ECCENTRIC DOMINATION NUMBER OF SOME CYCLE RELATED GRAPHS

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Abstract. In a graph $G$, a vertex $u$ is said to be an eccentric vertex of a vertex $v$ if the distance between $u$ and $v$ is equals to the eccentricity of vertex $v$. A dominating set $D$ of a graph $G = (V, E)$ is said to be an eccentric dominating set if for every $v \in V - D$, there exists at least one eccentric vertex of $v$ in $D$. The minimum cardinality of the minimal eccentric dominating sets of graph $G$ is said to be eccentric domination number, denoted by $\gamma_{ed}(G)$.

The eccentric domination numbers of some cycle related graphs have been investigated.

Keywords: dominating set; eccentric dominating set; eccentric domination number.

2010 AMS Subject Classification: 05C38, 05C69, 05C76.

1. INTRODUCTION

The domination in graphs is one of the most rapidly growing fields within and out side of graph theory. Many researchers have been attracted to work on it due to its diversified applications in the various fields like linear algebra and optimization, design and analysis of communication networks as well as surveillance.

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Received July 29, 2021
There are many domination models available in the literature. Total domination [2], independent domination [3], eccentric domination [5], equitable domination [6], restrained domination [7], global domination [11], steiner domination [12] are among worth to mention.

Throughout this work, a graph \( G = (V, E) \), we mean a finite, simple and connected graph with vertex set \( V = \{v_1, v_2, v_3, \cdots, v_n\} \) and edge set \( E \). For any graph theoretic terminology and notation we refer to West [13] while the terms related to domination are used in the sense of Haynes et al. [4].

2. Preliminaries

**Definition 2.1.** A set \( D \subseteq V \) of vertices in a graph \( G = (V, E) \) is said to be a dominating set if every vertex in \( V - D \) is adjacent to at least one vertex in \( D \).

**Definition 2.2.** A dominating set \( D \) is said to be a minimal dominating set if no proper subset \( D' \subset D \) is a dominating set. The set of all minimal dominating sets of a graph \( G \) is denoted by \( MDS(G) \). The minimum cardinality of a set from \( MDS(G) \) is called domination number of graph \( G \) and is denoted by \( \gamma(G) \).

**Definition 2.3.** Let \( G \) be a connected graph and \( v \) be a vertex of \( G \). The eccentricity of \( v \) is denoted by \( e(v) \) is defined by \( e(v) = \max\{d(u,v) : u \in V\} \). The radius of graph \( G \) is defined as \( \text{rad}(G) = \min\{e(v) : v \in V\} \) while the diameter of graph \( G \) is defined as \( \text{diam}(G) = \max\{e(v) : v \in V\} \).

In a graph \( G \), a vertex \( u \) is said to be an eccentric vertex of a vertex \( v \) if \( d(u,v) = e(v) = \text{eccentricity of vertex } v \). The eccentric set of a vertex \( v \) is denoted by \( E(v) \) and is defined as \( E(v) = \{u \in V(G) : d(u,v) = e(v)\} \).

**Definition 2.4.** A set \( D \subseteq V(G) \) is an eccentric dominating set of \( G \) if \( D \) is a dominating set of \( G \) and for every vertex \( v \in V - D \), there exists at least one eccentric vertex of \( v \) in \( D \).

An eccentric dominating set \( D \) of graph \( G \) is a minimal eccentric dominating set if no proper subset \( D' \subset D \) is an eccentric dominating set of graph \( G \).

The cardinality of a minimal eccentric dominating set of a graph \( G \) is called eccentric domination number of \( G \) which is denoted as \( \gamma_{ed}(G) \).
The concept of eccentric domination was introduced by Janakiraman et al [5].

3. SOME DEFINITIONS AND EXISTING RESULTS

Definition 3.1. [8] The m-shadow graph $D_m(G)$ of a connected graph $G$ is constructed by taking $m$ copies of $G$, say $G_1, G_2, \cdots, G_m$, then join each vertex $u$ in $G_i$ to the neighbours of the corresponding vertex $v$ in $G_j$, $1 \leq i, j \leq m$.

Definition 3.2. [10] The extended m-shadow graph $D^*_m(G)$ of a connected graph $G$ is constructed by taking $m$ copies of $G$, say $G_1, G_2, \cdots, G_m$, then join each vertex $u$ in $G_i$ to the neighbours of the corresponding vertex $v$ and with $v$ in $G_j$, $1 \leq i, j \leq m$.

Definition 3.3. [8] The $m$-splitting graph $Spl_m(G)$ of a graph $G$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to $v$ in $G$.

Definition 3.4. [9] The extended $m$-splitting graph $Spl^*_m(G)$ of a graph $G$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that $v_i, 1 \leq i \leq m$ is adjacent to each vertex that is adjacent to $v$ in $G$ and also adjacent to $v$.

Definition 3.5. [1]

The square of graph $G$, denoted by $G^2$, is defined to be the graph with the same vertex set as $G$ and in which two vertices $u$ and $v$ are joined by an edge if and only if in $G$ we have $1 \leq d(u, v) \leq 2$.

Janakiraman et al [5] have proved the following results.

Theorem 3.6. [5] For any complete graph $K_n$, $\gamma_{ed}(K_n) = 1$.

Theorem 3.7. [5] For any cycle $C_n$,

$$
\gamma_{ed}(C_n) = \begin{cases} 
\frac{n}{2} & \text{; } n \text{ is even} \\
\frac{n}{3} & \text{; } n = 3m \text{ and } n \text{ is odd} \\
\left\lfloor \frac{n}{3} \right\rfloor & \text{; } n = 3m + 1 \text{ and } n \text{ is odd} \\
\left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{; } n = 3m + 2 \text{ and } n \text{ is odd}
\end{cases}
$$
In the following Theorem 4.1, we have improved the results proved in Theorem 3.7.

4. MAIN RESULTS

Theorem 4.1. For any cycle $C_n$, $\gamma_{ed}(C_n) = \begin{cases} \frac{n}{2} & \text{; } n \text{ is even} \\ 2 \left\lceil \frac{n+3}{6} \right\rceil - 1 & \text{; } n \text{ is odd} \end{cases}$

Proof. Let $V = V(C_n) = \{v_1, v_2, \ldots, v_n\}$ is the set of all vertices of cycle $C_n$, $\forall n \geq 3$ and let $D$ is a minimal eccentric dominating set of $C_n$.

To dominate all vertices of $C_n$, we have to take one middle vertex from every three consecutive vertices of $C_n$ in $D$.

Case-1: $n$ is an even number.

Let $n = 2t$. Here, each vertex of cycle $C_n$ has exactly one eccentric vertex and for each $k, 1 \leq k \leq t$ the eccentric vertex of the vertex $v_k$ is $v_{k+t}$ and vice versa.

So, we have to choose, either $v_k$ or $v_{k+t}$, for $1 \leq k \leq t$, to construct the set $D$.

Case-(i): $n = 4$.

Then, $D = \{v_1, v_2\}$ is a minimal eccentric dominating set of $C_4$.

So, $\gamma_{ed}(C_4) = 2 = \frac{4}{2} = \frac{n}{2}$

Case-(ii): $n \geq 6$.

Here, we have to take one vertex from every two vertices, from $v_1$ to $v_{t+1}$ and one vertex from every two vertices, from $v_{t+2}$ to $v_n$.

Case-(a): $t$ is odd.

Then, $t + 1$ is an even number and $n - 1$ is an odd number.

Then, $D = \{v_1, v_3, v_5, \ldots, v_t\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \ldots, v_{n-1}\}$

$= \{v_1, v_3, v_5, \ldots, v_t, v_{t+2}, v_{t+4}, \ldots, v_{n-1}\}$

$= \{v_1, v_3, v_5, \ldots, v_{n-1}\}$ is a minimal eccentric dominating set of $C_n$.

Thus, $\gamma_{ed}(C_n) = |D| = \frac{n}{2}$.

Case-(b): $t$ is even.

Let, $t = 2u$, then $n = 2t = 2(2u) = 4u$.

Then, $D = \{v_1, v_3, v_5, \ldots, v_{t-1}\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \ldots, v_n\}$ is a minimal eccentric dominating set of $C_n$. 
Thus, \( \gamma_{ed}(C_n) = |D| = \left\lceil \frac{t-1}{2} \right\rceil + \left\lceil \frac{n-(t+1)}{2} \right\rceil = \left\lceil \frac{2u-1}{2} \right\rceil + \left\lceil \frac{4u-2u-1}{2} \right\rceil \)

\[= \left\lceil \frac{2u-1}{2} \right\rceil + \left\lceil \frac{2u-1}{2} \right\rceil = 2 \left\lceil \frac{2u-1}{2} \right\rceil = 2 \left\lceil u - \frac{1}{2} \right\rceil \]

\[= 2(u) \text{ (Since } u \text{ is an integer)} \]

\[= t = \frac{n}{2}. \]

**Case-2: n is an odd number.**

Let \( n = 2t + 1. \)

Here, each vertex has exactly two eccentric vertices and for each \( k, 1 \leq k \leq t \) the eccentric vertices of the vertex \( v_k \) are \( v_{k+t} \) and \( v_{k+t+1} \) and vice versa.

The eccentric vertices of the vertex \( v_{t+1} \) are \( v_{n} \) and \( v_{1} \) and vice versa.

**Case-(i): n = 3.**

Then, \( D = \{v_1\} \) is a minimal eccentric dominating set of \( C_3. \)

So, \( \gamma_{ed}(C_3) = 1 = 2(1) - 1 = 2 \left\lceil \frac{6}{6} \right\rceil - 1 = 2 \left\lceil \frac{3}{6} \right\rceil - 1 = 2 \left\lceil \frac{n+3}{6} \right\rceil - 1. \)

**Case-(ii): n = 5.**

Then, \( D = \{v_1,v_3,v_4\} \) is a minimal eccentric dominating set of \( C_5. \)

So, \( \gamma_{ed}(C_5) = 3 = 2(2) - 1 = 2 \left\lceil \frac{8}{6} \right\rceil - 1 = 2 \left\lceil \frac{5}{6} \right\rceil - 1 = 2 \left\lceil \frac{n+3}{6} \right\rceil - 1. \)

**Case-(iii): n \geq 7.**

Here, we have to choose, one vertex from every three vertices, from \( v_1 \) to \( v_{t+2} \) and one vertex from every three vertices, from \( v_{t+3} \) to \( v_{n} \), to construct the set \( D. \)

Then, \( D = \{v_1,v_4,v_7,\ldots,v_s\} \cup \{v_{t+3},v_{t+6},v_{t+9},\ldots,v_m\} \) is a minimal eccentric dominating set of \( C_n, \) where \( t \leq s \leq t+2 \) and \( n-2 \leq m \leq n. \)

Thus, \( \gamma_{ed}(C_n) = |D| = \left\lceil \frac{t+2}{3} \right\rceil + \left\lceil \frac{n-(t+2)}{3} \right\rceil = \left\lceil \frac{n-1}{2} + \frac{2}{3} \right\rceil + \left\lceil \frac{n-n-1}{2} - \frac{2}{3} \right\rceil \)

\[= \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil = \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n+3-6}{6} \right\rceil \]

\[= \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n+3}{6} - 1 \right\rceil = \left\lceil \frac{n+3}{6} \right\rceil + \left\lceil \frac{n+3}{6} \right\rceil - 1 \]
\begin{equation*}
= 2 \left\lceil \frac{n+3}{6} \right\rceil - 1.
\end{equation*}

Hence, \( \gamma_{ed}(C_n) = \begin{cases} 
\frac{n}{2} & ; n \text{ is even} \\
2 \left\lceil \frac{n+3}{6} \right\rceil - 1 & ; n \text{ is odd}
\end{cases} \)

\textbf{Illustration 4.2.} Minimal eccentric dominating set of cycle \( C_{11} = \{v_1, v_4, v_7, v_8, v_{11}\} \), is shown in Figure 1.

Where, \( \gamma_{ed}(C_{11}) = 5 = 2(3) - 1 = 2 \left\lceil \frac{14}{6} \right\rceil - 1 = 2 \left\lfloor \frac{11+3}{6} \right\rfloor - 1 \)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Minimal eccentric dominating set of cycle \( C_{11} \)}
\end{figure}

\textbf{Theorem 4.3.} \( \gamma_{ed}[D_m^*(C_n)] = \begin{cases} 
1 & ; n = 3 \\
n & ; n \text{ is even, } n > 3 \\
\left\lceil \frac{n}{2} \right\rceil & ; n \text{ is odd, } n > 3
\end{cases} \)

\textbf{Proof.} Let \( C^1_n, C^2_n, C^3_n, \ldots, C^n_m \) be the \( m \) copies of cycle \( C_n \) in the extended \( m \)-shadow graph of \( C_n \).

Let \( \{v^1_i, v^2_i, v^3_i, \ldots, v^n_i\} \) is the vertex set of cycle \( C^i_n, 1 \leq i \leq m \) and the set \( V \) is the set of all vertices of all copies of cycle \( C_n \).

Let the set \( D \) is a minimal eccentric dominating set of the extended \( m \)-shadow graph of \( C_n \), i.e. \( D_m^*(C_n) \).

Now, in the graph \( D_m^*(C_n) \), for \( 1 \leq i \leq m \) the vertices \( v^i_j \) are adjacent to the vertices \( v^k_{j-1} \) and \( v^k_{j+1} \), for all \( 1 \leq k \leq m \) and \( 2 \leq j \leq n - 1 \) and \( v^i_j \) are also adjacent to \( v^l_j \), \( 1 \leq i \neq l \leq m \).
For $j = 1$, the vertices $v_i^j$ are adjacent to the vertices $v_{k}^i$ and $v_{n}^i$, $1 \leq k \leq m$ and $v_i^j$ are also adjacent to the vertices $v_{l}^i$, $1 \leq i \neq l \leq m$.

And similarly for $j = n$, the vertices $v_i^n$ are adjacent to the vertices $v_{k}^i$ and $v_{n}^i$, $1 \leq k \leq m$ and $v_i^n$ are also adjacent to the vertices $v_{l}^i$, $1 \leq i \neq l \leq m$.

If $n$ is an even number and $n = 2t$ then the eccentric vertices of the vertices $v_k^i$ are $v_{k+t}^i$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and if $n$ is an odd number and $n = 2t + 1$ then the eccentric vertices of the vertices $v_k^i$ are $v_{k+t}^i$ and $v_{k+t+1}^i$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

Looking to the adjacency of vertices of the graph $D_m^*(C_n)$, it is possible to dominate all the vertices of the graph $D_m^*(C_n)$, just by considering all the vertices of any one copy of cycle $C_n$ and we can also choose the eccentric vertices from any one copy of cycle $C_n$.

So, without lose of generality, let we take the vertices from $C_n^1$ to construct the set $D$.

**Case-1:** $n = 3$

Then, $D = \{v_1^1\}$ is a minimal eccentric dominating set of the graph $D_m^*(C_3)$, as the vertex $v_1^1$ dominates all vertices of $V - D$ and distance between any two vertices in $D_m^*(C_3)$ is 1, so $v_1^1$ is also eccentric vertex of all vertices of $V - D$.

Thus, $\gamma_{ed}[D_m^*(C_3)] = 1$

**Case-2:** $n$ is an even number, $n > 3$.

Let $n = 2t$ then the eccentric vertices of the vertices $v_k^i$ are $v_{k+t}^i$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

So, for each $k$, $1 \leq k \leq t$, we have to choose one vertex from $v_k^i$ and one vertex from $v_{k+t}^i$, for $1 \leq i, j \leq m$, to construct the set $D$. Without lose of generality, we may choose the vertices from the vertex set of cycle $C_n^1$. Then, $D = \{v_1^1, v_2^1, v_3^1, \ldots, v_n^1\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[D_m^*(C_n)] = |D| = n$

**Case-3:** $n$ is an odd number, $n > 3$.

Let $n = 2t + 1$ then the eccentric vertices of the vertices $v_k^i$ are $v_{k+t}^i$ and $v_{k+t+1}^i$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices $v_{i+1}^j$ are $v_n^i$ and $v_1^j$ and vice versa, for $1 \leq i, j \leq m$. 
So, for each \( k \), \( 1 \leq k \leq t \), we have to choose one vertex from \( v^i_k \) and one vertex from \( v^j_{k+t} \) or \( v^j_{k+t+1} \), for \( 1 \leq i, j \leq m \), to construct the set \( D \). Without lose of generality, we may choose the vertices from the vertex set of cycle \( C^1_n \).

To construct the set \( D \), we have to choose one vertex for every two vertices of cycle \( C^1_n \). Then, \( D = \{v^1_1, v^1_3, v^1_5, \ldots, v^1_n\} \) is a minimal eccentric dominating set.

Thus, \( \gamma_{ed}[D^*_m(C_n)] = |D| = \frac{n+1}{2} = \left(\frac{n}{2} + \frac{1}{2}\right) = \left\lceil \frac{n}{2} \right\rceil \), since \( \left(\frac{n}{2} + \frac{1}{2}\right) \) is an integer.

Hence, \( \gamma_{ed}[D^*_m(C_n)] = \begin{cases} 
1 & ; n = 3 \\
 n & ; n \text{ is even, } n > 3 \\
 \left\lceil \frac{n}{2} \right\rceil & ; n \text{ is odd, } n > 3 
\end{cases} \)

\[\square\]

**Illustration 4.4.** Minimal eccentric dominating set of extended 3-Shadow graph of \( C_5 = \{v_1, v_3, v_5\} \), is shown in Figure 2.

Where, \( \gamma_{ed}[D^*_3(C_5)] = 3 = \left\lceil \frac{5}{2} \right\rceil \)

![Figure 2](image-url)

\[\text{Figure 2. Minimal eccentric dominating set of } D^*_3(C_5)\]

**Theorem 4.5.** \( \gamma_{ed}[Spl^*_m(C_n)] = \begin{cases} 
 n & ; n \text{ is even} \\
 \left\lceil \frac{n}{2} \right\rceil & ; n \text{ is odd} 
\end{cases} \)
Proof. Let \( v_1, v_2, v_3, \cdots, v_n \) is the vertex set of cycle \( C_n \).

Let \( v^i_1, v^i_2, \cdots, v^i_n \) are the \( m \) copies of the vertex set of cycle \( C_n \), \( 1 \leq i \leq m \) in the extended \( m \)-splitting graph of \( C_n \).

Let the set \( D \) is a minimal eccentric dominating set of the extended \( m \)-splitting graph of \( C_n \), i.e. \( Spl^*_m(C_n) \).

Now, in the graph \( Spl^*_m(C_n) \), for \( 1 \leq i \leq m \) the vertices \( v_j \) and \( v^i_j \) are adjacent to the vertices \( v_{j-1} \) and \( v_{j+1} \) of cycle \( C_n \), and \( v^i_j \) are also adjacent to \( v_j \), for all \( 2 \leq j \leq n-1 \).

For \( j = 1 \), the vertices \( v_1 \) and \( v^i_1 \) are adjacent to the vertices \( v_2 \) and \( v_n \) and \( v^i_1 \) are also adjacent to the vertex \( v_1 \), \( 1 \leq i \leq m \).

Similarly for \( j = n \), the vertices \( v_n \) and \( v^i_n \) are adjacent to the vertices \( v_{n-1} \) and \( v_1 \) and \( v^i_n \) are also adjacent to the vertex \( v_n \), \( 1 \leq i \leq m \).

Case-1: \( n \) is an even number.

Case-(i) \( n = 4 \)

For the graph \( Spl^*_m(C_4) \), the eccentric vertices of the vertices \( v_k \) and \( v^i_k \) are \( v_{k+2} \) and \( v^i_{k+2} \) and vice versa, for \( 1 \leq i, j \leq m \) and \( 1 \leq k \leq 2 \) and all the vertices \( v^i_k \) are the eccentric vertices to one another, for \( 1 \leq k \leq 4 \) and \( 1 \leq i \leq m \).

So, for each \( k, 1 \leq k \leq 2 \), we have to choose one vertex from \( v_k \) or \( v^i_k \) and one vertex from \( v_{k+2} \) or \( v^i_{k+2} \), for \( 1 \leq i, j \leq m \), to construct the set \( D \).

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle \( C_4 \) to construct the set \( D \). Then, \( D = \{v_1, v_2, v_3, v_4\} \) is a minimal eccentric dominating set.

Thus, \( \gamma_{ed}[Spl^*_m(C_4)] = |D| = 4 = n \).

Case-(ii) \( n > 4 \)

Let \( n = 2t \) then the eccentric vertices of the vertices \( v_k \) and \( v^i_k \) are \( v_{k+t} \) and \( v^i_{k+t} \) and vice versa, for \( 1 \leq i, j \leq m \) and \( 1 \leq k \leq t \).

So, for each \( k, 1 \leq k \leq t \), we have to choose one vertex from \( v_k \) or \( v^i_k \) and one vertex from \( v_{k+t} \) or \( v^i_{k+t} \), for \( 1 \leq i, j \leq m \), to construct the set \( D \).

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle \( C_n \) to construct the set \( D \). Then, \( D = \{v_1, v_2, v_3, \cdots, v_n\} \) is a minimal
Thus, $\gamma_{ed}[Spl_m^*(C_n)] = |D| = n$

**Case-2:** $n$ is an odd number.

**Case-(i) $n = 3$**

For the graph $Spl_m^*(C_3)$, all the vertices other than $v_k$ are the eccentric vertices of the vertex $v_k$, for $1 \leq k \leq 3$ and all the vertices $v_k^i$ are the eccentric vertices to one another, for $1 \leq k \leq 3$ and $1 \leq i \leq m$.

Moreover, for any $k$, $1 \leq k \leq 3$, the vertex $v_k$ dominates all vertices of the graph $Spl_m^*(C_3)$. So, to construct the set $D$, we have to choose one vertex from $v_k$ and one vertex from $v_k^i$, for $1 \leq k, l \leq 3$ and $1 \leq i \leq m$.

Then, without lose of generality, we may take, $D = \{v_1, v_1^1\}$, which is a minimal eccentric dominating set of $Spl_m^*(C_3)$.

Thus, $\gamma_{ed}[Spl_m^*(C_3)] = |D| = 3 = \left\lceil \frac{3}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$.

**Case-(ii) $n = 5$**

For the graph $Spl_m^*(C_5)$, the eccentric vertices of the vertices $v_k$ and $v_k^i$ are $v_{k+2}$, $v_{k+2}^j$, $v_k+3$ and $v_k^j+3$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq 2$ and the eccentric vertices of the vertices $v_3$ and $v_3^i$ are $v_5$, $v_5^j$, $v_1$ and $v_1^j$ and vice versa, for $1 \leq i, j \leq m$. Moreover, all the vertices $v_k^i$ are the eccentric vertices to one another, for $1 \leq k \leq 5$ and $1 \leq i \leq m$.

So, for each $k$, $1 \leq k \leq 2$, we can choose one vertex from $v_k$ or $v_k^i$ and one vertex from $v_{k+2}$ or $v_{k+2}^j$ or $v_{k+3}$ or $v_{k+3}^j$, for $1 \leq i, j \leq m$ and we can choose one vertex from $v_3$ or $v_3^i$ and one vertex from $v_5$ or $v_5^i$ or $v_1$ or $v_1^i$, for $1 \leq i, j \leq m$, to construct the set $D$.

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle $C_5$ to construct the set $D$. Moreover, to construct the set $D$, we have to choose one vertex for every two vertices of cycle $C_5$. Then, $D = \{v_1, v_3, v_5\}$ is a minimal eccentric dominating set.

Thus, $\gamma_{ed}[Spl_m^*(C_5)] = |D| = 3 = \left\lceil \frac{5}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$.

**Case-(iii) $n > 5$.**
Let \( n = 2t + 1 \) then for the graph \( Spl_m^r(C_n) \), the eccentric vertices of the vertices \( v_k \) and \( v'_k \) are \( v_{k+t}, v'_{k+t}, v_{k+t+1} \) and \( v'_{k+t+1} \) and vice versa, for \( 1 \leq i, j \leq m \) and \( 1 \leq k \leq t \) and the eccentric vertices of the vertices \( v_{t+1} \) and \( v'_{t+1} \) are \( v_n, v'_n, v_1 \) and \( v'_1 \) and vice versa, for \( 1 \leq i, j \leq m \).

So, for each \( k, 1 \leq k \leq t \), we can choose one vertex from \( v_k \) or \( v'_k \) and one vertex from \( v_{k+t} \) or \( v'_{k+t} \) or \( v_{k+t+1} \) or \( v'_{k+t+1} \), for \( 1 \leq i, j \leq m \) and we can choose one vertex from \( v_{t+1} \) or \( v'_{t+1} \) and one vertex from \( v_n \) or \( v'_n \) or \( v_1 \) or \( v'_1 \), for \( 1 \leq i, j \leq m \), to construct the set \( D \).

But to dominate all vertices and for minimum cardinality, we have to choose the vertices from the vertex set of cycle \( C_n \) to construct the set \( D \). Moreover, to construct the set \( D \), we have to choose one vertex for every two vertices of cycle \( C_n \). Then, \( D = \{v_1, v_3, v_5, \ldots, v_n\} \) is a minimal eccentric dominating set.

Thus, \( \gamma_{ed}[Spl_m^r(C_n)] = |D| = \frac{n+1}{2} = \left( \frac{n}{2} + \frac{1}{2} \right) = \left\lceil \frac{n}{2} \right\rceil \), since \( \left( \frac{n}{2} + \frac{1}{2} \right) \) is an integer.

Hence, \( \gamma_{ed}[Spl_m^r(C_n)] = \left\{ \begin{array}{ll} \frac{n}{2} & ; n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil & ; n \text{ is odd} \end{array} \right. \)

Illustration 4.6. Minimal eccentric dominating set of extended 2-Splitting graph of \( C_6 = \{v_1, v_2, v_3, v_4, v_5, v_6\} \), is shown in Figure 3.

Where, \( \gamma_{ed}[Spl_2^r(C_6)] = 6 \)

![Figure 3. Minimal eccentric dominating set of Extended 2-Splitting graph of \( C_6 \)](image-url)
Theorem 4.7. For any square of a cycle $C_n$,

$$\gamma_{ed}(C_n^2) = \begin{cases} \frac{n}{2} & ; n \equiv 2(\text{mod}4) \\ 2 \left\lfloor \frac{n+5}{10} \right\rfloor - 1 & ; n \equiv 1(\text{mod}4) \\ \left\lceil \frac{n+1}{6} \right\rceil + \left\lfloor \frac{n-3}{6} \right\rfloor & ; n \equiv 3(\text{mod}4) \\ \left\lceil \frac{n+2}{8} \right\rceil + \left\lfloor \frac{n-4}{8} \right\rfloor & ; n \equiv 0(\text{mod}4) \end{cases}$$

Proof. Let $V = V(C_n^2) = \{v_1, v_2, \cdots, v_n\}$ is the set of all vertices of graph $C_n^2$, $\forall n \geq 3$ and let $D$ is a minimal eccentric dominating set of $C_n^2$.

To dominate all vertices of $C_n^2$, we have to take one vertex from every five consecutive vertices of $C_n^2$ in $D$.

**Case-1: $n \equiv 2(\text{mod}4)$**

Let $n = 4u + 2$ and $t = 2u + 1$ then $n = 2t$, where $n$ is an even number and $t$ is an odd number. Here, each vertex has exactly one eccentric vertex and for each $k, 1 \leq k \leq t$ the eccentric vertex of the vertex $v_k$ is $v_{k+t}$ and vice versa.

So, we have to choose, either $v_k$ or $v_{k+t}$, for $1 \leq k \leq t$, to construct the set $D$. Moreover, we have to take one vertex from every two vertices, from $v_1$ to $v_{t+1}$ and one vertex from every two vertices, from $v_{t+2}$ to $v_n$.

Then, $D = \{v_1, v_3, v_5, \cdots, v_t\} \cup \{v_{t+2}, v_{t+4}, v_{t+6}, \cdots, v_{n-1}\} = \{v_1, v_3, v_5, \cdots, v_{n-1}\}$ is a minimal eccentric dominating set of $C_n^2$.

Thus, $\gamma_{ed}(C_n^2) = |D| = \frac{n}{2}$.

**Case-2: $n \equiv 1(\text{mod}4)$**

**Case-(i): $n = 5$**

Then, the graph $C_5^2$ is same as the complete graph $K_5$.

So, $\gamma_{ed}(C_5^2) = \gamma_{ed}(K_5) = 1$ (Using Theorem 3.6)

$$= 2(1) - 1 = 2 \left\lfloor \frac{10}{10} \right\rfloor - 1 = 2 \left\lfloor \frac{5+5}{10} \right\rfloor - 1 = 2 \left\lfloor \frac{n+5}{10} \right\rfloor - 1.$$
Let $n = 4u + 1$ and $t = 2u$ then $n = 2t + 1$, where $n$ is an odd number and $t$ is an even number.

Here, each vertex has exactly four eccentric vertices and for each $k, 1 \leq k \leq t - 1$ the eccentric vertices of the vertex $v_k$ are $v_{k+t-1}, v_{k+t}, v_{k+t+1}$ and $v_{k+t+2}$ and vice versa.

The eccentric vertices of the vertex $v_t$ are $v_{n-2}, v_{n-1}, v_n$ and $v_1$ and vice versa.

The eccentric vertices of the vertex $v_{t+1}$ are $v_{n-1}, v_n, v_1$ and $v_2$ and vice versa.

The eccentric vertices of the vertex $v_{t+2}$ are $v_n, v_1, v_2$ and $v_3$ and vice versa.

So, we have to choose, one vertex from every five vertices, from $v_1$ to $v_{t+3}$ and one vertex from every five vertices, from $v_{t+4}$ to $v_n$, to construct the set $D$.

Then, $D = \{v_1, v_6, v_{11}, \ldots, v_s\} \cup \{v_{t+4}, v_{t+9}, v_{t+14}, \ldots, v_m\}$ is a minimal eccentric dominating set of $C^2_n$, where $s \leq t + 3$ and $m \leq n$.

Thus, $\gamma_{ed}(C^2_n) = |D| = \left\lceil \frac{t+3}{5} \right\rceil + \left\lceil \frac{n-(t+3)}{5} \right\rceil = \left\lceil \frac{n-1}{2} + 3 \right\rceil + \left\lceil \frac{n-1}{2} - 3 \right\rceil$

\[
= \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{n-5}{10} \right\rceil = \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{(n+5)-10}{10} \right\rceil
\]

\[
= \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{n+5}{10} - 1 \right\rceil = \left\lceil \frac{n+5}{10} \right\rceil + \left\lceil \frac{n+5}{10} \right\rceil - 1
\]

\[
= 2 \left\lceil \frac{n+5}{10} \right\rceil - 1.
\]

**Case-3: $n \equiv 3(mod4)$**

**Case-(i): $n = 3$**

Then, the graph $C^2_3$ is same as the graph $C_3$.

So, $\gamma_{ed}(C^2_3) = \gamma_{ed}(C_3) = 1$ (Using Theorem 4.1)

\[
= 1 + 0 = \left\lceil \frac{4}{6} \right\rceil + \left\lceil \frac{0}{6} \right\rceil = \left\lceil \frac{3+1}{6} \right\rceil + \left\lceil \frac{3-3}{6} \right\rceil = \left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil.
\]

**Case-(ii): $n > 3$**

Let $n = 4u + 3$ and $t = 2u + 1$ then $n = 2t + 1$, where $n$ and $t$ are both odd numbers.

Here, each vertex has exactly two eccentric vertices and for each $k, 1 \leq k \leq t$ the eccentric vertices of the vertex $v_k$ are $v_{k+t}$ and $v_{k+t+1}$ and vice versa.

The eccentric vertices of the vertex $v_{t+1}$ are $v_n$ and $v_1$ and vice versa.
So, we have to choose, one vertex from every three vertices, from $v_1$ to $v_{t+1}$ and one vertex from every three vertices, from $v_{t+3}$ to $v_n$, to construct the set $D$.

Then, $D = \{v_1, v_4, v_7, \cdots, v_s\} \cup \{v_{t+3}, v_{t+6}, v_{t+9}, \cdots, v_m\}$ is a minimal eccentric dominating set of $C_n^2$, where $s \leq t + 1$ and $m \leq n$.

Thus, $\gamma_{ed}(C_n^2) = |D| = \left\lceil \frac{t+1}{3} \right\rceil + \left\lceil \frac{n-(t+2)}{3} \right\rceil = \left\lceil \frac{n-1}{2} + 1 \right\rceil + \left\lceil \frac{n-1}{2} - 2 \right\rceil = \left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil$.

Case-4: $n \equiv 0 (mod 4)$

Case-(i): $n = 4$

Then, the graph $C_4^2$ is same as the complete graph $K_4$.

So, $\gamma_{ed}(C_4^2) = \gamma_{ed}(K_4) = 1$ (Using Theorem 3.6)

$$= 1 + 0 = \left\lceil \frac{6}{8} \right\rceil + \left\lceil \frac{0}{8} \right\rceil = \left\lceil \frac{4+2}{8} \right\rceil + \left\lceil \frac{4-4}{8} \right\rceil = \left\lceil \frac{n+2}{8} \right\rceil + \left\lceil \frac{n-4}{8} \right\rceil.$$

Case-(ii): $n > 4$

Let $n = 4u$ and $t = 2u$ then $n = 2t$, where $n$ and $t$ are both even numbers.

Here, each vertex has exactly three eccentric vertices and for each $k, 1 \leq k \leq t - 1$ the eccentric vertices of the vertex $v_k$ are $v_{k+t-1}, v_{k+t}$ and $v_{k+t+1}$ and vice versa.

The eccentric vertices of the vertex $v_t$ are $v_{t-1}, v_n$ and $v_1$ and vice versa.

The eccentric vertices of the vertex $v_{t+1}$ are $v_n, v_1$ and $v_2$ and vice versa.

So, we have to choose, one vertex from every four vertices, from $v_1$ to $v_{t+1}$ and one vertex from every four vertices, from $v_{t+3}$ to $v_n$, to construct the set $D$.

Then, $D = \{v_1, v_5, v_9, \cdots, v_s\} \cup \{v_{t+3}, v_{t+7}, v_{t+11}, \cdots, v_m\}$ is a minimal eccentric dominating set of $C_n^2$, where $s \leq t + 1$ and $m \leq n$.

Thus, $\gamma_{ed}(C_n^2) = |D| = \left\lceil \frac{t+1}{4} \right\rceil + \left\lceil \frac{n-(t+2)}{4} \right\rceil = \left\lceil \frac{n+2}{4} \right\rceil + \left\lceil \frac{n-2}{4} \right\rceil = \left\lceil \frac{n+2}{8} \right\rceil + \left\lceil \frac{n-4}{8} \right\rceil.$
Hence, $\gamma_{ed}(C_n^2) =$ \[ \begin{array}{ll}
\frac{n}{2} & ; n \equiv 2(\text{mod}4) \\
2 \left\lceil \frac{n+5}{10} \right\rceil - 1 & ; n \equiv 1(\text{mod}4) \\
\left\lceil \frac{n+1}{6} \right\rceil + \left\lceil \frac{n-3}{6} \right\rceil & ; n \equiv 3(\text{mod}4) \\
\left\lceil \frac{n+2}{8} \right\rceil + \left\lceil \frac{n-4}{8} \right\rceil & ; n \equiv 0(\text{mod}4) 
\end{array} \]

Illustration 4.8. Minimal eccentric dominating set of $C_9^2 = \{v_1, v_6, v_8\}$, is shown in Figure 4.

Where, $\gamma_{ed}(C_9^2) = 3 = 2(2) - 1 = 2 \left\lceil \frac{14}{10} \right\rceil - 1 = 2 \left\lceil \frac{9+5}{10} \right\rceil - 1$

\[ \text{Figure 4. Minimal eccentric dominating set of cycle } C_9^2 \]

Theorem 4.9. $\gamma_{ed}[D_m(C_n)] =$ \[ \begin{array}{ll}
n & ; n = 3 \text{ or } n \text{ is an even number with } n \geq 6 \\
2 & ; n = 4 \\
3 & ; n = 5 \\
\left\lceil \frac{2n}{3} \right\rceil & ; n \text{ is an odd number with } n \geq 7
\end{array} \]

Proof. Let $C_n^1, C_n^2, C_n^3, \ldots, C_n^m$ are the $m$ copies of cycle $C_n$ in the $m$–shadow graph of $C_n$.

Let $\{v_1^i, v_2^i, v_3^i, \ldots, v_n^i\}$ is the vertex set of cycle $C_n^i, 1 \leq i \leq m$ and the set $V$ is the set of all vertices of all copies of cycle $C_n$. 

\[ \text{Illustration 4.8. Minimal eccentric dominating set of } C_9^2 = \{v_1, v_6, v_8\}, \text{ is shown in Figure 4.} \]

Where, $\gamma_{ed}(C_9^2) = 3 = 2(2) - 1 = 2 \left\lceil \frac{14}{10} \right\rceil - 1 = 2 \left\lceil \frac{9+5}{10} \right\rceil - 1$

\[ \text{Figure 4. Minimal eccentric dominating set of cycle } C_9^2 \]

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2 & ; n = 4 \\
3 & ; n = 5 \\
\left\lceil \frac{2n}{3} \right\rceil & ; n \text{ is an odd number with } n \geq 7
\end{array} \]

Proof. Let $C_n^1, C_n^2, C_n^3, \ldots, C_n^m$ are the $m$ copies of cycle $C_n$ in the $m$–shadow graph of $C_n$.

Let $\{v_1^i, v_2^i, v_3^i, \ldots, v_n^i\}$ is the vertex set of cycle $C_n^i, 1 \leq i \leq m$ and the set $V$ is the set of all vertices of all copies of cycle $C_n$. 

Let the set $D$ is a minimal eccentric dominating set of the $m$–shadow graph of $C_n$, i.e. $D_m(C_n)$. Now, in the graph $D_m(C_n)$, the vertices $v^i_k$ are adjacent to the vertices $v^i_{k-1}$ and $v^i_{k+1}$, for all $1 \leq i, j \leq m$ and $2 \leq k \leq n - 1$.

For $k = 1$, the vertices $v^i_1$ are adjacent to the vertices $v^j_2$ and $v^j_n$, $1 \leq i, j \leq m$.

And similarly for $k = n$, the vertices $v^i_n$ are adjacent to the vertices $v^j_{n-1}$ and $v^j_1$, $1 \leq i, j \leq m$.

**Case-1: $n = 3$**

Then, for the graph $D_m(C_3)$, the vertices $v^1_k, v^2_k, v^3_k, \ldots, v^m_k$ are the eccentric vertices to each other for $1 \leq k \leq 3$.

So, for each $k$, $1 \leq k \leq 3$, we have to choose one vertex from the vertices $v^1_k, v^2_k, v^3_k, \ldots, v^m_k$ to construct the set $D$. Moreover, any two vertices of any one copy of cycle $C_3$ will dominate to all remaining vertices of $D_m(C_3)$.

Without lose of generality, let we take the vertices from the vertex set of $C_3$ to construct the set $D$.

Then, $D = \{v^1_1, v^1_2, v^1_3\}$ is a minimal eccentric dominating set of the graph $D_m(C_3)$.

Thus, $\gamma_{ed}[D_m(C_3)] = 3 = n$

**Case-2: $n = 4$**

Then, for the graph $D_m(C_4)$, the vertices $v^1_k, v^2_k, v^3_k, \ldots, v^m_k, v^1_{k+2}, v^2_{k+2}, v^3_{k+2}, \ldots, v^m_{k+2}$ are the eccentric vertices to each other for $1 \leq k \leq 2$.

So, for each $k$, $1 \leq k \leq 2$, we have to choose one vertex from the vertices $v^1_k, v^2_k, v^3_k, \ldots, v^m_k, v^1_{k+2}, v^2_{k+2}, v^3_{k+2}, \ldots, v^m_{k+2}$ to construct the set $D$. Moreover, any two consecutive vertices of any one copy of cycle $C_4$ will dominate to all remaining vertices of $D_m(C_4)$.

Without lose of generality, let we take the vertices from the vertex set of $C_4$ to construct the set $D$.

Then, $D = \{v^1_1, v^1_2\}$ is a minimal eccentric dominating set of the graph $D_m(C_4)$.

Thus, $\gamma_{ed}[D_m(C_4)] = 2$

**Case-3: $n = 5$**

Then, for the graph $D_m(C_5)$, the vertices $v^1_k, v^2_k, v^3_k, \ldots, v^m_k$ are the eccentric vertices to each other for $1 \leq k \leq 5$. Moreover, the eccentric vertices of the vertices $v^i_k$ are $v^i_{k+2}$ and $v^i_{k+3}$ and vice versa, for $1 \leq k \leq 2$ and $1 \leq i, j \leq m$ and the eccentric vertices of the vertices $v^i_3$ are $v^i_5$ and $v^i_1$. 
and vice versa, for $1 \leq i, j \leq m$. Moreover, any three consecutive vertices of any one copy of cycle $C_5$ will dominate to all remaining vertices of $D_m(C_5)$.

Without lose of generality, let we take the vertices from $C^1_k$ to construct the set $D$.

Then, $D = \{v^1_1, v^1_2, v^1_3\}$ is a minimal eccentric dominating set of the graph $D_m(C_5)$.

Thus, $\gamma_{ed}[D_m(C_5)] = 3$

**Case-4: $n$ is an even number, with $n \geq 6$.**

Let $n = 2t$, then the eccentric vertices of the vertices $v^i_k$ are $v^i_{k+t}$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

So, for each $k, 1 \leq k \leq t$, we have to choose one vertex from $v^i_k$ and one vertex from $v^i_{k+t}$, for $1 \leq i, j \leq m$, to construct the set $D$.

Without lose of generality, we may choose the vertices from the vertex set of cycle $C^1_n$.

Then, $D = \{v^1_1, v^1_2, v^1_3, \ldots, v^1_n\}$ is a minimal eccentric dominating set, as they are dominating to all vertices and eccentric vertices of all vertices of the set $V - D$ of the graph $D_m(C_n)$.

Thus, $\gamma_{ed}[D_m(C_n)] = |D| = n$

**Case-5: $n$ is an odd number, with $n \geq 7$.**

Let $n = 2t + 1$ then the eccentric vertices of the vertices $v^i_k$ are $v^i_{k+t}$ and $v^i_{k+t+1}$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices $v^i_{t+1}$ are $v^i_n$ and $v^i_1$ and vice versa, for $1 \leq i, j \leq m$.

So, for each $k, 1 \leq k \leq t$, we can choose one vertex from $v^i_k$ and one vertex from $v^i_{k+t}$ or $v^i_{k+t+1}$, for $1 \leq i, j \leq m$, to construct the set $D$.

Without lose of generality, we may choose the vertices from the vertex set of cycle $C^1_n$.

To construct the set $D$, we have to choose two consecutive vertices for every three vertices of cycle $C^1_n$.

**Case-(i): $t = 3s$.**

Then, $n = 2t + 1 = 2(3s) + 1 = 6s + 1$

So, $D = \{v^1_1, v^1_2, v^1_4, v^1_5, v^1_6, \ldots, v^1_{6s-2}, v^1_{6s-1}, v^1_{6s+1}\}$ is a minimal eccentric dominating set.

Let $D' = \{v^1_1, v^1_4, v^1_7, \ldots, v^1_{6s-2}\} = \{v^1_1, v^1_4, v^1_7, \ldots, v^1_{3l-2}\}$ where, $l = 2s$ and so, $|D'| = l = 2s$.

Moreover, $|D| = 2|D'| + 1 = 2(2s) + 1 = 4s + 1 = 4 \left( \frac{n-1}{6} \right) + 1 = \frac{2n-2}{3} + 1 = \frac{2n-2+3}{3}$
Moreover, \( D = \{v_1^1, v_2^1, v_3^1, v_4^1, v_5^1, v_6^1, \ldots, v_{6s+1}^1, v_{6s+2}^1\} \) is a minimal eccentric dominating set.

Let \( D' = \{v_1^1, v_4^1, v_7^1, \ldots, v_{6s+1}^1\} = \{v_1^1, v_4^1, v_7^1, \ldots, v_{3l+1}^1\} \) where, \( l = 2s \) and so, \(|D'| = l + 1 = 2s + 1\).

Moreover, \(|D| = 2|D'| = 2(2s + 1) = 4s + 2 = 4 \left( \frac{n - 3}{6} \right) + 2 = \frac{2n - 6}{3} + 2 = \frac{2n - 6 + 6}{3} = \frac{2n}{3}\), since \( \frac{2n}{3} \) is an integer.

Thus, \( \gamma_{ed}[D_m(C_n)] = |D| = \left\lfloor \frac{2n}{3} \right\rfloor \).

Case-(ii): \( t = 3s + 1 \).

Then, \( n = 2t + 1 = 2(3s + 1) + 1 = 6s + 3 \)

So, \( D = \{v_1^1, v_2^1, v_4^1, v_5^1, v_7^1, v_8^1, \ldots, v_{6s+1}^1, v_{6s+2}^1\} \) is a minimal eccentric dominating set.

Let \( D' = \{v_1^1, v_4^1, v_7^1, \ldots, v_{6s+4}^1\} = \{v_1^1, v_4^1, v_7^1, \ldots, v_{3l+4}^1\} \) where, \( l = 2s \) and so, \(|D'| = l + 2 = 2s + 2\).

Moreover, \(|D| = 2|D'| = 2(2s + 2) = 4s + 4 = 4 \left( \frac{n - 5}{6} \right) + 4 = \frac{2n - 10}{3} + 4 = \frac{2n - 10 + 12}{3} = \frac{2n + 2}{3} = \frac{2n}{3} + \frac{2}{3} = \left\lfloor \frac{2n}{3} \right\rfloor \), since \( \left( \frac{2n}{3} + \frac{2}{3} \right) \) is an integer.

Thus, \( \gamma_{ed}[D_m(C_n)] = |D| = \left\lfloor \frac{2n}{3} \right\rfloor \).

Thus, for all odd \( n \geq 7 \), \( \gamma_{ed}[D_m(C_n)] = |D| = \left\lfloor \frac{2n}{3} \right\rfloor \).
Hence, $\gamma_{ed}[D_m(C_n)] = \begin{cases} 
  n ; & n = 3 \text{ or } n \text{ is an even number with } n \geq 6 \\
  2 ; & n = 4 \\
  3 ; & n = 5 \\
  \left\lceil \frac{2n}{3} \right\rceil ; & n \text{ is an odd number with } n \geq 7 
\end{cases}$

Illustration 4.10. Minimal eccentric dominating set of 3-Shadow graph of $C_9 = \{v_1, v_2, v_4, v_5, v_7, v_8\}$, is shown in Figure 5.

Where, $\gamma_{ed}[D_3(C_9)] = 6 = \left\lceil \frac{18}{3} \right\rceil = \left\lceil \frac{2(9)}{3} \right\rceil$

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.png}
\caption{Minimal eccentric dominating set of $D_3(C_9)$}
\end{figure}

\textbf{Theorem 4.11.} $\gamma_{ed}[Spl_m(C_n)] = \begin{cases} 
  n ; & n = 3 \text{ or } n \text{ is an even number with } n \geq 4 \\
  5 ; & n = 5 \text{ and } m = 1 \\
  6 ; & n = 5 \text{ and } m \geq 2 \\
  \left\lceil \frac{2n}{3} \right\rceil ; & n \text{ is an odd number with } n \geq 7 
\end{cases}$
Proof. Let \( v_1, v_2, v_3, \cdots, v_n \) is the vertex set of cycle \( C_n \).

Let \( v_1^i, v_2^i, v_3^i, \cdots, v_n^i \) are the \( m \) copies of the vertex set of cycle \( C_n \), \( 1 \leq i \leq m \) in the \( m \)-splitting graph of \( C_n \).

Let the set \( D \) is a minimal eccentric dominating set of the \( m \)-splitting graph of \( C_n \), i.e. \( Spl_m(C_n) \).

Now, in the graph \( Spl_m(C_n) \), for \( 1 \leq i \leq m \), the vertices \( v_k \) and \( v_k^i \) are adjacent to the vertices \( v_{k-1} \) and \( v_{k+1} \) of cycle \( C_n \), for all \( 2 \leq k \leq n - 1 \).

For \( k = 1 \), the vertices \( v_1 \) and \( v_1^i \) are adjacent to the vertices \( v_2 \) and \( v_n \), for \( 1 \leq i \leq m \).

Similarly for \( k = n \), the vertices \( v_n \) and \( v_n^i \) are adjacent to the vertices \( v_{n-1} \) and \( v_1 \), for \( 1 \leq i \leq m \).

Case-1: \( n = 3 \)

Then, for the graph \( Spl_m(C_3) \), the vertices \( v_k, v_k^1, v_k^2, v_k^3, \cdots, v_k^m \) are the eccentric vertices to each other for \( 1 \leq k \leq 3 \). Also, the eccentric vertices of \( v_1^i \) are \( v_2^i \) and \( v_3^i \), the eccentric vertices of \( v_2^i \) are \( v_3^i \) and \( v_1^i \) and the eccentric vertices of \( v_3^i \) are \( v_1^i \) and \( v_2^i \), for \( 1 \leq i, j \leq m \).

So, for each \( k, 1 \leq k \leq 3 \), we have to choose one vertex from the vertices \( v_k, v_k^1, v_k^2, v_k^3, \cdots, v_k^m \) to construct the set \( D \).

Moreover, any two vertices of cycle \( C_3 \) will dominate to all remaining vertices of \( Spl_m(C_3) \).

For minimum cardinality, we have to take the vertices from cycle \( C_3 \) to construct the set \( D \).

Then, \( D = \{v_1, v_2, v_3\} \) is a minimal eccentric dominating set of the graph \( Spl_m(C_3) \).

Thus, \( \gamma_{ed}[Spl_m(C_3)] = 3 = n \)

Case-2: \( n = 4 \)

Then, for the graph \( Spl_m(C_4) \), the eccentric vertices of the vertex \( v_k \) are \( v_k^i, v_k+k^2 \) and \( v_k^i+k+2 \) for \( 1 \leq k \leq 2 \) and \( 1 \leq i \leq m \). And, the eccentric vertices of the vertex \( v_k \) are \( v_k^i, v_k+k^2 \) and \( v_k^i+k-2 \) for \( 3 \leq k \leq 4 \) and \( 1 \leq i \leq m \).

The eccentric vertices of the vertices \( v_1^i \) and \( v_3^i \) are \( v_2^i \) and \( v_4^i \) for \( 1 \leq i, j \leq m \) and vice versa.

So, we have to choose one vertex from \( v_1^i \) or \( v_3^i \) and one vertex from \( v_2^i \) or \( v_4^i \) to construct the set \( D \) for \( 1 \leq i \leq m \).

Moreover, any two consecutive vertices of cycle \( C_4 \) will dominate to all remaining vertices of \( Spl_m(C_4) \).
Then, \( D = \{v_1, v_2, v_1^1, v_2^1\} \) is a minimal eccentric dominating set of the graph \( Spl_m(C_4) \).

Thus, \( \gamma_{ed}[Spl_m(C_4)] = 4 = n \)

**Case-3: \( n = 5 \)**

Then, for the graph \( Spl_m(C_5) \), the eccentric vertices of the vertex \( v_k \) are \( v_k^i, v_{k+2}^i, v_{k+2}^i, v_k^i \) and \( v_{k+3}^i \) for \( 1 \leq k \leq 2 \) and \( 1 \leq i \leq m \), the eccentric vertices of the vertex \( v_k \) are \( v_k^i, v_{k-2}^i, v_{k-2}^i, v_{k-3} \) and \( v_{k-3}^i \) for \( 4 \leq k \leq 5 \) and \( 1 \leq i \leq m \). And, the eccentric vertices of the vertex \( v_3 \) are \( v_3^i, v_1, v_1^i, v_5 \) and \( v_5^i \) for \( 1 \leq i \leq m \).

The eccentric vertices of the vertices \( v_k^i \) are \( v_{k-1}^i \) and \( v_{k+1}^i \) for \( 2 \leq k \leq 4 \) and \( 1 \leq i, j \leq m \). The eccentric vertices of the vertices \( v_1^i \) are \( v_5^i \) and \( v_2^i \), for \( 1 \leq i, j \leq m \). The eccentric vertices of the vertices \( v_5^i \) are \( v_4^i \) and \( v_1^i \) for \( 1 \leq i, j \leq m \).

Moreover, any three consecutive vertices of cycle \( C_5 \) will dominate to all remaining vertices of \( Spl_m(C_5) \).

**Case-(i): \( m = 1 \)**

Then, for the graph \( Spl_1(C_5) \), the set \( D = \{v_1, v_2, v_3, v_1^1, v_3^1\} \) is a minimal eccentric dominating set.

So, \( \gamma_{ed}[Spl_1(C_5)] = 5 \)

**Case-(ii): \( m \geq 2 \)**

Then, for the graph \( Spl_m(C_5) \), the set \( D = \{v_1, v_2, v_3, v_1^1, v_2^1, v_3^1\} \) is a minimal eccentric dominating set.

So, \( \gamma_{ed}[Spl_m(C_5)] = 6 \)

**Case-4: \( n = 6 \)**

Then, for the graph \( Spl_m(C_6) \), the eccentric vertices of the vertices \( v_k \) and \( v_k^i \) are \( v_{k+3} \) and \( v_{k+3}^i \) for \( 1 \leq k \leq 3 \) and \( 1 \leq i, j \leq m \). And the eccentric vertices of the vertices \( v_k \) and \( v_k^i \) are \( v_{k-3} \) and \( v_{k-3}^i \) for \( 4 \leq k \leq 6 \) and \( 1 \leq i, j \leq m \).

The vertices \( v_{k-1}^i \) and \( v_{k+1}^j \) are also eccentric vertices of the vertices \( v_k^i \), for \( 2 \leq k \leq 5 \) and \( 1 \leq i, j \leq m \). The vertices \( v_6^j \) and \( v_2^j \) are also eccentric vertices of the vertices \( v_1^i \) and the vertices \( v_5^j \) and \( v_1^i \) are also eccentric vertices of the vertices \( v_6^j \), for \( 1 \leq i, j \leq m \).

So, for minimal eccentric dominating set, we have to choose the vertices from cycle \( C_6 \) to construct the set \( D \). And for each \( k, 1 \leq k \leq 6 \), we have to choose vertex \( v_k \) for eccentric vertices.
of all remaining vertices of the graph $Spl_m(C_6)$.  

Then, $D = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_6)$.

Thus, $\gamma_{ed}[Spl_m(C_6)] = 6 = n$

**Case-5: $n = 7$**

Then, for the graph $Spl_m(C_7)$, the eccentric vertices of the vertices $v_k$ and $v'_k$ are $v_{k+3}, v'_{k+3}, v_{k+4}$ and $v'_{k+4}$, for $1 \leq k \leq 3$ and $1 \leq i, j \leq m$. And the eccentric vertices of the vertices $v_k$ and $v'_k$ are $v_{k-3}, v'_{k-3}, v_{k-4}$ and $v'_{k-4}$, for $5 \leq k \leq 7$ and $1 \leq i, j \leq m$. The eccentric vertices of the vertices $v_4$ and $v'_4$ are $v_7, v'_7, v_1$ and $v'_1$, for $1 \leq i, j \leq m$.

The vertices $v'_{k-1}$ and $v'_{k+1}$ are also eccentric vertices of the vertices $v'_k$, for $2 \leq k \leq 6$ and $1 \leq i, j \leq m$. The vertices $v'_2$ and $v'_3$ are also eccentric vertices of the vertices $v'_1$ and the vertices $v'_6$ and $v'_1$ are also eccentric vertices of the vertices $v'_7$, for $1 \leq i, j \leq m$.

So, for minimal eccentric dominating set, we have to choose two consecutive vertices from every three vertices of cycle $C_7$ to construct the set $D$. Then, $D = \{v_1, v_2, v_4, v_5, v_7\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_7)$.

So, $\gamma_{ed}[Spl_m(C_7)] = 5 = \left\lceil \frac{14}{3} \right\rceil = \left\lceil \frac{2(7)}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil$

**Case-6: $n$ is an even number, with $n \geq 8$.**

Let $n = 2t$, then the eccentric vertices of the vertices $v_k$ and $v'_k$ are $v_{k+t}$ and $v'_{k+t}$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$.

So, for each $k, 1 \leq k \leq t$, for eccentric vertices, we have to choose one vertex from $v_k$ or $v'_k$ and one vertex from $v_{k+t}$ or $v'_{k+t}$, for $1 \leq i, j \leq m$, to construct the set $D$.

For minimum cardinality, to dominate all remaining vertices, we have to choose the vertices from the vertex set of cycle $C_n$, to construct the set $D$.

Then, $D = \{v_1, v_2, v_3, \cdots, v_n\}$ is a minimal eccentric dominating set of the graph $Spl_m(C_n)$.

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = n$

**Case-7: $n$ is an odd number, with $n \geq 9$.**

Let $n = 2t + 1$ then the eccentric vertices of the vertices $v_k$ and $v'_k$ are $v_{k+t}, v'_{k+t}, v_{k+t+1}$ and $v'_{k+t+1}$ and vice versa, for $1 \leq i, j \leq m$ and $1 \leq k \leq t$ and the eccentric vertices of the vertices $v_{t+1}$ and $v'_{t+1}$ are $v_n, v'_n, v_1$ and $v'_1$ and vice versa, for $1 \leq i, j \leq m$. 
Moreover, for each $k, 1 \leq k \leq t$, we can choose one vertex from $v_k$ or $v'_k$ and one vertex from $v_{k+t}$ or $v'_{k+t}$ or $v_{k+t+1}$ or $v'_{k+t+1}$, for $1 \leq i, j \leq m$, to construct the set $D$.

For minimum cardinality, we have to choose the vertices from the vertex set of cycle $C_n$ to construct the set $D$. And we have to choose two consecutive vertices from every three vertices of cycle $C_n$.

Without lose of generality, we may choose first two consecutive vertices from every three vertices of cycle $C_n$.

**Case-(i):** $t = 3s$.

Then, $n = 2t + 1 = 2(3s) + 1 = 6s + 1$

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s-2}, v_{6s-1}, v_{6s+1}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s-2}\} = \{v_1, v_4, v_7, \cdots, v_{3l-2}\}$ where, $l = 2s$ and so, $|D'| = l = 2s$.

Moreover, $|D| = 2|D'| + 1 = 2(2s) + 1 = 4s + 1 = 4 \left( \frac{n - 1}{6} \right) + 1 = \frac{2n - 2}{3} + 1 = \frac{2n - 2 + 3}{3}$

So, for each $k, 1 \leq k \leq t$, we can choose one vertex from $v_k$ or $v'_k$ and one vertex from $v_{k+t}$ or $v'_{k+t}$ or $v_{k+t+1}$ or $v'_{k+t+1}$, for $1 \leq i, j \leq m$, to construct the set $D$.

For minimum cardinality, we have to choose the vertices from the vertex set of cycle $C_n$ to construct the set $D$. And we have to choose two consecutive vertices from every three vertices of cycle $C_n$.

Without lose of generality, we may choose first two consecutive vertices from every three vertices of cycle $C_n$.

**Case-(i):** $t = 3s$.

Then, $n = 2t + 1 = 2(3s) + 1 = 6s + 1$

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s+1}, v_{6s+2}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s+1}\} = \{v_1, v_4, v_7, \cdots, v_{3l+1}\}$ where, $l = 2s$ and so, $|D'| = l + 1 = 2s + 1$.

Moreover, $|D| = 2|D'| + 1 = 2(2s) + 1 = 4s + 1 = 4 \left( \frac{n - 1}{6} \right) + 1 = \frac{2n - 2}{3} + 1 = \frac{2n - 2 + 3}{3}$

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil \frac{2n}{3} \right\rceil$.

**Case-(ii):** $t = 3s + 1$.

Then, $n = 2t + 1 = 2(3s + 1) + 1 = 6s + 3$.

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s+1}, v_{6s+2}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s+1}\} = \{v_1, v_4, v_7, \cdots, v_{3l+1}\}$ where, $l = 2s$ and so, $|D'| = l + 1 = 2s + 1$.

Moreover, $|D| = 2|D'| + 1 = 2(2s) + 1 = 4s + 1 = 4 \left( \frac{n - 1}{6} \right) + 1 = \frac{2n - 2}{3} + 1 = \frac{2n - 2 + 3}{3}$

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil \frac{2n}{3} \right\rceil$.

**Case-(iii):** $t = 3s + 2$.

Then, $n = 2t + 1 = 2(3s + 2) + 1 = 6s + 5$.

So, $D = \{v_1, v_2, v_4, v_5, v_7, v_8, \cdots, v_{6s+4}, v_{6s+5}\}$ is a minimal eccentric dominating set.

Let $D' = \{v_1, v_4, v_7, \cdots, v_{6s+4}\} = \{v_1, v_4, v_7, \cdots, v_{3l+4}\}$ where, $l = 2s$ and so, $|D'| = l + 2 = 2s + 2$.
Moreover, $|D| = 2|D'| = 2(2s + 2) = 4s + 4 = 4\left(\frac{n - 5}{6}\right) + 4 = \frac{2n - 10}{3} + 4 = \frac{2n - 10 + 12}{3} = \frac{2n + 2}{3} = \frac{2n}{3} + \frac{2}{3} = \left\lceil\frac{2n}{3}\right\rceil$, since $\left(\frac{2n}{3} + \frac{2}{3}\right)$ is an integer.

Thus, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil\frac{2n}{3}\right\rceil$.

Thus, for all odd $n \geq 9$, $\gamma_{ed}[Spl_m(C_n)] = |D| = \left\lceil\frac{2n}{3}\right\rceil$.

Hence, $\gamma_{ed}[Spl_m(C_n)] = \begin{cases} 
  n & \text{; } n = 3 \text{ or } n \text{ is an even number with } n \geq 4 \\
  5 & \text{; } n = 5 \text{ and } m = 1 \\
  6 & \text{; } n = 5 \text{ and } m \geq 2 \\
  \left\lceil\frac{2n}{3}\right\rceil & \text{; } n \text{ is an odd number with } n \geq 7
\end{cases}$

Illustration 4.12. Minimal eccentric dominating set of 1-splitting graph of $C_5 = \{v_1, v_2, v_3, v_1', v_3\}$, is shown in Figure 6 and minimal eccentric dominating set of 2-splitting graph of $C_8 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, is shown in Figure 7.

Where, $\gamma_{ed}[Spl_1(C_5)] = 5$ and $\gamma_{ed}[Spl_2(C_8)] = 8$
Figure 7. Minimal eccentric dominating set of $Spl_2(C_8)$

5. Conclusion

The concept of eccentric domination relates a dominating set with the eccentricity of a vertex. We have investigated eccentric domination number of cycle as well as some cycle related graphs including $m$–shadow, extended $m$–shadow, $m$–splitting, extended $m$–splitting and square of cycle.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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