 G^+ is FI-cotorsion, so $\text{Ext}_R^1(M, G^+) = 0$ by hypothesis. From [6, p. 34] we have the isomorphism $\text{Ext}_R^1(M, G^+) \cong (\text{Tor}_1^R(M, G))^+$. Then $(\text{Tor}_1^R(M, G))^+ = 0$, and by [5] we have $\text{Tor}_1^R(M, G) = 0$. Hence, M is FI-flat, so ${}^{\perp}\mathcal{FC} \subseteq \mathcal{FF}$. Therefore, ${}^{\perp}\mathcal{FC} = \mathcal{FF}$. Now by definition, $\mathcal{FC} = \mathcal{FF}^{\perp}$ and so $(\mathcal{FF}, \mathcal{FC})$ is a cotorsion theory. □

The following proposition shows that $(\mathcal{FF}, \mathcal{FC})$ is an hereditary cotorsion theory.

Proposition 9. $(\mathcal{FF}, \mathcal{FC})$ is an hereditary cotorsion theory.

Proof. Consider an exact sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ with F and F'' are FI-flat. Applying the long exact sequence of functor $(N \otimes_R \cdot)$ where N is an FP-injective R -module, we obtain the exact sequence:

$$0 = \text{Tor}_2^R(N, F'') \longrightarrow \text{Tor}_1^R(N, F') \longrightarrow \text{Tor}_1^R(N, F) = 0.$$

Hence $\text{Tor}_1^R(N, F') = 0$ and F' is FI-flat, therefor, $(\mathcal{F}\mathcal{F}, \mathcal{F}\mathcal{C})$ is an hereditary cotorsion theory as claimed. \square

The next proposition is ensured the existence of the FI-flat cover and FI-cotorsion envelope of R -modules over an IF-ring.

Proposition 10. *Over an IF-ring R , every R -module has a $\mathcal{F}\mathcal{F}$ -cover and a $\mathcal{F}\mathcal{C}$ -envelope.*

Proof. As R is an IF-ring, every FI-cotorsion module is injective by Proposition 7. Hence every FI-cotorsion module is pure-injective. Now, by Proposition 9, ${}^\perp \mathcal{F}\mathcal{C} = \mathcal{F}\mathcal{F}$ and $({}^\perp \mathcal{F}\mathcal{C})^\perp = \mathcal{F}\mathcal{C}$. So, $(\mathcal{F}\mathcal{F}, \mathcal{F}\mathcal{C})$ is a cotorsion theory generated by $\mathcal{F}\mathcal{C} \subseteq \mathcal{P}\mathcal{I}$ where $\mathcal{P}\mathcal{I}$ denoted the class of pure-injective modules. Hence, $(\mathcal{F}\mathcal{F}, \mathcal{F}\mathcal{C})$ is complete, and hence perfect by [11, Theorem 2.8]. So $\mathcal{F}\mathcal{F}$ is a cover class and $\mathcal{F}\mathcal{C}$ is an envelope class over an IF-ring. \square

In the next corollary, we see that the cokernel of $\mathcal{F}\mathcal{C}$ -envelope is FI-flat and the kernel of the $\mathcal{F}\mathcal{F}$ -cover of R -module is FI-cotorsion.

Corollary 2. *Let R be an IF-ring, M and N are R -modules. Then there exist exact sequences*

$$0 \rightarrow M \rightarrow C(M) \rightarrow C(M)/M \rightarrow 0,$$

$$0 \rightarrow N' \rightarrow F(N) \rightarrow N \rightarrow 0$$

where $C(M)$ is the $\mathcal{F}\mathcal{C}$ -envelope of M , $F(N)$ is the $\mathcal{F}\mathcal{F}$ -cover of N , $C(M)/M$ is FI-flat and N' is FI-cotorsion.

Proof. The result follows from Proposition 10, since $(\mathcal{F}\mathcal{F}, \mathcal{F}\mathcal{C})$ is complete. \square

The following theorem answer the question: what is the FI-cotorsion envelope of FI-flat R -module M .

Theorem 3.1. *Let R be an IF-ring and M be an R -module, and consider $C(M)$ its FI-cotorsion envelope. Then M is FI-flat if and only if $C(M)$ is FI-flat.*

Proof. Let $\alpha : M \rightarrow C(M)$ be the FI-cotorsion envelope of M , then α is injective since the class of FI-cotorsion modules contains the class of injective modules by [11, Lemma 1.9]. So there exists an exact sequence $0 \rightarrow M \rightarrow C(M) \rightarrow C(M)/M \rightarrow 0$, where $C(M)/M$ is FI-flat by [12, Lemma 2.1.2]. Applying the long exact sequence of functor $(N \otimes_R \cdot)$ where N is FP-injective R -module, we obtain the exact sequence: $0 = \text{Tor}_2^R(N, C(M)/M) \rightarrow \text{Tor}_1^R(N, M) \rightarrow \text{Tor}_1^R(N, C(M)) \rightarrow \text{Tor}_1^R(N, C(M)/M) = 0$. Hence $\text{Tor}_1^R(N, M) = 0$ and M is FI-flat. This completes the proof. \square

Next theorem answer: what is the FI-flat cover of FI-cotorsion R -module M .

Theorem 3.2. *Let M be a R -module and $F(M)$ its FI-flat cover. Then M is FI-cotorsion if and only if $F(M)$ is FI-cotorsion.*

Proof. There exists an exact sequence $0 \rightarrow K \rightarrow F(M) \rightarrow M \rightarrow 0$. As the class of FI-flat modules is closed under extension and so K is FI-cotorsion by [12, Lemma 2.1.1], and Proposition 5 establish the result. \square

4. FI-COTORSION DIMENSIONS OF MODULES AND RINGS

4.1. FI-cotorsion dimension of modules. In this section we will investigate the FI-cotorsion dimension of modules and we study its properties and we give its characterization.

Definition 6. *Let R be a ring. The FI-cotorsion dimension of a module M , denoted $FI\text{-}cd_R(M)$, and defined to be the smallest positive integer n such that $\text{Ext}_R^{n+1}(N, M) = 0$ for any FI-flat module N .*

From the previous definition we obtain the following remarks.

Remark 2.

- (1) $FI\text{-}cd_R(M) = 0$ if and only if M is FI-cotorsion.
- (2) $cd_R(M) \leq FI\text{-}cd_R(M) \leq id_R(M)$, where $cd_R(M)$ is the cotorsion dimension of M .

The following proposition gives a characterization of FI-cotorsion dimension.

Proposition 11. *Let R be a ring and let M be a module and $n \geq 0$. Then the following conditions are equivalent:*

- (1) $FI\text{-}cd_R(M) \leq n$.
- (2) $Ext_R^{n+1}(N, M) = 0$ for any FI-flat R -module N .
- (3) $Ext_R^{n+i}(N, M) = 0$ for any FI-flat R -module N and any $i \geq 1$.
- (4) For any exact sequence $0 \rightarrow M \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow 0$, if M_i are FI-cotorsion for every $i \in \{0, 1, \dots, n-1\}$, then M_n is also FI-cotorsion.
- (5) $FI\text{-}cd_R(F(M)) \leq n$ where $F(M)$ is the FI-flat cover of M .

Proof. 1) \Leftrightarrow 2) Is by definition.

2) \Rightarrow 3) By induction on i .

3) \Rightarrow 4) If $Ext_R^{n+i}(N, M) = 0$ for any N FI-flat R -module. We consider an injective resolution of M with $(n-1)^{st}$ cosyzygy L^{n-1} . From [10, Theorem 9.7], we get: $0 = Ext_R^{n+1}(N, M) \cong Ext_R^1(N, L^{n-1})$. Then L^{n-1} is an FI-flat.

4) \Rightarrow 3) It follows that $Ext_R^{n+i}(N, M) \cong Ext_R^i(N, L^{n-1})$.

1) \Leftrightarrow 5) Suppose that K is the kernel of a FI-flat cover $f : F(M) \rightarrow M$ of M . As the class of FI-flat modules contains the class of projective modules, f is surjective by [11, Lemma 1.9]. So we have the exact sequence $0 \rightarrow K \rightarrow F(M) \rightarrow M \rightarrow 0$ where K is FI-cotorsion by [12, Lemma 2.1.1]. Let F be a FI-flat module, applying the long exact sequence of the functor $Hom_R(F, \cdot)$, we get the exact sequence:

$$0 = Ext_R^{n+1}(F, K) \rightarrow Ext_R^{n+1}(F, F(M)) \rightarrow Ext_R^{n+1}(F, M) \rightarrow Ext_R^{1+2}(F, k) = 0$$

where both ends vanish since K is FI-cotorsion. So $Ext_R^{n+1}(F, F(M)) \cong Ext_R^{n+1}(F, M)$, establish the result. \square

Theorem 4.1. *The following conditions are identical for an R -module M :*

- (1) $FI\text{-}cd_R(M)$.
- (2) $\inf\{m \mid \text{there exists an exact sequence } 0 \rightarrow M \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{m-1} \rightarrow M_m \rightarrow 0 \text{ such that every } M_i \in \mathcal{FC} \text{ for } 0 \leq i \leq m\}$.

(3) The integer k such that M admits a minimal FI-cotorsion resolution. Equivalently, there exists an exact sequence $0 \xrightarrow{\alpha_{-1}} M (= M_{-1}) \xrightarrow{\alpha_0} M_0 \xrightarrow{\alpha_1} M_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{k-1}} M_{k-1} \xrightarrow{\alpha_k} M_k \rightarrow 0$ such that every M_i is a FI-cotorsion envelope of $N_i = \text{coker}(\alpha_{i-1})$, $M_i \neq 0$, for every $0 \leq i \leq k$, $M_{-1} = M$, $M_{-2} = 0$.

Proof. (1) = (2) Let (1) = h , $l = (2)$, suppose that $h \neq l$ and suppose with out loss of generality that $h > l$. From the exact sequence $0 \rightarrow M \rightarrow M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_{l-1} \rightarrow M_l \rightarrow 0$ such that every $M_i \in \mathcal{FC}$ for $0 \leq i \leq l$, we get the following short exact sequences:

$$\begin{aligned} 0 \longrightarrow M \longrightarrow M_0 \longrightarrow C_1 \longrightarrow 0, \\ \\ 0 \longrightarrow C_1 \longrightarrow M_1 \longrightarrow C_2 \longrightarrow 0, \\ \\ \dots \\ 0 \longrightarrow C_{l-2} \longrightarrow M_{l-2} \longrightarrow C_{l-1} \longrightarrow 0, \\ \\ 0 \longrightarrow C_{l-1} \longrightarrow M_{l-1} \longrightarrow M_l \longrightarrow 0 \end{aligned}$$

Applying the long exact sequence of the functor $\text{Hom}_R(F, \cdot)$, we get:

$$\begin{aligned} 0 = \text{Ext}_R^l(F, M_0) \rightarrow \text{Ext}_R^l(F, C_1) \rightarrow \text{Ext}_R^{l+1}(F, M) \rightarrow \text{Ext}_R^{l+1}(F, M_0) = 0, \\ \\ 0 = \text{Ext}_R^{l-1}(F, M_1) \rightarrow \text{Ext}_R^{l-1}(F, C_2) \rightarrow \text{Ext}_R^l(F, C_1) \rightarrow \text{Ext}_R^l(F, M_1) = 0, \\ \\ \dots \\ 0 = \text{Ext}_R^2(F, M_{l-2}) \rightarrow \text{Ext}_R^2(F, C_{l-1}) \rightarrow \text{Ext}_R^3(F, C_{l-2}) \rightarrow \text{Ext}_R^3(F, M_{l-2}) = 0, \\ \\ 0 = \text{Ext}_R^1(F, M_l) \rightarrow \text{Ext}_R^2(F, C_{l-1}) \rightarrow \text{Ext}_R^2(F, M_{l-1}) = 0 \end{aligned}$$

where every ends vanish since every M_i is FI-cotorsion for every $0 \leq i \leq l$ and F is a FI-flat. From the last sequence we have $\text{Ext}_R^2(F, C_{l-1}) = 0$. That is $0 = \text{Ext}_R^2(F, C_{l-1}) \cong \text{Ext}_R^3(F, C_{l-2}) \cong \dots \cong \text{Ext}_R^{l-1}(F, C_2) \cong \text{Ext}_R^l(F, C_1) \cong \text{Ext}_R^l(F, C_1) \cong \text{Ext}_R^{l+1}(F, M)$. That is $\text{Ext}_R^{l+1}(F, M) = 0$, and this contradicts the fact h is smallest integer such that $\text{Ext}_R^{h+1}(F, M) = 0$, hence $h = l$.

(1) \leq (3) Obvious.

(1) \geq (3) Let $FI\text{-}cd_R(M) = h < \infty$ and suppose by contradiction that $h < k$. There exists an exact sequence $0 \rightarrow N_h \rightarrow M_h \rightarrow N_{h+1} \cong M_h/N_h \rightarrow 0$ where M_h is a FI-cotorsion envelope of N_h and N_{h+1} is FI-flat by Proposition 9 and [12, Lemma 2.1.2] since the class of FI-cotorsion is closed under extensions. By Proposition 11 (5), N_h is FI-cotorsion using the exact sequence $0 \rightarrow M \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{h-1} \rightarrow N_h \rightarrow 0$. That is $Ext_R^1(N_{h+1}, N_h) = 0$ and the short exact sequence split. Hence $M_h = N_h \oplus M_h/N_h$. As M_h is a FI-cotorsion envelope of N_h , $M \cap M_h/N_h \neq \emptyset$, and this contradicts the definition of direct sum. Whence $M_h/N_h = 0$ and so $N_{h+1} = 0$, it follows that $M_{h+1} = 0$, contradiction. Therefore, (1) \geq (3) and so (1) = (3). \square

Proposition 12. *Let R be a ring and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ an exact sequence of R -modules. If two of $FI\text{-}cd_R(A)$, $FI\text{-}cd_R(B)$ and $FI\text{-}cd_R(C)$ are finite, so is the third. Moreover,*

$$(1) \quad FI\text{-}cd_R(A) \leq \sup\{FI\text{-}cd_R(B), FI\text{-}cd_R(C) + 1\}.$$

$$(2) \quad FI\text{-}cd_R(B) \leq \sup\{FI\text{-}cd_R(A), FI\text{-}cd_R(C)\}.$$

$$(3) \quad FI\text{-}cd_R(C) \leq \sup\{FI\text{-}cd_R(B), FI\text{-}cd_R(A) - 1\}.$$

Proof. We prove only assertion 1 and the other assertion are proved by the same method. Suppose that $FI\text{-}cd_R(B) = n$ and $FI\text{-}cd_R(C) = m$. We can suppose that $n \geq m + 1$. From Proposition 1 we have $Ext_R^{n+1}(F, B) = Ext_R^n(F, C) = 0$ for any FI-flat R -module F . Using the long exact sequence of the functor $(Hom_R(F, \cdot))$ we obtain: $0 = Ext_R^n(F, C) \rightarrow Ext_R^{n+1}(F, A) \rightarrow Ext_R^{n+1}(F, B) = 0$. Then $Ext_R^{n+1}(F, A) = 0$, so $FI\text{-}cd_R(A) \leq n = \sup\{FI\text{-}cd_R(B), FI\text{-}cd_R(C) + 1\}$. \square

Proposition 13. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of modules, where B is FI-cotorsion. Then if A is FI-cotorsion, then C is FI-cotorsion. If not*

$$FI\text{-}cd_R(A) = FI\text{-}cd_R(C) + 1.$$

Proof. Let F be an FI-flat R -module. Using the long exact sequence of the functor $(Hom_R(F, \cdot))$ to the exact sequence we obtain:

$$(*) \quad 0 = Ext_R^n(F, B) \rightarrow Ext_R^n(F, C) \rightarrow Ext_R^{n+1}(F, A) \rightarrow Ext_R^n(F, B) = 0.$$

The first and the last term are zero since B is FI-cotorsion and F is FI-flat so:

$$\text{Ext}_R^n(F, C) \cong \text{Ext}_R^{n+1}(F, A)$$

From Definition 6 $FI\text{-}cd_R(A) = FI\text{-}cd_R(C) + 1$.

Now, if A is FI-cotorsion and for $n = 1$ in (*) we obtain:

$$\text{Ext}_R^n(F, C) = \text{Ext}_R^{n+1}(F, A) = 0$$

So C is FI-cotorsion. □

Corollary 3. *Let $\{M_i\}_{i \in I}$ be a family of modules. Then $FI\text{-}cd_R(\prod M_i) = \sup\{FI\text{-}cd_R(M_i) | M_i R\text{-module}\}$.*

Proof. Follows from [10, Theorem 7.14] since $\text{Ext}_R^n(F, \prod_{i \in I} M_i) \cong \prod_{i \in I} \text{Ext}_R^n(F, M_i)$. Then we can deduce the result using Proposition 11. □

Corollary 4. *Let R be a Noetherian ring and let M be a R modules of finite flat dimension. If $cd(M) = n$, then $FI\text{-}cd_R(M) = n$.*

Proof. By induction, we can suppose that M is flat. Since $cd_R(M) = n$, there exists an exact sequence $0 \xrightarrow{\alpha_{-1}} M (= N_{-1}) \xrightarrow{\alpha_0} N_0 \xrightarrow{\alpha_1} N_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} N_{n-1} \xrightarrow{\alpha_n} N_n \rightarrow 0$ such that every N_i is a cotorsion envelope of $L_i = \text{coker}(\alpha_{i-1})$, for every $0 \leq i \leq n$. Since N_i is a cotorsion envelope of L_i , $\text{Ext}(N_i/L_i, C) = 0$ for every cotorsion module C and hence N_i/L_i is flat for every $0 \leq i \leq n$ by [12, Theorem 3.4.2]. As every $\text{coker}(\alpha_i)$ is flat, N_i is flat as well for every $0 \leq i \leq n$. Hence every N_i is FI-cotorsion by Proposition 8. Thus $FI\text{-}cd_R(M) \leq n$ by Theorem 4.1. And we have $cd_R(M) \leq FI\text{-}cd_R(M)$ and Remark 2, complete the proof. □

Corollary 5. *For a R -module M over a coherent ring, the following are equivalent:*

- 1) $FI\text{-}cd_R(M) \leq n$;
- 2) $FI\text{-}cd_R(\text{Hom}_R(P, M)) \leq n$ for every projective R -module P .

4.2. FI-cotorsion dimension of rings. We end this paper by studying the FI-cotorsion dimension of rings.

Definition 7. *For any ring R the global FI-cotorsion dimension of R , denoted by $FI\text{-}Cdim(R)$, is the supremum of FI-cotorsion dimensions of all R -modules, denoted:*

$$FI\text{-}Cdim(R) = \sup\{FI\text{-}cd_R(M) | M \text{ is an } R\text{-module}\}.$$

Remark 3.

For any ring R we have:

$$CD(R) \leq FI\text{-}Cdim(R) \leq gldim(R).$$

The following result gives a characterization of the FI-cotorsion dimension of rings.

Proposition 14. *Let R be a ring and $n \geq 0$ an integer. Then the following conditions are equivalent:*

- (1) $FI\text{-}Cdim(R) \leq n$;
- (2) $Ext_R^{n+1}(N, M) = 0$ for any R -module M and any FI-flat R -module N ;
- (3) $Ext_R^{n+j}(N, M) = 0$ for any integer $j \geq 1$ for any R -module M and any FI-flat R -module N ;
- (4) $pd_R(N) \leq n$ for any FI-flat R -module N ;
- (5) $FI\text{-}cd(M) \leq n$ for any R -module M .

Proof. The proof is obvious it follows from the definition and Proposition 11.

Proposition 15. *Let R be a ring.*

- (1) $FI\text{-}Cdim(R) = \sup\{pd(M)/M \text{ is a FI-flat module}\}$;
- (2) If $FI\text{-}Cdim(R) < \infty$, then $FI\text{-}Cdim(R) = \sup\{FI\text{-}cd(N)/N \text{ is a projective module}\}$.

Proof. 1) Follows from the definition and Proposition 14.

2) Let M be a FI-flat module. As $FI\text{-}Cdim(R) < \infty$, we can suppose that $pd(M) = n < \infty$ by 1). So there exists an exact sequence

$$0 \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow M \longrightarrow 0$$

where every N_i is projective. By Proposition 12, $FI\text{-}cd(M) \leq \sup\{FI\text{-}cd(N_i)\} \leq \sup\{FI\text{-}cd(N)/N \text{ is a projective module}\}$ for every i . Since M was arbitrary, $FI\text{-}Cdim(R) \leq \sup\{FI\text{-}cd(N)/N \text{ is a projective module}\}$. On other hand, we have $FI\text{-}Cdim(R) = \sup\{FI\text{-}cd(L)/R\text{-module}L\} \geq \sup\{FI\text{-}cd(N)/N \text{ is a projective module}\}$. This complete the result. \square

In the following theorem we characterize rings of $FI\text{-}Cdim(R) \leq 1$.

Theorem 4.2. *Let R be a ring. Then the following are equivalent:*

- (1) $FI\text{-}Cdim(R) \leq 1$,
- (2) Every quotient module of a FI-cotorsion R -module is FI-cotorsion,
- (3) Every quotient module of an injective R -module is FI-cotorsion,
- (4) Every FI-flat R -module has an projective dimension at most 1.

Proof. 1) \Rightarrow 2) Let N be a submodule of FI-cotorsion R -module M , and let F be an FI-flat R -module. applying the long exact sequence of the functor $Hom_R(F, \cdot)$ to the short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$, we get:

$$Ext_R^1(F, M) \rightarrow Ext_R^1(F, M/N) \rightarrow Ext_R^2(F, N).$$

The first term and the last term vanish since M is FI-cotorsion and by hypothesis hence $Ext_R^1(F, M/N) = 0$ and N is FI-cotorsion.

2) \Rightarrow 3) Obvious since every injective R -module is FI-cotorsion.

3) \Rightarrow 4) Let F be an FI-flat R -module, for any R -module M applying the long exact sequence of the functor $Hom_R(F, \cdot)$ to the exact sequence $0 \rightarrow M \rightarrow E \rightarrow C \rightarrow 0$, where E is injective, we get:

$$Ext_R^1(F, C) \rightarrow Ext_R^2(F, M) \rightarrow Ext_R^2(F, E) = 0.$$

By hypothesis $Ext_R^1(F, C) = 0$ which implies that $Ext_R^2(F, M) = 0$ hence $pd(F) \leq 1$.

4) \Rightarrow 2) Let M/N be a quotient of an FI-cotorsion R -module M , then there exists an exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$. Applying the long exact sequence of the functor $Hom_R(F, \cdot)$, where F is an FI-flat R -module we get: $Ext_R^{n-1}(F, N) \rightarrow Ext_R^n(F, M/N) \rightarrow Ext_R^2(F, M)$. We have $Ext_R^n(F, M) = 0$ since M is FI-cotorsion and $Ext_R^{n-1}(F, N) = 0$ since $pd(F) \leq 1$ by hypothesis for every n . Therefore $Ext_R^n(F, M/N) = 0$ and hence M/N is FI-cotorsion.

2) \Rightarrow 1) Let M be R -module. Then there exists a FI-cotorsion resolution: $0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0$ where N is FI-cotorsion. By hypothesis N/M is FI-cotorsion and so $FI\text{-}cd_R(M) \leq 1$. That is $FI\text{-}Cdim(R) \leq 1$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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