RESULTS ON UNICITY OF MEROMORPHIC FUNCTION WITH ITS SHIFT AND $q$-DIFFERENCE

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Abstract. This paper is insisted to studying the problems on sharing value for the derivative of meromorphic function with its shift and $q$-difference. The results in this paper improve and generalize the recent results to C. Meng and G. Liu (2020).

Keywords: Nevanlinna theory; uniqueness; value sharing; meromorphic functions.

2010 AMS Subject Classification: 30D35.

1. INTRODUCTION

In what follows, we assume that the reader is familiar with standard notations and main results of Nevanlinna theory [37]. As usual the abbreviation CM means “Counting Multiplicity”, while IM stands for “Ignoring Multiplicity”.

Let $f$ and $g$ are two non constant meromorphic functions. Let $k$ be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero point of $f$ with multiplicity $k$ is counted $k$ times in the set. If these zeros are only once counted, then we denote the set by $\overline{E}(a, f)$. If $E(a, f) = E(a, g)$, so that $f$ and $g$ share the value $a$ CM, and $f$ and $g$ share $a$ IM if $\overline{E}(a, f) = \overline{E}(a, g)$.

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Received July 03, 2021
For a complex number \( a \in \mathbb{C} \cup \infty \), we denote by \( E_k(a, f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point with multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a, f) = E_k(a, g) \) for a complex number \( a \in \mathbb{C} \cup \infty \) we say that \( f \) and \( g \) share the value \( a \) with weight \( k \) ([16], page 195).

The definition implies that if \( f \) and \( g \) share a value \( a \) with weight \( k \), then \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) (\( \leq k \)) if and only if it is a zero of \( g - a \) with multiplicity \( m \) (\( \leq k \)) and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m \) (\( > k \)) if and only if it is a zero of \( g - a \) with multiplicity \( n \) (\( > k \)), where \( m \) is not necessarily equal to \( n \). We write \( f \) and \( g \) share \((a, k)\) to mean that \( f \) and \( g \) share the value \( a \) with weight \( k \). Clearly if \( f \) and \( g \) share \((a, k)\) then \( f \) and \( g \) share \((a, p)\) for all integer \( p, 0 \leq p \leq k \). Also we note that \( f \) and \( g \) share a value \( a \) IM or CM if and only if \( f \) and \( g \) share \((a, 0)\) or \((a, \infty)\) respectively.

We denotes \( E_k(a, f) \) the set of all \( a \) points of \( f \) with multiplicities not exceeding \( k \), where an \( a \) point is counted accordingly and the set of distinct \( a \) points of \( f \) with multiplicities not greater than \( k \) is \( \bar{E}_k(a, f) \).

And \( N_k(r, \frac{1}{f-a}) \) the counting function for zeros of \( f - a \) with multiplicity less than or equal to \( k \), and by \( \bar{N}_k(r, \frac{1}{f-a}) \) the corresponding one for which multiplicity is not counted. Let \( N_k(r, \frac{1}{f-a}) \) be the counting function for zeros of \( f - a \) with multiplicity at least \( k \) and \( \bar{N}_k(r, \frac{1}{f-a}) \) the corresponding one for which multiplicities is not counted.

Meromorphic functions sharing values with their derivatives has become a subject of great interest in uniqueness theory. The paper by Rubel and Yang is the starting point of this topic, along with the following.

2. Preliminaries and Lemmas

Theorem 2.1. ([33], page 101) Let \( f \) be a non-constant entire function. If \( f \) and \( f' \) share two distinct finite values CM, then \( f = f' \).

The function \( f = e^z \int_0^z e^{-\xi} (1 - e') d\xi \) from [4] shows clearly that \( f \) and \( f' \) share 1 CM but \( f \neq f' \). In a special case, we recall a well-known conjecture by Brück:
Conjecture 2.1. ([4], page 22) Let \( f \) be a non-constant entire function such that hyper-order \( \rho_2(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \) is not a positive integer or infinity. If \( f \) and \( f' \) share the finite value \( a \) CM, then \( \frac{f'-a}{f-a} = c \), where \( c \) is nonzero constant.

The conjecture has been verified in the special cases when \( a = 0 \) [4], or when \( f \) is of finite order [12], or when \( \rho_2(f) < \frac{1}{2} \) [7]. Many results have been obtained for this and related topics (See [1, 5, 11, 17, 18],[23]-[28],[34, 35, 38, 39, 41, 43],[45]-[48], and the references therein). Heittokangas et al. considered analogues of Brück’s conjecture for meromorphic functions concerning their shifts, and proved the following theorem.

Theorem 2.2. ([15], Theorem 1, page 353) Let \( f \) be a meromorphic function of order
\[
\rho(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} < 2
\]
and let \( c \in \mathbb{C} \). If \( f(z) \) and \( f(z+c) \) share the values \( a \in \mathbb{C} \) and \( \infty \) CM, then
\[
\frac{f(z+c)-a}{f(z)-a} = \tau,
\]

Since then, many mathematicians considered this topic (See [6, 8, 10, 19, 22, 30, 42] and the references therein). In 2018, Qi, Li and Yang considered the value sharing problem related to \( f'(z) \) and \( f(z+c) \), where \( c \) is a complex number. They obtained the following result.

Theorem 2.3. ([29], Theorem 1.5, page 570) Let \( f \) be a non-constant meromorphic function of finite order and \( n \geq 9 \) be an integer. If \( [f'(z)]^n \) and \( f^n(z+c) \) share \( (1, 2), (\infty, 0) \) and \( \infty \) CM, then \( f'(z) = tf(z+c) \), for a constant \( t \) that satisfies \( t^n = 1 \).

It is natural to ask whether the \( f' \) can be extended to \( f^{(k)} \) in Theorem 2.3. Here \( f^n \) denotes the \( n^{th} \) power of the function \( f \) and \( f^{(k)} \) stands for the \( k^{th} \) derivative of \( f \), where \( k \) is a non-negative integer. Considering this question, C. Meng and G. Liu proved the following results.

Theorem 2.4. Let \( f \) be a non-constant meromorphic function of finite order and \( n \) be a positive integer. If one of the following conditions is satisfied:

(I) \([f^{(k)}(z)]^n \) and \( f^n(z+c) \) share \( (1, 2), (\infty, 0) \) and \( n \geq 2k+8 \);

(II) \([f^{(k)}(z)]^n \) and \( f^n(z+c) \) share \( (1, 2), (\infty, \infty) \) and \( n \geq 2k+7 \);

(III) \([f^{(k)}(z)]^n \) and \( f^n(z+c) \) share \( (1, 0), (\infty, 0) \) and \( n \geq 3k+14 \);
then \( f^{(k)}(z) = tf(z+c) \), for a constant \( t \) that satisfies \( t^n = 1 \).

If they consider entire function instead of meromorphic function, the counting functions related to the poles of \( [f^{(k)}(z)]^n \) and \( f^n(z+c) \) can be neglected. Arguing similarly as in Theorem 2.4, one can see that \( k \) is not related to the coefficient of \( N_{k+1} r \left( \frac{1}{r} \right) \). So obtained the following corollary.

**Corollary 2.1.** Let \( f \) be a non-constant entire function of finite order and \( n \geq 5 \) be an integer. If \( [f^{(k)}(z)]^n \) and \( f^n(z+c) \) share \((1,2)\), then \( f^{(k)}(z) = tf(z+c) \), for a constant \( t \) that satisfies \( t^n = 1 \).

If the shifts \( f(z+c) \) in Theorem 2.3 and 2.4 are replaced by \( q \)-difference \( f(qz) \), where \( q \in \mathbb{C} \setminus \{0\} \), they obtained:

**Theorem 2.5.** Let \( f \) be a non-constant meromorphic function of zero order and \( n \) be a positive integer. If one of the following conditions is satisfied:

(I) \( [f^{(k)}(z)]^n \) and \( f^n(qz) \) share \((1,2)\), \((\infty,0)\) and \( n \geq 2k+8 \);

(II) \( [f^{(k)}(z)]^n \) and \( f^n(qz) \) share \((1,2)\), \((\infty,\infty)\) and \( n \geq 2k+7 \);

(III) \( [f^{(k)}(z)]^n \) and \( f^n(qz) \) share \((1,0)\), \((\infty,0)\) and \( n \geq 3k+14 \);

then \( f^{(k)}(z) = tf(qz) \), for a constant \( t \) that satisfies \( t^n = 1 \).

**Corollary 2.2.** Let \( f \) be a non-constant entire function of zero order and \( n \geq 5 \) be an integer. If \( [f^{(k)}(z)]^n \) and \( f^n(qz) \) share \((1,2)\), then \( f^{(k)}(z) = tf(qz) \), for a constant \( t \) that satisfies \( t^n = 1 \).

We present some lemmas which will be needed later on. We will denote by \( H \) the following function:

\[
H = \left( \frac{F''}{F'} - \frac{2F''}{F-1} \right) \left( \frac{G''}{G'} - \frac{2G''}{G-1} \right)
\]

where \( F \) and \( G \) are non-constant meromorphic functions. From above, it can be easily calculated that the possible poles of \( H \) occur at (i) multiple zeros of \( F \) and \( G \), (ii) those \( 1 \) points of \( F \) and \( G \) whose multiplicities are different, (iii) those poles of \( F \) and \( G \) whose multiplicities are different,
(iv) zeros of $F'$ which are not the zeros of $F(F - 1)$ and zeros of $G'$ which are not the zeros of $G(G - 1)$. And we define the following notations which are used in the proof.

$$N_2 \left( r, \frac{1}{f} \right) = \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{G} \right),$$

where a simple zero point of $f$ is counted once and a multiple zero point of $f$ is counted twice.

Let $z_0$ be a zero of $f - 1$ of multiplicity $p$ and a zero of $g - 1$ of multiplicity $q$. We denote by $N_{E}^{(1)} \left( r, \frac{1}{f - 1} \right)$ the counting function of those 1-points of $f$ where $p = q = 1$; by $N_{L} \left( r, \frac{1}{f - 1} \right)$ the counting function of the 1-points of $f$ whose multiplicities are greater than 1-points of $g$; each point in these counting functions is counted only once. We are ignoring $g$ in the notations above only because the reader can interpret from the context with which function $g$ we are comparing the function $f$.

**Lemma 2.1.** ([2], Lemma 2.13, page 13) Let $F$, $G$ be two non-constant meromorphic functions. If $F$, $G$ share $(1, 2)$ and $(\infty, k)$, where $0 \leq k \leq \infty$, and $H \neq 0$, then

$$T \left( r, F \right) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + \overline{N} \left( r, F \right) + \overline{N} \left( r, G \right) + \overline{N} \left( r, \infty; F, G \right) + S \left( r, F \right) + S \left( r, G \right),$$

where $\overline{N} \left( r, \infty; F, G \right)$ denotes the reduced counting function of those poles of $F$ whose multiplicities differ from the multiplicities of the corresponding poles of $G$.

**Lemma 2.2.** ([36], Lemma 2, page 108) Let $f$ be a non-constant meromorphic function, and let $a_1, a_2, \ldots, a_n$ be finite complex numbers, $a_n \neq 0$. Then

$$T \left( r, a_n f^n + \ldots + a_2 f^2 + a_1 f + a_0 \right) = nT \left( r, f \right) + S \left( r, f \right).$$

**Lemma 2.3.** ([19], Theorem 2.1, page 109) Let $f$ be a meromorphic function of finite order $\rho(f)$, and let $c$ be a nonzero constant. Then

$$T \left( r, f(z + c) \right) = T \left( r, f(z) \right) + O \left( r^{\rho(f) - 1 + \varepsilon} \right) + O \left( \log r \right),$$

for an arbitrary $\varepsilon > 0$.

We mention that Lemma 2.3 holds also for $c = 0$ as in the case $T \left( r, f(z + c) \right) = T \left( r, f(z) \right)$. 
Lemma 2.4. ([48], Lemma 2.1, page 4) Let $f$ be a non-constant meromorphic function, $p$, $k$ be positive integers, then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + kN(r, f) + S(r, f),$$

where $N_p\left(r, \frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $f^{(k)}$ where a zero of multiplicity $m$ is counted $m$ times if $m \leq p$ and $p$ times if $m > p$.

We point out that in Lemma 2.4 one obviously has that

$$N\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right).$$

Lemma 2.5. ([13], Theorem 2.1, page 465) Let $f$ be a non-constant meromorphic function of finite order, and let $c \in \mathbb{C}$ and $\delta \in (0, 1)$. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r, f)}{r^\delta}\right) = S(r, f).$$

Lemma 2.6. ([43], Lemma 3.3, page 349) Suppose that two non-constant meromorphic functions $F$ and $G$ share $1$ and $\infty$ IM. Let $H$ be given as above. If $H \not\equiv 0$, then

$$T(r, F) + T(r, G) \leq 3N(r, F) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_1\left(r, \frac{1}{F-1}\right)$$

$$+ 2N_2\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right)$$

$$+ S(r, F) + S(r, G).$$

Lemma 2.7. ([44], Theorem 1.1, page 538) Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C}\{0\}$. Then

$$T(r, f(qz)) = (1 + o(1))T(r, f(z))$$

and

$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic density 1.

Lemma 2.8. ([3], Theorem 1.1, page 457) Let $f$ be a zero-order meromorphic function, and $q \in \mathbb{C}\{0\}$. Then

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f)$$

on a set of logarithmic density 1.
3. Main Results

In this paper, by considering the difference operator $\Delta_c F$ in Theorem 2.4 and 2.5, we obtain analogous results which are more general.

**Theorem 3.1.** Let $f$ be a non-constant meromorphic function of finite order and $n$ be a positive integer. If one of the following conditions is satisfied:

1. $[f^{(k)}(z)]^n$ and $\Delta_c F$ share $(1, 2), (\infty, 0)$ and $n \geq k + 6$;
2. $[f^{(k)}(z)]^n$ and $\Delta_c F$ share $(1, 2), (\infty, \infty)$ and $n \geq k + 5$;
3. $[f^{(k)}(z)]^n$ and $\Delta_c F$ share $(1, 0), (\infty, 0)$ and $n \geq 2k + 12$;

where $\Delta_c F = f^n(z + n) - f^n(z)$ then $f^{(k)}(z) = t f(z + c)$, for a constant $t$ that satisfies $t^n = \frac{1}{2}$.

**proof.** Let

$$F = \Delta_c F = f^n(z + n) - f^n(z)$$

(1). Suppose $[f^{(k)}(z)]^n$ and $\Delta_c F$ share $(1, 2), (\infty, 0)$ and $n \geq 2k + 8$. Then it follows directly from the assumption of the theorem that $F$ and $G$ share $(1,2)$ and $(\infty,0)$. Let $H$ be defined as above. Suppose that $H \neq 0$. It follows from Lemma 2.1 that

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + N_\delta(r, \infty; F, G) + S(r, F) + S(r, G).$$

According to Lemma 2.2 and Lemma 2.3, we have

$$T(r, F) = nT(r, f(z + \eta)) + nT(r, f(z)) + S(r, f) = 2nT(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + S(r, f).$$

It is obvious that

$$N_2\left(r, \frac{1}{F}\right) = 2\overline{N}\left(r, \frac{1}{f^n(z + \eta) - f^n(z)}\right) \leq 2\overline{N}\left(r, \frac{1}{f(z + \eta)}\right) + 2\overline{N}\left(r, \frac{1}{f(z)}\right) \leq 4T(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + S(r, f).$$

$$\overline{N}(r, F) = \overline{N}(r, f^n(z + \eta) - f^n(z)) \leq 2T(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + S(r, f).$$

(5)
(6) \( \mathcal{N}_*(r, \infty; F, G) \leq \mathcal{N}(r, F) \leq 2T(r, f) + O(r^{\rho(f)-1+\varepsilon}) + S(r, f). \)

Since \( E(\infty, f^{(k)}) = E(\infty, f) \), we have

(7) \( \mathcal{N}(r, G) = \mathcal{N}(r, f^{(k)}(z)) = \mathcal{N}(r, f) \leq T(r, f). \)

Lemma 2.4 gives

(8) \( N_2 \left( r, \frac{1}{G} \right) = 2N \left( r, \frac{1}{f^{(k)}} \right) \leq 2N_{k+1} \left( r, \frac{1}{f} \right) + 2kN(r, f) + S(r, f) \)

\[ \leq (2 + 2k)T(r, f) + S(r, f). \]

Combining (2)-(8), we deduce

\[ T(r, F) \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + \mathcal{N}(r, F) + \mathcal{N}(r, G) \]

\[ + \mathcal{N}_*(r, \infty; F, G) + S(r, f) + S(r, g), \]

\[ 2nT(r, f) \leq 4T(r, f) + (2 + 2k)T(r, f) + 2T(r, f) + T(r, f) + 2T(r, f) + O(r^{\rho(f)-1+\varepsilon}) + S(r, f), \]

\[ 2nT(r, f) \leq (2k + 11)T(r, f) + O(r^{\rho(f)-1+\varepsilon}) + S(r, f). \]

(9) \( (2n - 2k - 11)T(r, f) \leq O(r^{\rho(f)-1+\varepsilon}) + S(r, f), \)

which contradicts \( n \geq \frac{2k+12}{2} \geq k + 6 \). Therefore \( H \equiv 0 \), that is

\[ \frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1}. \]

By integrating twice, we get

(10) \( \frac{1}{F - 1} = \frac{A}{G - 1} + B, \)

where \( A \neq 0 \) and \( B \) are constants. From (10) we have

(11) \[ G = \frac{(B - A)F + (A - B - 1)}{BF - (B + 1)} \]

Suppose that \( B \neq 0, -1 \). From (11), we have

(12) \[ \mathcal{N} \left( r, \frac{1}{F - \frac{B+1}{B}} \right) = \mathcal{N}(r, G) \]
From the second fundamental theorem and Lemma 2.3, we have
\[ 2nT(r, f) = T(r, F) + S(r, f) \]
\[ \leq N(r, F) + N\left(\frac{1}{F}\right) + N\left(\frac{1}{F - \frac{B+1}{B}}\right) + S(r, f) \]
\[ \leq N(r, \Delta_c f) + N\left(\frac{1}{\Delta_c f}\right) + N(r, f) + S(r, f) \]
\[ \leq 2T(r, f) + 2T(r, f) + T(r, f) + S(r, f) \]
\[ \leq 5T(r, f) + O(r^{\rho(f)^{-1+\varepsilon}}) + S(r, f), \]
which contradicts \( n \geq k + 6 \). Suppose that \( B = -1 \). From (11) we have
\[ G = \frac{(A + 1)F - A}{F} \]
If \( A \neq -1 \), we obtain from (14) that
\[ 2nT(r, f) = T(r, F) + S(r, f) \leq N(r, F) + N\left(\frac{1}{F}\right) + N\left(\frac{1}{F - \frac{A}{A+1}}\right) + S(r, f) \]
\[ \leq N(r, \Delta_c f) + N\left(\frac{1}{\Delta_c f}\right) + N\left(\frac{1}{f^{(k)}}\right) + S(r, f) \]
\[ \leq N(r, \Delta_c f) + N\left(\frac{1}{\Delta_c f}\right) + N_{k+1}\left(\frac{1}{f}\right) + kNr, f + S(r, f) \]
\[ \leq (k + 5)T(r, f) + O(r^{\rho(f)^{-1+\varepsilon}}) + S(r, f), \]
which contradicts with \( n \geq k + 6 \). Hence \( A = -1 \). From (14) we get \( FG = 1 \), that is
\[ \left[f^n(z + \eta) - f^n(z)\right] \cdot [f^{(k)}(z)]^n = 1 \]
Since \([f^{(k)}(z)]^n\) and \([f(z + \eta) - f^n(z)]\) share \((\infty, 0)\), from (17) we get
\[ N(r, f^{(k)}) = 0, \quad T(r, f^{(k)}) = T(r, f(z + \eta)) + S(r, f), \]
and
\[ [f^{(k)}(z)]^{2n} = \frac{[f^{(k)}(z)]^n}{f^n(z + \eta) - f^n(z)} = \frac{[f^{(k)}(z)]^n}{f^n(z)} \cdot \frac{f^n(z + \eta) - f^n(z)}{f^n(z)}. \]
From Lemma 2.5 and logarithmic derivative lemma, we get
\[ 2nT(r, f^{(k)}) = T(r, [f^{(k)}]^{2n}) = m(r, [f^{(k)}]^{2n}) + N(r, [f^{(k)}]^{2n}) = m(r, [f^{(k)}(z)]^{2n}) = S(r, f). \]

that is
\[ (20) \quad T(r, f^{(k)}) = S(r, f) \]

By (18) and (20), we know that
\[ (21) \quad T(r, f(z + \eta)) = T(r, f^{(k)}) = S(r, f). \]

which is a contradiction with Lemma 2.3.

Suppose that \( B = 0 \). From (11), we have
\[ (22) \quad G = AF - (A - 1) \]

If \( A \neq 1 \), from (22) we obtain
\[ (23) \quad \overline{N}\left(r, \frac{1}{F - \frac{A - 1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right). \]

From the second fundamental theorem, Lemma 2.3 and Lemma 2.4, we have
\[ 2nT(r, f) = T(r, F) + S(r, f) \]
\[ \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \frac{A - 1}{A}}\right) + S(r, f) \]
\[ \leq \overline{N}(r, \Delta_c f) + \overline{N}\left(r, \frac{1}{\Delta_c f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + + S(r, f) \]
\[ \leq \overline{N}(r, \Delta_c f) + \overline{N}\left(r, \frac{1}{\Delta_c f}\right) + N_{k+1}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \]
\[ \leq (k + 5)T(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + S(r, f), \]

which contradicts with \( n \geq k + 6 \). Thus \( A = 1 \). From (22) we have \( F = G \), that is \( f^n(z + \eta) - f^n(z) = [f^{(k)}(z)]^n \). Hence \( f^{(k)}(z) = tf(z + \eta) \), for a constant \( t \) with \( t^n = \frac{1}{2} \). We can get the conclusion of Theorem 3.1.

(II). Suppose \( [f^{(k)}(z)]^n \) and \( f^n(z + \eta) - f^n(z) \) share \( (1, 2), (\infty, \infty) \) and \( n \geq 2k + 7 \). Then
it follows directly from the assumption of the theorem that $F$ and $G$ share $(1,2)$ and $(\infty,\infty)$. Let $H$ be defined as above. Suppose that $H \not\equiv 0$. It follows from Lemma 2.1 that

\begin{equation}
T(r,F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G).
\end{equation}

According to Lemma 2.2 and Lemma 2.3, we have

\begin{equation}
T(r,F) = nT(r,f(z+\eta)) + nT(r,f(z)) + S(r,f) = 2nT(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f).
\end{equation}

It is obvious that

\begin{equation} N_2\left(r, \frac{1}{F}\right) = 4T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{equation}

\begin{equation} \overline{N}(r,F) = 2T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{equation}

\begin{equation} \overline{N}(r,G) = \overline{N}(r,f) \leq T(r,f), \end{equation}

\begin{equation} \overline{N}_*(r,\infty;F,G) = 0 \end{equation}

Lemma 2.4 gives

\begin{equation} N_2\left(r, \frac{1}{G}\right) = (2k+2)T(r,f) + S(r,f). \end{equation}

Combining (25)-(31), we deduce

\begin{equation} (2n-2k-9)T(r,f) \leq O(r^{\rho(f)-1+\varepsilon}) + S(r,f), \end{equation}

which contradicts with $n \geq k+5$. Therefore $H \equiv 0$. Similar to the proof in (I), we can get the conclusion of Theorem 3.1.

(III). Suppose $[f^{(k)}(z)]^n$ and $f^n(z+\eta) - f^n(z)$ share $(1,0), (\infty,0)$ and $n \geq 3k+14$. Then it follows directly from the assumption of the theorem that $F$ and $G$ share $(1,0)$ and $(\infty,0)$. Let
be defined as above. Suppose that \( H \neq 0 \). It follows from Lemma 2.6 that

\[
T(r, F) + T(r, G) \leq 3N(r, F) + N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_E^1 \left( r, \frac{1}{F - 1} \right) + 2N_E^2 \left( r, \frac{1}{F - 1} \right) + 3N_L \left( r, \frac{1}{F - 1} \right) + N_L \left( r, \frac{1}{G - 1} \right) + S(r, F) + S(r, G).
\]  

Since

\[
N_E^1 \left( r, \frac{1}{F - 1} \right) + 2N_E^2 \left( r, \frac{1}{F - 1} \right) + N_L \left( r, \frac{1}{F - 1} \right) + 2N_L \left( r, \frac{1}{G - 1} \right) \leq N \left( r, \frac{1}{G - 1} \right) \leq T(r, G) + O(1),
\]

we get from (33) and (34) that

\[
T(r, F) \leq 3N(r, F) + N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + 2N_L \left( r, \frac{1}{F - 1} \right) + N_L \left( r, \frac{1}{G - 1} \right) + S(r, F) + S(r, G).
\]  

According to Lemma 2.2 and Lemma 2.3, we have

\[
T(r, F) = 2nT(r, f) + O(r^{\rho(f) - 1 + \varepsilon}) + S(r, f).
\]

It is obvious that

\[
N(r, F) = 2T(r, f) + O(r^{\rho(f) - 1 + \varepsilon}) + S(r, f),
\]

\[
N_2 \left( r, \frac{1}{F} \right) = 4T(r, f) + O(r^{\rho(f) - 1 + \varepsilon}) + S(r, f),
\]

\[
N_L \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{F}{F'} \right) \leq N \left( r, \frac{F'}{F} \right) + S(r, f)
\]

\[
\leq N(r, F) + N \left( r, \frac{1}{F} \right) + S(r, f)
\]

\[
= N(r, \Delta_c f) + N \left( r, \frac{1}{\Delta_c f} \right) + S(r, f)
\]

\[
\leq 4T(r, f) + O(r^{\rho(f) - 1 + \varepsilon}) + S(r, f).
\]
Lemma 2.4 gives

\( N_2 \left( r, \frac{1}{G} \right) \leq (2k+2)T(r,f) + S(r,f), \)

\[
N_L \left( r, \frac{1}{G-1} \right) \leq N \left( r, \frac{G}{G'} \right) \leq N \left( r, \frac{G'}{G} \right) + S(r,f) \\
\leq \overline{N}(r,G) \overline{N} \left( r, \frac{1}{G} \right) + S(r,f) \\
\leq \overline{N}(r,f) + \overline{N} \left( r, \frac{1}{f(k)} \right) + S(r,f) \\
\leq \overline{N}(r,f) + N_{k+1} \left( r, \frac{1}{f} \right) + k\overline{N}(r,f) + S(r,f) \\
\]

\( (41) \quad \leq (k+2)T(r,f) + S(r,f). \)

Combining (35)-(41), we deduce

\( (42) \quad (2n - 3k - 22)T(r,f) \leq O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \)

which contradicts with \( n \geq \frac{3k+23}{2} \geq 2k + 12 \). Therefore \( H \equiv 0 \). Similar to the proof of (I), we can get the conclusion of Theorem 3.1.

\( \square \)

**Corollary 3.1.** Let \( f \) be a non-constant meromorphic function of zero order and \( n \) be a positive integer. If one of the following conditions is satisfied:

(I) \( [f^{(k)}](z)^n \) and \( \Delta_c f(qz) \) share \( (1,2), (\infty,0) \) and \( n \geq k + 6; \)

(II) \( [f^{(k)}](z)^n \) and \( \Delta_c f(qz) \) share \( (1,2), (\infty,\infty) \) and \( n \geq k + 5; \)

(III) \( [f^{(k)}](z)^n \) and \( \Delta_c f(qz) \) share \( (1,0), (\infty,0) \) and \( n \geq 2k + 12; \)

where \( \Delta_c F = f^n(qz + n) - f^{(n)}(qz) \), then \( f^{(k)}(z) = tf(qz) \), for a constant \( t \) that satisfies \( t^n = \frac{1}{\pi}. \)

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.
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