STUDY OF L-FUNCTION USING WEAKLY WEIGHTED SHARING DEFINED OVER AN EXTENDED SELBERG CLASS

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Abstract. In this article, the concept of weakly weighted sharing is used to prove the uniqueness results of a meromorphic function and an L-function described in Selberg class S. Our result will improve the result due to D.C. Pramanik and Ja. Roy [4].

Keywords: Meromorphic function; L-function; Selberg class; Weakly weighted sharing.

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1. INTRODUCTION

We use the basic notations of Nevanilinna theory as described in [1, 8, 9, 10]. We define, $F = \{ h_1 : h_1 \text{ is a non-constant meromorphic function} \}$, where meromorphic function is always defined in the complex plane. A meromorphic function $a$ is a small function with respect to $h_1 \in F$, if either $a \equiv \infty$ or $T(r,a) = S(r,h_1)$. $S(h_1)$ is the set of all small functions with respect to $h_1$ that are specified in the complex plane.

In 19th century, A. Selberg introduced a class known as Selberg Class in order to understand the value distribution of L-functions, since then Selberg Class has been an important field of research. For the concept of L-function, we encourage the reader to refer [6].

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Definition 1. [9] For \( a \in \mathbb{C} \cup \{\infty\} \), the quantity
\[
\delta(a, h_1) = 1 - \limsup_{r \to \infty} \frac{N(r, a; h_1)}{T(r, h_1)},
\]
is called the deficiency of ‘a’ for the function \( h_1 \).

Definition 2. [9] The quantity
\[
\Theta(a, h_1) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a; h_1)}{T(r, h_1)},
\]
for \( a \in \mathbb{C} \cup \{\infty\} \) is the ramification index of \( a \) for the function \( h_1 \).

The first comprehensive review of the principle of weighted sharing to prove the uniqueness of meromorphic functions was proposed by Indrajit Lahiri in 2001. We steer the reader to [2] (see p.195, Definition 7) for a discussion on weighted sharing.

The argument of Shanhua Lin and Weichuan Lin in 2006 touched upon the principle of weakly weighted sharing. For the definition, reader can refer [3] (see p.274, Definition 4).

We denote with the notation \( \overline{N}_L(r, 1; H_1) \), the reduced counting function which are 1–points of \( H_1 \) whose multiplicities are greater than 1–points of \( L \) when \( H_1 \) and \( L \) share \( \text{“IM”} \). \( \overline{N}_L(r, 1; L) \) is described similarly. We denote the reduced counting function for the poles for the function \( h_1 \) by \( N_1(r, h_1) = \overline{N}(r, \infty; h_1) \). Similarly, \( N_1 \left( r, \frac{1}{h_1} \right) = \overline{N}(r, 0; h_1), N_1(r, L) = \overline{N}(r, L), N_1 \left( r, \frac{1}{L} \right) = \overline{N}(r, 0; L) \).

In 2006, S. Lin and W. Lin (see [3], p. 272, Theorem 1 – 3) defined and used the concept of weakly weighted sharing of functions to prove the uniqueness of a meromorphic functions and its derivatives for the first time. By contributing three theorems, they proved the uniqueness of \( h_1 \) and \( h_1^{(n)} \) when they both share “(a,m)”, “(a,1)”, “(a,0)” with some relative conditions.

Later in 2011, H-Y. Xu and Y. Hu (see [7], p. 104, Theorem 1 – 3) generalized the theorems proved by S. Lin and W. Lin by proving the uniqueness of a non-constant meromorphic function \( h_1 \) and \( L(h_1) = h_1^{(n_1)} + a_{n_1 - 1} h_1^{(n_1 - 1)} + \ldots + a_0 h_1 \) where \( a_i \neq 0, \infty \in S(h_1) \) for \( 0 \leq i \leq n_1 - 1 \), sharing “(a,k_1)”, “(a,1)”, “(a,0)” with some conditions of suitability.

Recently, in 2019, D. C. Pramanik and Ja. Roy (see [4], p. 44, Theorem 7) considered the issue more broadly with consideration of non-constant homogeneous differential polynomials \( P[h_1] \) and \( P[g] \), where \( f \) and \( g \) are two functions having non-constant meromorphic properties.
and showed that if $P[h_1]$ and $P[g]$ share “$(a,l)$” with some appropriate conditions, then $P[h_1] = P[g]$.

Inspired by such study, it is normal to inquire that what will be the relation between a non-constant meromorphic function $h_1$ and an $L$-function $L$ when they share “$(a,l)$” and weakly weighted sharing is taken into account where $a \in S(h_1) \cap S(L)$, $a \not\equiv 0$, $\infty$. As an answer for this, we have proved the result as stated in the Section 3 of this paper.

2. Preliminaries

We highlight only those lemmas that are needed to prove our conclusion.

We consider

\[
\Psi = \left( \frac{H_1''}{H_1'} - 2 \frac{H_1'}{H_1 - 1} \right) - \left( \frac{H_2''}{H_2'} - 2 \frac{H_2'}{H_2 - 1} \right).
\]

Here, the notations $H_1$ and $H_2$ are used and they are considered to be non-constant meromorphic functions. For Second Fundamental Theorem (SFT), we redirect the reader to [1].

If suppose $h_1 \in F$ defined in $\mathbb{C}$ and $p \in \mathbb{Z}^+$, then we have

\[
N(r,0;h_1^{(p)}) \leq N(r,0;h_1) + pN_1(r,h_1) + O(\log(T(r,h_1)) + \log r),
\]

refer [9].

Suppose,

(1) $H_1$ and $H_2$ share “$(1,l)$” then

\[
\bar{N}_L(r,1;H_1) \leq \frac{1}{2} \bar{N}(r,0;H_1) + \frac{1}{2} \bar{N}(r,\infty;H_1) + S(r,H_1),
\]

refer [7], [p. 106, Lemma 4].

(2) $H_1$ and $H_2$ share “$(1,0)$” then

\[
\bar{N}_L(r,1;H_1) \leq \bar{N}(r,0;H_1) + \bar{N}(r,\infty;H_1) + S(r,H_1),
\]

refer [7], [p. 106, Lemma 6].

Also, letting a non-negative integer or $\infty$ as ‘$l$’, $H_1$ and $H_2$ sharing “$(1,l)$”, if $\Psi$ as defined in (2.1) is not equal to zero then we have the following cases.
(1) Whenever $2 \leq l \leq \infty$,
\[
T(r, H_1) \leq N_2(r, \infty; H_1) + N_2(r, \infty; H_2) + N_2(r, 0; H_1)
\]
(2.5)
\[+ N_2(r, 0; H_1) + S(r, H_1) + S(r, H_2).\]

(2) Whenever $l = 1$,
\[
T(r, H_1) \leq N_2(r, \infty; H_1) + N_2(r, \infty; H_2) + N_2(r, 0; H_1) + N_2(r, 0; H_2)
\]
(2.6)
\[+ \overline{N}_L(r, 1; H_1) + S(r, H_1) + S(r, H_2).\]

(3) Whenever $l = 0$,
\[
T(r, H_1) \leq N_2(r, \infty; H_1) + N_2(r, \infty; H_2) + N_2(r, 0; H_1) + N_2(r, 0; H_1)
\]
(2.7)
\[+ 2 \overline{N}_L(r, 1; H_1) + \overline{N}_L(r, 1; H_2) + S(r, H_1) + S(r, H_2).\]

Similarly, we can define for $T(r, H_2)$, refer [3] [p. 273, Lemma 3].

3. MAIN RESULTS

**Theorem 1.** Considering, $h_1 \in F$ defined in $\mathbb{C}$ and $L$ be an $L$-function, $a \in S(h_1) \cap S(L)$, where $a \neq 0, \infty$. If $h_1$ and $L$ share "$(a, l)$" with one of the conditions mentioned below

(i) $l \geq 2$ and

\[
\min \left\{ \frac{2}{p+1} \delta_{p+1}(0, h_1) + \frac{4+2p}{p+1} \Theta(\infty, f), \frac{2}{p+1} \delta_{p+1}(0, L) + \frac{4+2p}{p+1} \Theta(\infty, L) \right\} > \frac{3p+7}{p+1},
\]
(3.1)

(ii) $l = 1$ and

\[
\min \left\{ \frac{5}{2p+2} \delta_{p+1}(0, h_1) + \frac{5p+9}{2p+2} \Theta(\infty, f), \frac{5}{2p+2} \delta_{p+1}(0, L) + \frac{5p+9}{2p+2} \Theta(\infty, L) \right\} > \frac{3p+12}{2p+2},
\]
(3.2)

(iii) $l = 0$ and

\[
\min \left\{ \frac{5}{p+1} \delta_{p+1}(0, h_1) + \frac{5p+7}{p+1} \Theta(\infty, f), \frac{5}{p+1} \delta_{p+1}(0, L) + \frac{5p+7}{p+1} \Theta(\infty, L) \right\} > \frac{4p+11}{p+1},
\]
(3.3)

then $f \equiv L$.

**Proof.** Let

\[H_1 = \frac{h_1(p)}{a}, \quad H_2 = \frac{L(p)}{a}.\]
Since $h_1^{(p)}$ and $L^{(p)}$ share $(a, l)$, it follows that $H_1$, $H_2$ share $(1, l)$ except at the zeros and poles of $a$.

Suppose that $\Psi \neq 0$. Now we will consider the cases as below:

**Case 1:** If $2 \leq l \leq \infty$. From (2.5), we obtain

\[
(3.4) \quad T(r, h_1^{(p)}) \leq 2N(r, \infty; h_1^{(p)}) + 2N(r, \infty; L^{(p)}) + N(r, 0; h_1^{(p)}) + N(r, 0; L^{(p)})
\]

\[
+ S(r, h_1) + S(r, L).
\]

From (2.2) and (3.4), we obtain

\[
(3.5) \quad (p + 1)T(r, h_1) \leq (2 + p)N_1(r, h_1) + (2 + p)N_1(r, L) + N_{p+1} \left(r, \frac{1}{h_1}\right)
\]

\[
+ N_{p+1} \left(r, \frac{1}{L}\right) + S(r, h_1) + S(r, L)
\]

So,

\[
T(r, h_1) \leq \left(\frac{2 + p}{p + 1}\right)N_1(r, h_1) + \left(\frac{2 + p}{p + 1}\right)N_1(r, L) + \frac{1}{p + 1}N_{p+1} \left(r, \frac{1}{h_1}\right)
\]

\[
+ \frac{1}{p + 1}N_{p+1} \left(r, \frac{1}{L}\right) + S(r, h_1) + S(r, L).
\]

**Similarly,**

\[
T(r, L) \leq \left(\frac{2 + p}{p + 1}\right)N_1(r, L) + \left(\frac{2 + p}{p + 1}\right)N_1(r, h_1) + \frac{1}{p + 1}N_{p+1} \left(r, \frac{1}{L}\right)
\]

\[
+ \frac{1}{p + 1}N_{p+1} \left(r, \frac{1}{h_1}\right) + S(r, L) + S(r, h_1).
\]

Now, from (3.5) and (3.6), we obtain

\[
T(r, h_1) + T(r, L) \leq \left(\frac{4 + 2p}{p + 1}\right)N_1(r, h_1) + \left(\frac{4 + 2p}{p + 1}\right)N_1(r, L)
\]

\[
+ \frac{2}{p + 1}N_{p+1} \left(r, \frac{1}{h_1}\right) + \frac{2}{p + 1}N_{p+1} \left(r, \frac{1}{L}\right) + S(r, L) + S(r, h_1)
\]

\[
\left\{\frac{2}{p + 1} \delta_{p+1}(0, h_1) + \left(\frac{4 + 2p}{p + 1}\right) \Theta(\infty, f) - \left(\frac{3p + 7}{p + 1}\right)\right\} T(r, h_1)
\]

\[
+ \left\{\frac{2}{p + 1} \delta_{p+1}(0, L) + \left(\frac{4 + 2p}{p + 1}\right) \Theta(\infty, L) - \left(\frac{3p + 7}{p + 1}\right)\right\} T(r, L)
\]

\[
\leq S(r, h_1) + S(r, L),
\]
which conflicts our assumption (3.1). Therefore \( \Psi \equiv 0 \) and now from (2.1) we obtain
\[
\frac{1}{H_2 - 1} = \frac{I_1}{H_1 - 1} + I_2,
\]
where \( I_1 \neq 0 \) and \( I_2 \) are constants. This gives
\[
H_2 = \frac{(I_2 + 1)H_1 + (I_1 - I_2 - 1)}{I_2 H_1 + (I_1 - I_2)}, \tag{3.7}
\]
\[
H_1 = \frac{(I_2 - I_1)H_2 + (I_1 - I_2 - 1)}{I_2 H_2 + (I_2 + 1)} \tag{3.8}.
\]

Next we consider three subcases:

**Subcase 1:** \( I_2 \neq 0, -1 \). Then from (3.8),
\[
N \left( r, -\frac{I_1 + I_2 + 1}{I_2 + 1}; H_2 \right) = N(r, \infty; H_1).
\]

By using the Second Fundamental Theorem (SFT) of Nevanlinna and (2.2),
\[
T(r, H_2) < N(r, \infty; H_2) + N(r, 0; H_2) + \overline{N} \left( r, \frac{I_2 + 1}{I_2}; H_2 \right) + S(r, H_2), \tag{3.9}
\]
\[
T(r, L) \leq \frac{1}{(p + 1)} N(r, \infty; f) + \frac{1}{(p + 1)} N_{p+1} \left( r, \frac{1}{L} \right) + N_1(r, L) + S(r, h_1) + S(r, L).
\]

If \( I_1 - I_2 - 1 \neq 0 \), then it follows from (3.7) that
\[
N \left( r, -\frac{-I_1 + I_2 + 1}{I_2 + 1}; H_1 \right) = N(r, 0; H_2).
\]

Applying SFT of Nevanlinna and (2.2), subsequently we obtain
\[
T(r, H_1) < \overline{N}(r, \infty; H_1) + \overline{N}(r, 0; H_1) + \overline{N} \left( r, -\frac{-I_1 + I_2 + 1}{I_2 + 1}; H_1 \right) + S(r, H_1) \tag{3.10}
\]
\[
T(r, f) \leq N_1(r, h_1) + \frac{1}{(p + 1)} N_{p+1} \left( r, \frac{1}{h_1} \right) + \frac{1}{(p + 1)} \overline{N}(r, 0; L)
\]
\[
+ S(r, h_1).
\]

From (3.9) and (3.10), we obtain
\[
T(r, h_1) + T(r, L) \leq \frac{2}{(p + 1)} N_{p+1} \left( r, \frac{1}{h_1} \right) + 2N_1(r, h_1)
\]
\[
+ \frac{1}{(p + 1)} N_{p+1} \left( r, \frac{1}{L} \right) + N_1(r, L) + S(r, h_1) + S(r, L),
\]
which again contradicts (3.1).

Therefore \( I_1 - I_2 - 1 = 0 \). Then from (3.7),

\[
\overline{N}(r, 0; H_1 + \frac{1}{I_2}) = \overline{N}(r, \infty; H_2).
\]

By applying the SFT of Nevanlinna and (2.2) we have

\[
T(r, H_1) < \overline{N}(r, \infty; H_1) + \overline{N}(r, 0; H_1) + \overline{N}(r, 0; H_1 + \frac{1}{I_2}) + S(r; H_1),
\]

\[
(p + 1)T(r, h_1) \leq N_1(r, h_1) + pN_1(r, h_1) + N_{p+1}(r, \frac{1}{h_1}) + N_1(r, L)
\]

\[
+ S(r, h_1) + S(r, L).
\]

So

\[
T(r, h_1) \leq N_1(r, h_1) + \frac{1}{(p + 1)}N_{p+1}(r, \frac{1}{h_1}) + \frac{1}{(p + 1)}N_1(r, L)
\]

\[
+ S(r, h_1) + S(r, L).
\]

From (3.9) and (3.11) we obtain

\[
T(r, h_1) + T(r, L) \leq \frac{1}{(p + 1)}N_{p+1}(r, \frac{1}{h_1}) + \left(\frac{p + 2}{p + 1}\right)N_1(r, h_1)
\]

\[
+ \frac{1}{(p + 1)}N_{p+1}(r, \frac{1}{L}) + \left(\frac{2 + p}{p + 1}\right)N_1(r, L)
\]

\[
+ S(r, h_1) + S(r, L),
\]

which violates assumption (3.1).

Subcase 2: \( I_2 = -1 \). Then from (3.7) and (3.8) we obtain

\[
H_2 = \frac{I_1}{I_1 + 1 - H_1}, \quad H_1 = \frac{(1+I_1)H_2 - I_1}{H_2}.
\]

If \( I_1 + 1 \neq 0 \), then

\[
\overline{N}(r, I_1 + 1; H_1) = \overline{N}(r, \infty; H_2), \quad \overline{N}(r, \frac{I_1}{I_1 + 1}; H_2) = \overline{N}(r, 0; H_1).
\]

By similar argument as in previous subcase, we arrive at a contradiction. Hence \( I_1 + 1 = 0 \), then \( H_1H_2 = 1 \).
Subcase 3: $I_2 = 0$. Then (3.7) and (3.8) gives $H_2 = \frac{H_1 + h_{I_1} - 1}{I_1}$ and $H_1 = I_1H_2 + 1 - I_1 \neq 0$. $N(r, 0; I_1 - 1 + H_1) = N(r, 0; H_2)$ and $N(r, \frac{H_1 + h_{I_1} - 1}{I_1}; H_2) = N(r, 0; H_1)$. Proceeding in the same direction as in Subcase 1 we obtain a paradox. Therefore $I_1 - 1 = 0$, then $H_1 = H_2$.

Case 2: For $l = 1$, from (2.6) we have

$$T(r, H_1) \leq 2\bar{N}(r, \infty; H_1) + 2\bar{N}(r, \infty; H_2) + N(r, 0; H_1) + N(r, 0; H_2)$$

$$+ \bar{N}_L(r, 1; H_1) + S(r, H_1) + S(r, H_2).$$

Now, by using (2.2) and (2.3), we obtain

$$T(r, H_1) \leq 2\bar{N}(r, \infty; H_1) + 2\bar{N}(r, \infty; H_2) + pN_1(r, h_1) + N_{p+1}(r, \frac{1}{h_1})$$

$$+ pN_1(r, L) + N_{p+1}(r, \frac{1}{L}) + \frac{1}{2}\bar{N}(r, 0; H_1) + \frac{1}{2}\bar{N}(r, \infty; H_1)$$

$$+ \frac{1}{2}\bar{N}(r, \infty; H_1) + S(r, h_1) + S(r, L).$$

Likewise,

$$T(r, L) \leq \left(\frac{3p + 5}{2p + 2}\right)N_1(r, h_1) + \frac{3}{2(p + 1)}N_{p+1}(r, \frac{1}{h_1})$$

$$+ \left(\frac{2 + p}{p + 1}\right)N_1(r, L) + \frac{1}{(p + 1)}N_{p+1}(r, \frac{1}{L})$$

$$+ S(r, h_1) + S(r, L).$$

From (3.12) and (3.13), we obtain

$$T(r, h_1) + T(r, L) \leq \left(\frac{3p + 5}{2p + 2}\right)N_1(r, h_1) + \frac{3}{2(p + 1)}N_{p+1}(r, \frac{1}{h_1}) + \left(\frac{2 + p}{p + 1}\right)N_1(r, L)$$

$$+ \frac{1}{(p + 1)}N_{p+1}(r, \frac{1}{L}) + \left(\frac{3p + 5}{2p + 2}\right)N_1(r, L) + \frac{3}{2(p + 1)}N_{p+1}(r, \frac{1}{L})$$

$$+ \left(\frac{2 + p}{p + 1}\right)N_1(r, h_1) + \frac{1}{(p + 1)}N_{p+1}(r, \frac{1}{h_1}) + S(r, h_1) + S(r, L),$$
that conflicts with our assumption (3.2). Following the procedure of case (i), we obtain the result for this case.

**Case 3:** $l = 0$. Now, from (2.7), we have

\[(3.14) \quad T(r, H_1) \leq 2\overline{N}(r, \infty; H_1) + 2\overline{N}(r, \infty; H_2) + N(r, 0; H_1) + N(r, 0; H_2)
\]
\[+ 2\overline{N}_L(r, 1; H_1) + \overline{N}_L(r, 1; H_2) + S(r, H_1) + S(r, H_2).\]

Using (2.2), (2.4) and (3.14) we obtain

\[(3.15) \quad T(r, h_1) \leq \frac{4p + 5}{p + 1} N_1(r, h_1) + \frac{4}{p + 1} N_{p+1}\left( r, \frac{1}{h_1} \right) + \frac{2 + p}{p + 1} N_1(r, L)
\]
\[+ \frac{1}{p + 1} N_{p+1}\left( r, \frac{1}{L} \right) + S(r, h_1) + S(r, L).\]

Similarly,

\[(3.16) \quad T(r, L) \leq \frac{4p + 5}{p + 1} N_1(r, L) + \frac{4}{p + 1} N_{p+1}\left( r, \frac{1}{L} \right) + \frac{2 + p}{p + 1} N_1(r, h_1)
\]
\[+ \frac{1}{p + 1} N_{p+1}\left( r, \frac{1}{h_1} \right) + S(r, h_1) + S(r, L).\]

From (3.15) and (3.16), we obtain

\[T(r, h_1) + T(r, L) \leq \frac{4p + 5}{p + 1} N_1(r, h_1) + \frac{4}{p + 1} N_{p+1}\left( r, \frac{1}{h_1} \right) + \frac{2 + p}{p + 1} N_1(r, L)
\]
\[+ \frac{1}{p + 1} N_{p+1}\left( r, \frac{1}{L} \right) + \frac{4p + 5}{p + 1} N_1(r, L) + \frac{4}{p + 1} N_{p+1}\left( r, \frac{1}{L} \right)
\]
\[+ \frac{2 + p}{p + 1} N_1(r, h_1) + \frac{1}{p + 1} N_{p+1}\left( r, \frac{1}{h_1} \right) + S(r, h_1) + S(r, L)\]
\[
\left\{ \frac{5}{p+1} \delta_{p+1}(0,h_1) + \left( \frac{5p+7}{p+1} \right) \Theta(\infty,f) - \frac{4p+11}{p+1} \right\} T(r,h_1) \\
+ \left\{ \frac{5}{p+1} \delta_{p+1}(0,L) + \left( \frac{5p+7}{p+1} \right) \Theta(\infty,L) - \frac{4p+11}{p+1} \right\} T(r,L) \\
\leq S(r,h_1) + S(r,L),
\]

that conflicts with our assumption (3.3).

We obtain the requisite inference for this case in the same way as in Case 1. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES