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THE EDGE HOP DOMINATION NUMBER OF A GRAPH

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Abstract. Let $G=(V,E)$ be a graph. A set $S \subseteq E(G)$ is called an edge hop dominating set if $S=E(G)$ or for every $g \in E(G) \setminus S$, there exists $h \in S$ such that $d(g,h)=1$. The minimum cardinality of an *edge hop domination set* of G is called the *edge hop domination number* of G is denoted by $\gamma_{eh}(G)$. The edge hop domination number of some standard graphs are determined. It is proved that for any two connected graphs H and K of orders n_1 and n_2 respectively, $\gamma_{eh}(H+K)=3$. Also it is proved that for any two connected graphs of sizes $m_1 \geq 3$ and $m_2 \geq 3$ respectively, $\gamma_{eh}(H \circ K) \leq m_1$.

Keywords: distance; hop domination number; edge hop domination number.

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1. INTRODUCTION

For notation and graph theory terminology we in general,follow [6,8]. Specifically, let $G=(V,E)$ be a graph with vertex set V of *order* $n=|V|$ and edge set E of *size* $m=|E|$. Let v be a vertex in $V(G)$. Then the *open neighborhood* of v is the set $N(v)=\{u \in V(G)/uv \in E\}$, and the *closed neighborhood* of v is $N[v]=\{v\} \cup N(v)$. The *degree* of a vertex v is $deg(v)=|N(v)|$.

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If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then e is called a *pendant edge* or *end edge*, u is a *leaf* or *end vertex* and v is a *support vertex* of u. A vertex of degree $n-1$ is called a *universal vertex*.

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. A u-v path of length $d(u,v)$ is called a u-v *geodesic*. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. $e(v) = max\{d(v,u) : u \in V(G)\}$. The minimum eccentricity among the vertices of G is the *radius*, *radG* or $r(G)$ and the maximum eccentricity is its *diameter*, *diamG*. We denote $rad(G)$ by r and *diamG* by d. Two vertices u and v of G are *antipodal* if $d(u,v) = diamG$ or $d(G)$. A *double star* is a tree with diameter 3. It is denoted by $K_{2,n,m}$. The vertex set of $K_{2,n,m}$, where uv is the internal edge of $K_{2,n,m}$. Therefore $K_{2,n,m} = K_{1,n} \cup K_{1,m} \cup \{uv\}$, where the centre vertex of $K_{1,n}$ is u and the centre vertex of $K_{1,m}$ is v. The distance concepts has applications in social network. For example if one is locating an emergency facility like police station, fire station, hospital, school, college, library, ambulance depot, emergency care center, etc., then the primary aim is to minimize the distance between the facility and the location of a possible emergency. For edges $e, f \in E(G)$, the distance $d(e,f)$ is defined as $d(e,f) = min\{d(x,y) : x$ is an end edge of e and y is an end edge of f. A x-y path of length $d(e,f)$ is called an e-f geodesic joining the edges e and f. If e and f are adjacent if and only if $d(e,f) = 0$ and if e and f has a common edge, then $d(e,f) = 1$. $W_n = K_1 + C_{n-1}$ is called wheel graph. We denote $V(C_{n-1}) = \{v_1, v_2, ..., v_{n-1}\}$. The *helm graph* H_n is a graph obtained from a wheel graph by attaching a pendent edge at each vertex of the cycle C_{n-1}. Denote the pendent vertices of H_n by $\{u_1, u_2, ..., u_{n-1}\}$. A *sunflower graph* SF_n is the graph obtained from helm graph by introducing the edges $u_i v_{i+1}$ $(1 \leq i \leq n-2)$ and $u_{n-1}v_1$. The *triangular book* with n pages is defined as n copies of cycle C_3 sharing a common edge. The common edge is called the base of the book. A *quadrilateral book* consists of r quadrilaterals sharing a common edge uv. That is, it is a cartesian product of a star and a single edge. It is denoted by $Q_{r,2}$. A *banana tree graph* is obtained by connecting one leaf of each of copies of a star graph with a single root vertex that is distinct from all the stars.

A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex $v \in V(G) \setminus D$ is adjacent to some vertex in D. A dominating set D is said to be *minimal* if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. The domination number of a graph was studied in [4,5,7,9-11,16-19]. A set $S \subseteq V(G)$ of a graph G is a *hop dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of G is called the *hop domination number* and is denoted by $\gamma_h(G)$. Any hop dominating set of order $\gamma_h(G)$ is called γ_h-*set* of G. The hop domination number of a graph was studied in [1-3,12-14]. The join $G+H$ of two graphs G and H is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The corona product $K \circ H$ is defined as the graph obtained from K and H by taking one copy of K and $|V(K)|$ copies of H and then joining by an edge, all the vertices from the i^{th}-copy of H to the i^{th}-vertex of K, where $i = 1, 2, ..., |V(H)|$. The join and corona concept was studied in [15]. In this paper, we introduce the concept of the edge hop domination number of a graph. Hop dominating concept have interesting application in social network. If we apply edge hop dominating concept in the social network then the effectiveness can be increased.

2. THE EDGE HOP DOMINATION NUMBER OF A GRAPH

Definition 2.1. Let $G = (V,E)$ be a graph. A set $S \subseteq E(G)$ is called an edge hop dominating set if $S = E(G)$ or for every $g \in E(G) \setminus S$, there exists $h \in S$ such that $d(g,h) = 1$. The minimum cardinality of an *edge hop domination set* of G is called the *edge hop domination number* of G is denoted by $\gamma_{eh}(G)$.

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1v_2, v_2v_3, v_5v_6\}$ is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 3$.

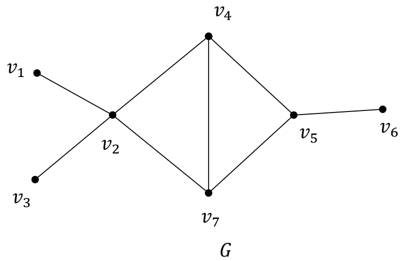

Figure 2.1

Remark 2.3. There can be more than one γ_{eh}-sets of G. For the graph G given in Figure 2.1, S_1 = $\{v_2v_4, v_2v_7, v_4v_7\}$, S_2 = $\{v_4v_5, v_4v_7, v_5v_7\}$, S_3 = $\{v_1v_2, v_2v_4, v_4v_5\}$, S_4 = $\{v_1v_2, v_2v_7, v_5v_7\}$, S_5 = $\{v_2v_3, v_2v_7, v_5v_7\}$, S_6 = $\{v_2v_3, v_2v_4, v_4v_5\}$, S_7 = $\{v_2v_7, v_5v_6, v_5v_7\}$, S_8 = $\{v_2v_4, v_4v_5, v_5v_6\}$ are the γ_{eh}-sets of G such that $\gamma_{eh}(S_i)$ = 3 for all i $(1 \leq i \leq 8)$.

Theorem 2.4. For every connected graph G of size $m \geq 2$, $2 \leq \gamma_{eh}(G) \leq m$.

Proof. Since any edge hop dominating set of G contains at least two edges, $\gamma_{eh}(G) \geq 2$. Since $E(G)$ is an edge hop dominating set of G, $\gamma_{eh}(G) \leq m$. Thus $2 \leq \gamma_{eh}(G) \leq m$. □

Remark 2.5. The bound in Theorem 2.4 is sharp. For the graph $G = P_4$, $\gamma_{eh}(G) = 2$ and for $G = K_{1,m}$, $\gamma_{eh}(G) = m$. Also the bound in Theorem 2.4 can be strict. For the graph G given in Figure 2.1, $\gamma_{eh}(G) = 3$. Thus $2 < \gamma_{eh}(G) < m$.

Theorem 2.6. For the complete graph $G = K_n$ $(n \geq 3)$, $\gamma_{eh}(G) = 3$.

Proof. Let $V = \{v_1, v_2, ..., v_n\}$ be the vertex set of G. Let S_1 = $\{v_1v_2, v_2v_3, v_1v_3\}$. Then S_1 is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq 3$. We prove that $\gamma_{eh}(G) = 3$. On the contrary, suppose that $\gamma_{eh}(G) = 2$. Let $S^{'} = \{f, h\}$ be a γ_{eh}-set of G. Since G is complete, $g \in E \setminus S^{'}$ such that g is adjacent to both f and h. Then $d(g, f) = d(g, h) = 0$, which is a contradiction to $S^{'}$ a γ_{eh}-set of G. Therefore $\gamma_{eh}(G) = 3$. □

Theorem 2.7. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$, $\gamma_{eh}(G)$ = $\begin{cases} s & if \quad \text{r= 1} \\ r & otherwise \end{cases}$

Proof. If $r = 1$, then $S = E$ is the unique γ_{eh}-set of G so that $\gamma_{eh}(G) = s$. So, let $2 \leq r \leq s$. Let U = $\{u_1, u_2, ..., u_r\}$ and W = $\{w_1, w_2, ..., w_s\}$ are the two bipartite sets of G. Let $S = \{w_1u_1, w_1u_2, ..., w_1u_r\}$. Then S is an edge hop dominating set of G and so $\gamma_{eh}(G) \leq r$. We prove that $\gamma_{eh}(G) = r$. On the contrary, suppose that $\gamma_{eh}(G) \leq r - 1$. Then there exists a γ_{eh}-set S' of G such that $|S^{'}| \leq r - 1$. Let $g \in E \setminus S^{'}$. Then g is not adjacent to any edge of $S^{'}$. Let $g = uw$. where $u \in U$ and $w \in W$. Then u and w are adjacent to elements of $V(S^{'})$. Let $g_1 = ux$ and $g_2 = wy$ such that $g_1, g_2 \notin S^{'}$, where $x, y \in S^{'}$. Then $d(h_1, g_1) = d(h_2, g_2) = 0$ for $h_1, h_2 \in S^{'}$, which is a contradiction to $S^{'}$ a γ_{eh}-set of G. Therefore $\gamma_{eh}(G) = r$. □

Theorem 2.8. For the cycle $G = C_n$ $(n \geq 3)$, $\gamma_{eh}(G) =$

$$\begin{cases} 2 & \text{if } n = 4, 5 \\ 3 & \text{if } n = 3 \\ 2r & \text{if } n = 6r \\ 2r+1 & \text{if } n = 6r\text{+}1 \text{ or } 6r\text{+}3 \\ 2r+2 & \text{if } n = 6r\text{+}2 \text{ or } 6r\text{+}4 \text{ or } 6r\text{+}5\ ;\ r \geq 1 \end{cases}$$

Proof. Let $G = C_n$ be $v_1, v_2, v_3, ..., v_n, v_1$.

Case 1:

Case 1a: $n = 4$. Then $S_1 = \{v_1v_2, v_2v_3\}$, $S_2 = \{v_2v_3, v_3v_4\}$, $S_3 = \{v_3v_4, v_1v_4\}$ and $S_4 = \{v_1v_2, v_1v_4\}$ are the only minimum edge hop dominating sets of G so that $\gamma_{eh}(G) = 2$.

Case 1b: $n = 5$. Then $S_1 = \{v_1v_2, v_2v_3\}$, $S_2 = \{v_2v_3, v_3v_4\}$, $S_3 = \{v_3v_4, v_4v_5\}$, $S_4 = \{v_1v_5, v_4v_5\}$ and $S_5 = \{v_1v_2, v_1v_5\}$ are the only minimum edge hop dominating sets of G and so $\gamma_{eh}(G) = 2$.

Case 2: $n = 3$. Then $S_1 = \{v_1v_2, v_2v_3, v_1v_3\}$ is the unique edge hop dominating set of G and so $\gamma_{eh}(G) = 3$.

Case 3: $n = 6r$. Let $S = \{v_1v_2, v_4v_5, ..., v_{6r-5}v_{6r-4}, v_{6r-2}v_{6r-1}\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq | S | = 2r$. We prove that $\gamma_{eh}(G) = 2r$. If $r = 1$, then the result is obvious. So, let $r \geq 2$. On the contrary, suppose that $\gamma_{eh}(G) \leq 2r - 1$. Then there exists a γ_{eh}-set S' of G such that $|S^{'}| \leq 2r - 1$. Hence there exists $g \in E \setminus S^{'}$ such that $d(g, h) \geq 1$, where $h \in S^{'}$. Therefore $S^{'}$ is not an edge hop dominating set of G, which is a contradiction.

Case 4:

Case 4a: $n = 6r + 1$. Let $Y = S \cup \{v_{6r-1}v_{6r}\}$. Then as in Case 3. We can prove that Y is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r + 1$.

Case 4b: $n = 6r + 3$. Let $T = S \cup \{v_{6r+1}v_{6r+2}\}$. Then as in Case 3. We can prove that T is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r + 1$.

Case 5:

Case 5a: $n = 6r + 2$. Let $T^{'} = T \cup \{v_1v_{6r+2}\}$. Then as in Case 3, we can prove that $T^{'}$ is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r + 2$.

Case 5b: $n = 6r+4$. Let $W = T \cup \{v_1v_{6r+4}\}$. Then as in Case 3, we can prove that W is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r+2$.

Case 5b: $n = 6r+5$. Let $X = \{v_1v_2, v_7v_8, ..., v_{6r+1}v_{6r+2}\} \cup \{v_2v_3, v_8v_9, ..., v_{6r+2}v_{6r+3}\}$. Then as in Case 3, we can prove that X is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r+2$. □

Theorem 2.9. For the path $G = P_n$ $(n \geq 3)$, $\gamma_{eh}(G) =$

$$\begin{cases} 2 & \text{if } n = 3 \text{ or } 4 \text{ or } 5 \\ 2r & \text{if } n = 6r \text{ or } 6r{+}1 \\ 2r+1 & \text{if } n = 6r{+}2 \\ 2r+2 & \text{if } n = 6r{+}3 \text{ or } 6r{+}4 \text{ or } 6r{+}5;\ r \geq 1 \end{cases}$$

Proof. Let $G = P_n$ be $v_1, v_2, v_3, ..., v_n$.

Case 1:

Case 1a: $n = 3$. Then $S = \{v_1v_2, v_2v_3\}$ is the unique minimum edge hop dominating set of G and so $\gamma_{eh}(G) = 2$.

Case 1b: $n = 4$. Then $S_1 = \{v_1v_2, v_2v_3\}$ and $S_2 = \{v_2v_3, v_3v_4\}$ are the only minimum edge hop dominating sets of G and so $\gamma_{eh}(G) = 2$.

Case 1c: $n = 5$. Then $S_1 = \{v_1v_2, v_2v_3\}$, $S_2 = \{v_2v_3, v_3v_4\}$, $S_1 = \{v_1v_2, v_4v_5\}$ and $S_4 = \{v_3v_4, v_4v_5\}$ are the only minimum edge hop dominating sets of G and so $\gamma_{eh}(G) = 2$.

Case 2: $n = 6r$ or $6r+1$. Let $S = \{v_3v_4, v_9v_{10}, ..., v_{6r-3}v_{6r-2}\} \cup \{v_4v_5, v_{10}v_{11}, ..., v_{6r-2}v_{6r-1}\}$. Then S is an edge hop dominating set of G and so $\gamma_{eh}(G) \leq 2r$. We prove that $\gamma_{eh}(G) = 2r$. If $r = 1$, then result is obvious. So, let $r \geq 2$. On the contrary, suppose that $\gamma_{eh}(G) \leq 2r-1$. Then there exists a γ_{eh}-set $S^{'}$ of G such that $|S'| \leq 2r-1$. Hence there exists $g \in E \setminus S^{'}$ such that $d(g,h) \geq 1$, where $h \in S^{'}$. Therefore $S^{'}$ is not an edge hop dominating set of G, which is a contradiction.

Case 3: $n = 6r+2$. Let $T = \{v_1v_2, v_4v_5, v_7v_8, ..., v_{6r+1}v_{6r+2}\}$. Then as in Case 2, we can prove that T is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r+1$.

Case 4:

Case 4a: $n = 6r+3$ or $6r+4$. Let $T^{'} = T \cup \{v_{6r+2}v_{6r+3}\}$. Then as in Case 2. We can prove that $T^{'}$ is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r+2$.

Case 4b: $n = 6r+5$. Let $W = T \cup \{v_{6r+4}v_{6r+5}\}$. Then as in Case 2. We can prove that W is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2r+2$. □

Theorem 2.10. For the wheel $G = W_n$ $(n \geq 4)$, $\gamma_{eh}(G) = = \begin{cases} 2 & \text{if } n = 7 \\ 3 & otherwise \end{cases}$

Proof. Let $V(K_1) = \{u\}$ and $V(C_{n-1}) = \{v_1, v_2, ..., v_{n-1}\}$.

Case 1: $n = 7$. Then $S_1 = \{v_1v_2, v_4v_5\}$, $S_2 = \{v_2v_3, v_5v_6\}$ and $S_3 = \{v_3v_4, v_1v_6\}$ are the γ_{eh}-sets of G so that $\gamma_{eh}(G) = 2$.

Case 2:

Case 2a: $n = 4$. Then $G = K_4$. By Theorem 2.6, $\gamma_{eh}(G) = 3$.

Case 2b: $n = 5$ or 6. Then $S_4 = \{uv_1, v_1v_2, uv_2\}$ is the γ_{eh}-set of G so that $\gamma_{eh}(G) = 3$.

Case 2c: $n \geq 8$. Then $S_5 = \{uv_1, uv_2, v_1v_2\}$ is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq 3$. We prove that $\gamma_{eh}(G) = 3$. On the contrary, suppose that $\gamma_{eh}(G) = 2$. Let $S' = \{f, g\}$ be a γ_{eh}-set of G. First assume that f and g are adjacent. If $f, g \in E(C_{n-1})$ then without loss of generality, let us assume that $f = v_1v_2$ and $g = v_2v_3$. Since u is adjacent to each vertex of G, then $d(v_1v_2, uv_2) = d(v_2v_3, uv_2) = 0$, which is a contradiction. Next we assume that f and g are non-adjacent. If $f \in E(C_{n-1})$ and $g \notin E(C_{n-1})$, then without loss of generality let us assume that $f = v_1v_2$ and $g = uv_3$. This implies $d(uv_1, v_1v_2) = d(uv_3, uv_1) = 0$ which is a contradiction, S' a γ_{eh}-set of G. Therefore $\gamma_{eh}(G) = 3$. □

Theorem 2.11. Let $G = K_{2,m,n}$ be a double star. Then $\gamma_{eh}(G) = = \begin{cases} 2 & \text{if } n = 1 \text{ or m} = 1 \\ 3 & otherwise \end{cases}$

Proof. Let $V(G) = \{u, v\} \cup \{u_1, u_2, ..., u_n\} \cup \{v_1, v_2, ..., v_m\}$.

Case 1:

Case 1a: $n = 1$. Then $S = \{u_1u, uv\}$ is the unique minimum edge hop dominating set of G and so $\gamma_{eh}(G) = 2$.

Case 1b: $m = 1$. Then $S = \{uv, vv_1\}$ is the unique minimum edge hop dominating set of G and so $\gamma_{eh}(G) = 2$.

Case 2: $n \geq 2$ and $m \geq 2$. Let $S = \{uu_1, vv_1\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq 3$. We prove that $\gamma_{eh}(G) = 3$. On the contrary, suppose that $\gamma_{eh}(G) = 2$. Let $S' = \{f, g\}$

be a γ_{eh}-set of G. First assume that f and g are adjacent. Without loss of generality, let us assume that $f = uu_1$ and $g = uv$. This implies $d(uv, vv_1) = 0$, which is a contradiction. Next assume that f and g are non-adjacent. Without loss of generality, let us assume that $f = uu_1$ and $g = vv_1$. This implies $d(uu_1, uv) = d(uv, vv_1) = 0$ which is a contradiction, $S^{'}$ a γ_{eh}-set of G. Therefore $\gamma_{eh}(G) = 3$. □

Theorem 2.12. For the helm graph $G = H_n$ $(n \geq 3)$, $\gamma_{eh}(G) = 3$.

Proof. Let x be the central vertex of G and $v_1, v_2, ..., v_{n-1}, v_1$ be the cycle. Let $\{u_1v_1, u_2v_2, ..., u_{n-1}v_{n-1}\}$ be the set of all end edges of G. Let $S = \{xv_1, v_1v_2, xv_2\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq 3$. We prove that $\gamma_{eh}(G) = 3$. On the contrary, suppose that $\gamma_{eh}(G) = 2$. Then there exists a γ_{eh}-set $S^{'}$ of G such that $S^{'} = \{e, f\}$. Suppose that e and f are adjacent. Then there exist at least one edge $h \in E(G) \setminus S^{'}$ such that h is incident with exactly one vertex of $V(S^{'})$. Hence it follows that $d(e,h) = 0$ and $d(f,h) = 0$, which is a contradiction. Suppose that e and h are not adjacent. Then there exists at least one edge $h^{'} \in E(G) \setminus S^{'}$ such that either $d(e,h^{'}) = d(e,f) = 0$ or $d(e,h^{'}) = d(e,f) = 2$, which is a contradiction. Therefore $\gamma_{eh}(G) = 3$. □

Theorem 2.13. For the sunflower graph $G = SF_n$ $(n \geq 3)$, $\gamma_{eh}(G) = 3$.

Proof. Let $S = \{uu_1, u_1u_2, u_2u\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq |S| = 3$. We prove that $\gamma_{eh}(G) = 3$. On the contrary, suppose that $\gamma_{eh}(G) = 2$. Let $S^{'} = \{g, h\}$ is a γ_{eh}-set of G. First assume that g and h are adjacent. Without loss of generality, let us assume that $g = uu_1$ and $h = u_1u_2$. This implies $d(uu_1, uu_2) = d(u_1u_2, uu_2) = 0$, which is a contradiction. Next assume that g and h are not adjacent. Without loss of generality, let us assume that $g = uu_1$ and $h = v_1u_2$. This implies $d(uu_1, u_1v_1) = d(uu_1, u_1u_2) = d(v_1u_2, uu_2) = 0$, which is a contradiction. Therefore $\gamma_{eh}(G) = 3$. □

Theorem 2.14. For the banana graph $G = B_{m,n}$ $(m \leq n)$, $\gamma_{eh}(G) = m$.

Proof. Let u be the central vertex and take m copies of a n-star graph with a single root vertex that is distinct for all stars. Let $S = \{uu_1, uu_2, ..., uu_m\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq |S| = m$. We prove that $\gamma_{eh}(G) = m$. On the contrary, suppose that

$\gamma_{eh}(G) \leq m-1$. Then there exists a γ_{eh}-set S' of G such that $| S' | \leq m-1$. Let $g \in E \setminus S'$. Then g is not adjacent to any edge of S'. Let $g_1 = ux$ and $g_2 = wy$ such that $g_1, g_2 \notin S$ where $x, y \in S'$. Then $d(h_1, g_1) = d(h_2, g_2) = 0$ for $h_1, h_2 \in S$, which is a contradiction. Therefore $\gamma_{eh}(G) = m$. □

Theorem 2.15. For the triangular graph $G = K_2 \vee \overline{K}_{n-2}$ $(n \geq 3)$, $\gamma_{eh}(G) = 3$.

Proof. Let $V(K_2) = \{x, y\}$ and $V(K_{n-2}) = \{v_1, v_2, ..., v_{n-2}\}$. Let $S = \{xy, xv_1, yv_1\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \leq 3$. We prove that $\gamma_{eh}(G) = 3$. On the contrary, suppose that $\gamma_{eh}(G) = 2$. Then there exists a γ_{eh}-set S' of G such that $| S' | = 2$. Then $G[S']$ is connected. Hence there exists at least one edge $e \in E(G) \setminus S'$ such that $d(e, f) = 0$, where $f \in S'$, which is a contradiction. Therefore $\gamma_{eh}(G) = 3$. □

Theorem 2.16. For the Quadrilateral book graph $G = Q_{n-2,2}$, $\gamma_{eh}(G) = 2$.

Proof. Let $P_i = u_i, v_i$ $(1 \leq i \leq n-2)$ be a copy of path on two vertices. Let $V(K_2) = \{x, y\}$. Then the Quadrilateral book graph $G = Q_{n-2,2}$ is obtained from P_i $(1 \leq i \leq n-2)$ and K_2 by joining x with each u_i $(1 \leq i \leq n-2)$ and y with each v_i $(1 \leq i \leq n-2)$. Let $S = \{xu_1, u_1v_1\}$ be a γ_{eh}-set of G. Then for every $e \in E(G) \setminus S$, there exist $f \in S$ such that $d(e, f) = 1$. Therefore S is an edge hop dominating set of G. Hence $\gamma_{eh}(G) = 2$. □

3. THE EDGE HOP DOMINATION OF JOIN AND CORONA OF GRAPHS

Theorem 3.1. Let H and K be two connected graphs of orders $n_1 \geq 2$ and $n_2 \geq 2$ respectively. Then $\gamma_{eh}(H+K) = 3$.

Proof. Let $V(H+K) = \{u_1, u_2, ..., u_{n_1}, v_1, v_2, ..., v_{n_2}\}$. Let $S = \{u_1v_1, u_1u_2, u_2v_1\}$. Then S is an edge hop dominating set of $H+K$ so that $\gamma_{eh}(H+K) \leq 3$. We prove that $\gamma_{eh}(H+K) = 3$. On the contrary, suppose that $\gamma_{eh}(H+K) = 2$. Then there exists a γ_{eh}-set S' of $H+K$ such that $|S'| \leq 2$. Let $S' = \{g, h\}$ be a γ_{eh}-set of $H+K$. Then $d_{H+K}(g, f) = d_{H+K}(h, f) = 0$ for $f \in E \setminus S'$, which is a contradiction to S' a γ_{eh}-set of $H+K$. Hence $\gamma_{eh}(H+K) = 3$. □

Theorem 3.2. Let $G = K_{1,n_1} \circ K_{n_2}$ $n_1 \geq 3$ and $n_2 \geq 3$. Then $\gamma_{eh}(G) = n_1 + 1$.

Proof. Let $V(G) = \{x, u_1, u_2, ..., u_{n_1}, u_{1,1}, u_{1,2}, ..., u_{1,n_1}, u_{2,1}, u_{2,2}, ..., u_{2,n_2} u_{n_1,1}, u_{n_1,2}, ...,$ $u_{n_1,n_2}, x_{1,1}, x_{1,2}, ..., x_{1,n_2}\}$. Let $S = \{xu_1, xu_2, ..., xu_n, u_i u_{i,j}\}$; $(1 \le i \le n_1)$ and $(1 \le j \le n_2)$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \le |S| = n_1 + 1$. We prove that $\gamma_{eh}(G) = n_1 + 1$. On the contrary, suppose that $\gamma_{eh}(G) \le n_1$. Then there exist a γ_{eh}-set S' of G such that $f \in S$ and $f \notin S'$. First assume that $f \in \{xu_1, xu_2, ..., xu_{n_1}\}$. Without loss of generality, let us assume $f = xu_1$. Then $d_G(f, u_j u_{j+1}) = 0$ for $u_j u_{j+1} \in E \setminus S'$ $(1 \le j \le m-1)$. Next assume that $f = \{u_i u_{i,j}\}$ for $(1 \le i \le n_1)$ and $(1 \le j \le n_2)$. Without loss of generality, let us assume that $f = u_1 u_{1,1}$. Then $d_G(f, xx_{i,j}) = 0$ for $xx_{i,j} \in E \setminus S'$ $(1 \le i \le n_1)$ and $(1 \le j \le n_2)$. Hence S' is not a γ_{eh}-set of G, which is a contradiction. Therefore $\gamma_{eh}(G) = n_1 + 1$. □

Theorem 3.3. Let $G = K_{1,n_1} \circ K_1$ $n_1 \ge 3$. Then $\gamma_{eh}(G) = 2$.

Proof. Let $V(G) = \{x, u_1, u_2, ..., u_{n_1}, x_{1,1}, u_{1,1}, u_{2,1}, ..., u_{n_1,1}\}$. Let $S = \{xu_i, u_i u_{i,1}\}$ for $(1 \le i \le n_1)$. Then $d_G(xu_i, u_{i+1} u_{i+1,1}) = 1$ for $u_{i+1} u_{i+1,1} \in E \setminus S$ $(1 \le i \le n_1 - 1)$ and $d_G(u_i u_{i,1}, xu_{i+1}) = d_G(u_i u_{i,1}, xx_{1,1}) = 1$ for $xu_{i+1}, xx_{1,1} \in E \setminus S$ $(1 \le i \le n_1 - 1)$. Hence S is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 2$. □

Theorem 3.4. Let $G = P_{n_1} \circ C_{n_2}$ $n_1, n_2 \ge 3$. $\gamma_{eh}(G) = \begin{cases} 3 & \text{if } n_1 = 3 \\ 3r & \text{if } n_1 = 4r \\ 3r+1 & \text{if } n_1 = 4r+1 \\ 3r+2 & \text{if } n_1 = 4r+2 \\ 3r+2 & \text{if } n_1 = 4r+3 \end{cases}$

Proof. Let $V(G) = \{u_1, u_2, ..., u_{n_1}, u_{1,1}, u_{1,2}, ..., u_{1,n_2}, u_{2,1}, u_{2,2}, ..., u_{2,n_2}, ..., u_{2,n_2}, ...,$ $u_{n_1,1}, u_{n_1,2}, ..., u_{n_1,n_2}\}$.

Case 1: $n_1 = 3$. Then $S = \{u_1 u_2, u_2 u_3, u_1 u_{1,1}\}$ is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 3$.

Case 2: $n_1 = 4r$ and $r = 1, 2, 3...$

Let $S = \{u_1 u_2, u_5 u_6, ..., u_{4r-3} u_{4r-2}\} \cup \{u_2 u_3, u_6 u_7, ..., u_{4r-2} u_{4r-1}\} \cup \{u_3 u_4, u_7 u_8, ..., u_{4r-1} u_{4r}\}$. Then S is an edge hop dominating set of G so that $\gamma_{eh}(G) \le |S| = 3r$. We have to prove that $\gamma_{eh}(G) = 3r$. On the contrary, suppose that $\gamma_{eh}(G) \le 3r - 1$. Let f be an edge of G such that $f \in S$ and $f \notin S'$. First assume that $f \in \{u_1 u_2, u_5 u_6, ..., u_{4r-3}, u_{4r-2}\}$. Without loss of generality, let us assume $f = u_1u_2$. Then $d_G(f, u_{1,i}u_{1,i+1}) = 0$, $(1 \leq i \leq n_2 - 1)$ for $u_{1,i}u_{1,i+1} \notin E \setminus S'$. Next assume that $f = \{u_2u_3, u_6u_7, ..., u_{4r-2}u_{4r-1}\} \cup \{u_3u_4, u_7u_8, ..., u_{4r-1}u_{4r}\}$. Without loss of generality, let us assume that $f = u_2u_3$. Then $d_G(f, u_{2,i}u_{2,i+1}) = 0$, $(1 \leq i \leq n_2 - 1)$ for $u_{2,i}u_{2,i+1} \in E \setminus S'$. Therefore S' is not a γ_{eh}-set of G, which is a contradiction. Therefore $\gamma_{eh}(G) = 3r$.

Case 3: $n_1 = 4r + 1$ and $r = 1, 2, 3...$

Let $S_1 = S \cup \{u_{4r}u_{4r+1}\}$. Then as in Case 2, we can prove that S_1 is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 3r + 1$.

Case 4: $n_1 = 4r + 2$ and $r = 1, 2, 3...$

Let $T = S_1 \cup \{u_{4r+1}u_{4r+2}\}$. Then as in Case 2, we can prove that T is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 3r + 2$.

Case 5: $n_1 = 4r + 3$ and $r = 1, 2, 3...$

Let $W = T \cup \{u_{4r+2}u_{4r+3}\}$. Then as in Case 2, we can prove that W is a γ_{eh}-set of G so that $\gamma_{eh}(G) = 3r + 3$. □

Theorem 3.5. Let H and K be two connected graphs of sizes $m_1 \geq 3$ and $m_2 \geq 3$ respectively. Then $\gamma_{eh}(H \circ K) \leq m_1$.

Proof. Let $G = H \circ K$ and $S = E(H)$. Let $e \in E(H) \setminus S$. If e is incident with a vertex of H, then there exists an edge f in H, which is independent of e such that $d_G(e, f) = 1$. If e is not incident with a vertex of H. Then there exists $f \in H$ such that $d_G(e, f) = 1$. Therefore S is an edge hop dominating set of G. Hence $\gamma_{eh}(H \circ K) \leq m_1$. □

Remark 3.6. The bound in Figure 3.5 is sharp. For the graph $G = K_3 \circ K_1$, $\gamma_{eh}(G) = 3$. Thus $\gamma_{eh}(G) = m_1 = 3$. Also the bound in Theorem 3.5 is srict. For the graph $G = C_4 \circ P_3$, $\gamma_{eh}(G) = 3$ and $m_1 = 4$. Thus $\gamma_{eh}(G) < m_1$.

4. CONCLUSION

In this article we introduced the concept of the edge hop domination number of a connected graphs of size $m \geq 2$. It can be further investigated to find out under which conditions the lower and upper bounds of the edge hop domination number are sharp.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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