ON SOFT QUASI $R_0$-SPACES

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Abstract. Separation axioms is one of the important topic of research in general topology. In 1961, Davis introduced a separation axiom called $R_0$ spaces which is a weaker than both $T_0$ and $T_1$. In the present paper, we studied this axiom in soft bitopological spaces obtain some of its properties.

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1. INTRODUCTION

It appears to be easy to understand that a mathematical theory is based on various abstract thoughts. In such cases one has full freedom to establish certain environment depending on neglecting relevent facts, for example in physics we often neglect the frictional effect of air on a free falling body but this fact is fully impossible in real life. Similarly as we know in vacuum 1 kg of cotton and 1 kg of iron from a height touches the ground at the same time. However this will be impossible in our real life problem due to the presence of frictional forces. Similarly other branches like economics, engineering, social sciences are full of uncertainties.

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As we know, we had four mathematical tools to deal with uncertainties, namely probability theory, fuzzy set theory, rough set theory and theory of interval in mathematics. In order to overcome the choice of degree of membership in fuzzy set theory when the facts are concerned with uncertainties, Molodtsov [12] introduced the concept of soft set theory and investigated its applications in game theory, smoothness of function, operation research, Perron integration, probability theory and theory of measurement. However afterwards Maji et al. [11] defined various operations on soft set to study some of the fundamental properties. Pei, Miao [14] and Chen [2] stressed that error in the paper of Maji et al. [11] and accordingly introduced new notions and properties. Now, in present days investigation of different properties and applications of soft set theory have attracted many researchers from various backgrounds. It is pertinent to mention that Shabir and Naz [16] in the year 2011 introduced the term of soft topology and studied some introductory results. In the same year, Cagman et al. [1] introduced soft topology into different perspective. Till then various researchers have studied various foundation results in soft topology [6, 5, 16, 20]. It is relevent to note that Kandil et al. [7, 8, 9] intoduced the concept of soft ideal and studied some weaker notions and properties. Recently, Yuksel et al. [18] applied soft set theory to determine prostrate cancer risk. Senel and Cagman [15] introduced soft topological subspace and studied some properties. Kelly [10] introduced the concept of bitopological spaces. The study of quasi open sets was initiated by Dutta [4]. The present paper extended and studied the concept of soft quasi open sets in soft bitopological spaces and utilizes soft quasi open set to introduce and investigate the properties of soft quasi $R_0$ bitopological spaces.

2. Preliminaries

Let $U$ is an initial universe set, $E$ be a set of parameters, $P(U)$ denote the power set of $U$ and $A \subseteq E$.

**Definition 2.1.** [12] A pair $(F,A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \rightarrow P(U)$. In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For all $e \in A$, $F(e)$ may be considered as the set of $e$-approximate elements of the soft set $(F,A)$.
**Definition 2.2.** [11] For two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\), we say that \((F,A)\) is a soft subset of \((G,B)\), denoted by \(F(A) \subseteq G(B)\), if:

1. \(A \subseteq B\) and
2. \(F(e) \subseteq G(e), \forall e \in E\).

**Definition 2.3.** [11] Two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\) are said to be soft equal denoted by \((F,A) = (G,B)\), if

\[(F,A) \subseteq (G,B) \text{ and } (G,B) \subseteq (F,A)\]

**Definition 2.4.** [11] The complement of a soft set \((F,A)\) denoted by \((F,A)^c\), is defined by \((F,A)^c = (F^c,A)\) where \(F^c : A \rightarrow P(U)\) is a mapping given by \(F^c(e) = U - F(e), \forall e \in E\).

**Definition 2.5.** [11] Let a soft set \((F,A)\) over \(U\).

1. Null soft set denoted by \(\phi\), if \(\forall e \in A, F(e) = \phi\).
2. Absolute soft set denoted by \(\tilde{U}\) if \(\forall e \in A, F(e) = U\).

Clearly, \(\tilde{U}^c = \phi\) and \(\phi^c = \tilde{U}\).

**Definition 2.6.** [16] The difference of two soft sets \((F,A)\) and \((G,B)\) over the common universe \(U\) denoted by \((F,A) - (G,B)\) is the soft set \((H,C)\), where \(H(e) = F(e) - G(e), \forall e \in C\).

**Definition 2.7.** [11] Union of two soft sets \((F,A)\) and \((G,B)\) over the common universe \(U\), is the soft set \((H,C)\), where \(C = A \cup B\) and \(\forall e \in C\),

\[
H(e) = \begin{cases} 
F(e) & \text{if } e = A - B \\
G(e) & \text{if } e = B - A \\
F(e) \cup G(e) & \text{if } e \in A \cap B
\end{cases}
\]

**Definition 2.8.** [14] Intersection of two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\), is the soft set \((H,C)\), where \(C = A \cap B\) and \(H(e) = F(e) \cap G(e), \forall e \in E\).

Let \(X\) and \(Y\) be an initial universe sets and \(E\) and \(K\) be the nonempty sets of parameters, \(S(X,E)\) denotes the family of all soft sets over \(X\) and \(S(Y,K)\) denotes the family of all soft sets over \(Y\).

**Definition 2.9.** [19] Let \(\{(F_j,E) : j \in J\}\) be a nonempty family of soft sets over a common universe \(U\). Then:
(1) Intersection of this family, denoted by $\bigcap_{j \in J}$, is defined by $\bigcap_{j \in J} (F_j, E) = (\bigcap_{j \in J} F_j, E)$, where $(\bigcap_{j \in J} F_j)(e) = \bigcap_{j \in J} (F_j(e))$, $\forall e \in E$.

(2) Union of this family, denoted by $\bigcup_{j \in J}$, is defined by $\bigcup_{j \in J} (F_j, E) = (\bigcup_{j \in J} F_j, E)$, where $(\bigcup_{j \in J} F_j)(e) = \bigcup_{j \in J} (F_j(e))$, $\forall e \in E$.

**Definition 2.10.** [16] A subfamily $\tau$ of $S(X, E)$ is called a soft topology on $X$ if:

1. $\emptyset, X$ belongs to $\tau$.
2. The union of any number of soft sets in $\tau$ belongs to $\tau$.
3. The intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called a soft topological space. The members of $\tau$ are called soft open sets in $X$ and their complements called soft closed sets in $X$.

**Lemma 2.1.** [16] Let $(X, \tau, E)$ be a soft topological space. Then the collection $\tau_\alpha = \{F(\alpha) : (F, E) \in \tau\}$ for each $\alpha \in E$, defines a topology on $X$.

**Definition 2.11.** [16] In a soft topological space $(X, \tau, E)$ the intersection of all soft closed super sets of $(F, E)$ is called the soft closure of $(F, E)$. It is denoted by $\text{Cl}(F, E)$.

**Definition 2.12.** [19] In a soft topological space $(X, \tau, E)$ the union of all soft open subsets of $(F, E)$ is called soft interior of $(F, E)$. It is denoted by $\text{Int}(F, E)$.

**Lemma 2.2.** [1, 16, 19] Let $(X, \tau, E)$ be a soft topological space and let $(F, E), (G, E) \in S(X, E)$.

Then:

1. $(F, E)$ is soft closed if and only if $(F, E) = \text{Cl}(F, E)$.
2. If $(F, E) \subseteq (G, E)$, then $\text{Cl}(F, E) \subseteq \text{Cl}(G, E)$.
3. $(F, E)$ is soft open if and only if $(F, E) = \text{Int}(F, E)$.
4. If $(F, E) \subseteq (G, E)$, then $\text{Int}(F, E) \subseteq \text{Int}(G, E)$.
5. $(\text{Cl}(F, E))^c = \text{Int}((F, E)^c)$.
6. $(\text{Int}(F, E))^c = \text{Cl}((F, E)^c)$.

**Definition 2.13.** [6] Let $(X, \tau, E)$ be a soft topological space over $X$ and $Y$ be a nonempty subset of $X$. Then $\tau_Y = \{(F_Y, E) : (F, E) \in \tau\}$ is said to be the soft relative topology on $Y$ and $(Y, \tau_Y, E)$ is called a soft subspace of $(X, \tau, E)$.
Lemma 2.3. [6] Let \((Y, \tau_Y, E)\) be a soft subspace of a soft topological space \((X, \tau, E)\) and \((F, E)\) be a soft open set in \(Y\). If \(\tilde{Y} \in \tau\) then \((F, E) \in \tau\).

Lemma 2.4. [16] Let \((Y, \tau_Y, E)\) be a soft topological subspace of a soft topological space \((X, \tau, E)\) and \((F, E)\) be a soft set over \(X\), then:

1. \((F, E)\) is soft open in \(Y\) if and only if \((F, E) = \tilde{Y} \cap (G, E)\) for some soft open set \((G, E)\) in \(X\).
2. \((F, E)\) is soft closed in \(Y\) if and only if \((F, E) = \tilde{Y} \cap (G, E)\) for some soft closed set \((G, E)\) in \(X\).

Lemma 2.5. [15] Let \((X, \tau, E)\) be a soft topological space and \((Y, \tau_Y, E)\) be a soft subspace of \((X, \tau, E)\), then a soft closed set \((F_Y, E)\) of \(Y\) is soft closed in \(X\) if and only if \(\tilde{Y}\) is soft closed in \(X\).

Lemma 2.6. [19] The soft set \((F, E) \in S(X, E)\) is called a soft point if there exists \(x \in X\) and \(e \in E\) such that \(F(e) = \{x\}\) and \(F(e^c) = \emptyset\) for each \(e^c \in E - \{e\}\) and the soft point \((F, E)\) is denoted by \(x_e\). We denote the family of all soft points over \(X\) by \(SP(X, E)\).

Definition 2.14. [19] The soft point \(x_e\) is said to be in the soft set \((G, E)\), denoted by \(x_e \in (G, E)\) if \(x_e \subseteq (G, E)\).

Lemma 2.7. [3, 13] Let \((F, E), (G, E) \in S(X, E)\) and \(x_e \in SP(X, E)\). Then we have:

1. \(x_e \in (F, E)\) if and only if \(x_e \notin (F, E)^c\).
2. \(x_e \in (F, E) \cup (G, E)\) if and only if \(x_e \in (F, E)\) or \(x_e \in (G, E)\).
3. \(x_e \in (F, E) \cap (G, E)\) if and only if \(x_e \in (F, E)\) and \(x_e \in (G, E)\).
4. \((F, E) \subseteq (G, E)\) if and only if \(x_e \in (F, E)\) implies \(x_e \in (G, E)\).

Definition 2.15. [17] Let \((X, \tau, E)\) be a soft topological space, \((F, E)\) be a soft set and \(x_e\) be a soft point over \(X\), then \((F, E)\) is called soft neighborhood of \(x_e\), if there exists soft open set \((G, E)\) such that \(x_e \in (G, E) \subseteq (F, E)\).

3. Soft Quasi Open Sets

Definition 3.1. A soft bitopological space \((X, \tau_1, \tau_2, E)\) is a quadral with a non empty set of parameter \(E\) and with two soft topologies \(\tau_1\) and \(\tau_2\) defined on \(X\).

Definition 3.2. Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space over \(X\) and \(Y\) be a non-empty subset of \(X\). Then \(\tau_{1Y} = \{((F, E) \cap \tilde{Y}) : (F, E) \in \tau_1\}\) and \(\tau_{2Y} = \{((F, E) \cap \tilde{Y}) : (F, E) \in \tau_2\}\) are
said to be the soft relative topology on $Y$ and $(Y, \tau_{1Y}, \tau_{2Y}, E)$ is called a soft relative bitopological space.

**Definition 3.3.** A subset $(F, E)$ of a soft bitopological space $(X, \tau_1, \tau_2, E)$ is said to be soft quasi open if it is a union of a $\tau_1$-soft open and a $\tau_2$-soft open set.

**Remark 3.1.** In a soft bitopological space $(X, \tau_1, \tau_2, E)$, every $\tau_1$-soft open (resp. $\tau_2$-soft open) set is soft quasi open.

However, the converse need not be true. For,

**Example 3.1.** Let $X = \{\alpha, b\}$, $E = \{e_1, e_2\}$, $\tau_1 = \{\phi, \bar{X}, \{\alpha\}, \phi\}$, $\tau_2 = \{\phi, \bar{X}, \{\phi, \{\alpha\}\}\}$ be soft topology on $X$, then $\{\alpha\}, \{\beta\}$ is a soft quasi open set in a soft bitopological space $(X, \tau_1, \tau_2, E)$ but it is neither $\tau_1$-soft open nor $\tau_2$-soft open.

**Definition 3.4.** The complement of soft quasi open set in a soft bitopological space is called soft quasi closed. The family of all soft quasi open (resp. soft quasi closed) sets in $(X, \tau_1, \tau_2, E)$ will be denoted by $qO(X)$ (resp. $qC(X)$).

**Theorem 3.1.** Any union of soft quasi open sets in a soft bitopological space is soft quasi open.

*Proof.* Evident  

**Remark 3.2.** The intersection of two soft quasi open sets in a soft bitopological space may not be soft quasi open. For,

**Example 3.2.** Let $X = \{\alpha, \beta, \gamma\}$, $E = \{e_1, e_2\}$, $\tau_1 = \{\phi, \bar{X}, \{\alpha\}, \{\beta\}\}$, $\tau_2 = \{\phi, \bar{X}, \{\gamma\}, \{\beta\}\}$ be soft topology on $X$, then $\{\alpha\}, \{\beta\}, \{\gamma\}, \{\beta\}$ are soft quasi open sets in soft bitopological space $(X, \tau_1, \tau_2, E)$ but their intersection $\{\phi, \{\beta\}\}$ is not soft quasi open.

**Definition 3.5.** Let $(A, E)$ be a soft subset of a soft bitopological space $(X, \tau_1, \tau_2, E)$, then the smallest soft quasi closed set which contains $(A, E)$ is called soft quasi closure of $(A, E)$. It is denoted by $qCl(A, E)$.
Theorem 3.2. Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space and \((A,E)\) be a soft subset of \(X\) then,

1. A soft point \(x_e \in qCl(A,E)\) if and only if for any soft quasi open set \((U,E)\) containing \(x_e\), \((U,E) \cap (A,E) \neq \emptyset\).
2. \((A,E)\) is soft quasi closed if and only if \((A,E) = qCl(A,E)\).
3. \(qCl(A,E) = \tau_1 - Cl(A,E) \cap \tau_2 - Cl(A,E)\).

Proof. Evident \(\Box\)

Theorem 3.3. Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space. If \((O,E)\) is soft biopen and \((A,E)\) is soft quasi open in \(X\), then \((O,E) \cap (A,E)\) is soft quasi open.

Proof. Since \((A,E)\) is soft quasi open by Definition 3.3, let \((W,E)\) be \(\tau_1\)-soft open and \((V,E)\) be a \(\tau_2\)-soft open set such that \((A,E) = (W,E) \cup (V,E)\). Then, \((O,E) \cap (A,E) = ((O,E) \cap (W,E)) \cup ((O,E) \cap (V,E))\). Since \((O,E)\) is soft biopen, \((O,E) \cap (W,E)\) is \(\tau_1\)-soft open and \((O,E) \cap (V,E)\) is \(\tau_2\)-soft open. Therefore by Definition 3.3, \((O,E) \cap (A,E)\) is soft quasi open. \(\Box\)

Theorem 3.4. Let \((Y,(\tau_1)_Y, (\tau_2)_Y, E)\) be a soft subspace of a soft bitopological space \((X, \tau_1, \tau_2, E)\). If \((A,E)\) is soft quasi open in \(X\) then \((A,E) \cap \tilde{Y}\) is soft quasi open in \(Y\).

Proof. Since \((A,E)\) is soft quasi open in \(X\), by virtue of Definition 3.3, there is a \(\tau_1\)-soft open set \((W,E)\) and a \(\tau_2\)-soft open set \((V,E)\) such that \((A,E) = (W,E) \cup (V,E)\). Then \((A,E) \cap \tilde{Y} = ((W,E) \cap \tilde{Y}) \cup ((V,E) \cap \tilde{Y})\). It is clear that \((W,E) \cap \tilde{Y}\) is \((\tau_1)_Y\)-soft open and \((V,E) \cap \tilde{Y}\) is \((\tau_2)_Y\)-soft open. Consequently by Definition 3.3, it results that \((A,E) \cap \tilde{Y}\) is soft quasi open in \(Y\). \(\Box\)

Remark 3.3. Let \((Y,(\tau_1)_Y, (\tau_2)_Y, E)\) be a soft subspace of a soft bitopological space \((X, \tau_1, \tau_2, E)\). If \((A,E)\) is soft quasi open in \(Y\) and \(Y\) is even if \(\tau_1\)-soft open (resp. \(\tau_2\)-soft open) then \((A,E)\) may not be soft quasi open in \(X\).

Theorem 3.5. Let \((Y,(\tau_1)_Y, (\tau_2)_Y, E)\) be a soft biopen subspace of a soft bitopological space \((X, \tau_1, \tau_2, E)\). If \((A,E)\) is soft quasi open in \(Y\), then \((A,E)\) is soft quasi open in \(X\).

Proof. Let \((W,E)_Y\) be \((\tau_1)_Y\)-soft open and \((V,E)_Y\) be \((\tau_2)_Y\)-soft open such that \((A,E) = (W,E)_Y \cup (V,E)_Y\). Since \((W,E)_Y \subseteq \tilde{Y}, (V,E)_Y \subseteq \tilde{Y}\) and \(\tilde{Y}\) is soft biopen, \((W,E)_Y\) is \((\tau_1)_Y\)-soft open and \((V,E)_Y\) is \((\tau_2)_Y\)-soft open. Consequently, \((A,E)\) is soft quasi open in \(X\). \(\Box\)
4. **Soft Quasi R₀-Space**

**Definition 4.1.** A soft bitopological space \((X, τ_1, τ_2, E)\) is soft quasi \(R₀\)-space if for each soft quasi open set \((F, E)\) and a soft point \(x_e \in (F, E)\) implies \(qCl\{x_e\} \subseteq (F, E)\).

**Definition 4.2.** A soft bitopological space \((X, τ_1, τ_2, E)\) is termed as pairwise soft-\(R₀\), if for every \(τ_1\)-soft open set \((O, E)\) and a soft point \(x_e \in (O, E)\) implies that \(τ_j – Cl\{x_e\} \subseteq (O, E), i, j = 1, 2, \ldots i \neq j\).

**Theorem 4.1.** Every pairwise soft \(R₀\)-space \((X, τ_1, τ_2, E)\) is soft quasi \(R₀\).

**Proof.** Let \((G, E)\) be a soft quasi open set and a soft point \(x_e \in (G, E)\). Then by Definition 3.3, there exists a \(τ_1\)-soft open set \((W, E)\) and a \(τ_2\)-soft open set \((V, E)\) such that \((G, E) = (W, E) \cup (V, E)\). If \(x_e \in (W, E)\) then \(τ_2 – Cl\{x_e\} \subseteq (W, E)\), since \(X\) is pairwise soft \(R₀\). And so, \(q(Cl\{x_e\}) \subseteq (W, E)\), in view of Definition 3.2(3). Consequently, \(q(Cl\{x_e\}) \subseteq (G, E)\). The case if \(x_e \in (V, E)\) is similar. Hence, the space \((X, τ_1, τ_2, E)\) is soft quasi \(R₀\). \(□\)

**Definition 4.3.** In a soft bitopological space \((X, τ_1, τ_2, E)\) the soft quasi kernel of a soft point \(x_e\) of \(X\) denoted by \(q(Ker\{x_e\})\) is defined as follows:

\[ q(Ker\{x_e\}) = \cap\{(G, E) : (G, E)\text{ is soft quasi open, } x_e \in (G, E)\} \]

**Lemma 4.1.** \(q(Ker\{x_e\}) = \{y_e : x_e \in qCl\{y_e\}\}\). *First, we prove,*

**Proof.** Results from Definition 3.3 and Theorem 3.2(1). \(□\)

The following theorem now obtain several characterization of soft quasi \(R₀\)-spaces.

**Theorem 4.2.** In a soft bitopological space \((X, τ_1, τ_2, E)\), the following statements are equivalent:

1. \((X, τ_1, τ_2, E)\) is soft quasi \(R₀\)-spaces.
2. For \(x_e \in X\), \(qCl\{x_e\} \subseteq qKer\{x_e\}\).
3. For \(x_e, y_e \in X, y_e \in qKer\{x_e\} \iff x_e \in qKer\{y_e\}\).
4. For \(x_e, y_e \in X, y_e \in qCl\{x_e\} \iff x_e \in qCl\{y_e\}\).
5. For any soft quasi closed set \((F, E)\) and a soft point \(x_e \notin (F, E)\), there exists a soft quasi open set \((G, E)\) such that \(x_e \notin (G, E)\) and \((F, E) \subseteq (G, E)\).
(6) For any soft quasi closed set \((F,E),(F,E) = \cap\{(G,E) : (G,E) \text{ is soft quasi open}, (F,E) \subseteq (G,E)\}\).

(7) Each soft quasi open set \((G,E)\) is a union of soft quasi closed sets contained in \((G,E)\).

(8) For each soft quasi closed set \((F,E)\) and a soft point \(x_e \notin (F,E)\) implies \(qCl\{x_e\} \cap (F,E) = \phi\).

Proof. (1)⇒(2) For \(x_e \in X\), \(qKer\{x_e\} = \cap\{(G,E) : (G,E) \text{ is soft quasi open}, x_e \in (G,E)\}\). Since \(X\) is soft quasi \(R_0\) each soft quasi open set \((G,E)\) containing \(x_e\) contains \(qCl\{x_e\}\). Hence, \(qCl\{x_e\} \subseteq qKer\{x_e\}\).

(2)⇒(3) If \(y_e \in qKer\{x_e\}\), then \(x_e \in qCl\{y_e\}\). Now by (2), \(x_e \in qKer\{y_e\}\). Analogously, \(x_e \in qKer\{y_e\}\) implies \(y_e \in qKer\{x_e\}\).

(3)⇒(4) For \(x_e,y_e \in X\), \(y_e \in qCl\{x_e\} \iff x_e \in qKer\{y_e\}\). But by (3) \(x_e \in qKer\{y_e\} \iff y_e \in qKer\{x_e\}\). Since, \(y_e \in qKer\{x_e\} \iff x_e \in qCl\{y_e\}\), (4) holds.

(4)⇒(5) Let \((F,E)\) be a soft quasi closed set and \(x_e \notin (F,E)\). Then \(y_e \subseteq (F,E) \Rightarrow qCl\{y_e\} \subseteq (F,E)\) \Rightarrow \(x_e \notin qCl\{y_e\}\) \Rightarrow \(y_e \notin qCl\{x_e\}\) by (4). Therefore there exists a soft quasi open set \((G,E)_{y_e}\) such that \(y_e \in (G,E)_{y_e}\) and \(x_e \notin (G,E)_{y_e}\). Let \((G,E) = \cup_{y_e \in (F,E)}(G,E)_{y_e}\). Then \((G,E)\) is soft quasi open such that \(x_e \notin (G,E)\) and \((F,E) \subseteq (G,E)\).

(5)⇒(6) Let \((F,E)\) be a soft quasi closed set and suppose that \((H,E) = \cap\{(G,E) : (G,E)\text{ is soft quasi open}, (F,E) \subseteq (G,E)\}\). Clearly, \((F,E) \subseteq (H,E)\). Let \(x_e \notin (F,E)\) then by (1), there exists a soft quasi open set \((G,E)\), such that \(x_e \notin (G,E)\) and \((F,E) \subseteq (G,E)\). Therefore, \(x_e \notin (H,E)\). Hence, \((F,E) = (H,E)\).

(6)⇒(7) Obvious.

(7)⇒(8) Let \((F,E)\) be a soft quasi closed set and \(x_e \notin (F,E)\). Then \((F,E)^c = (G,E)\) (say) is soft quasi open and contains \(x_e\). By (7), there exists a soft quasi closed set \((H,E)\) such that \(x_e \in (H,E) \subseteq (G,E)\). Therefore, \(qCl\{x_e\} \subseteq (G,E)\). Hence, \(qCl\{x_e\} \subseteq (G,E) \cap (F,E) = \phi\).

(8)⇒(1) Obvious. \(\square\)

**Theorem 4.3.** Let \((X,\tau_1,\tau_2,E)\) be a soft quasi \(R_0\)-space and \(x,e,y \in X\). Then either \(qCl\{x_e\} = qCl\{y_e\}\) or \(qCl\{x_e\} \cap qCl\{y_e\} = \phi\).

**Proof.** Suppose that \(qCl\{x_e\} \cap qCl\{y_e\} \neq \phi\), and let \(z_e \in qCl\{x_e\} \cap qCl\{y_e\}\). Now, \(z_e \in qCl\{x_e\} \Rightarrow qCl\{z_e\} \subseteq qCl\{x_e\}\). By Theorem 4.2(4), \(z_e \in qCl\{x_e\} \Rightarrow x_e \in qCl\{z_e\} \Rightarrow qCl\{x_e\} \subseteq qCl\{z_e\} \Rightarrow qCl\{z_e\} \subseteq qCl\{y_e\}\). Therefore, \(qCl\{z_e\} \subseteq qCl\{y_e\}\). Hence, \(qCl\{x_e\} \cap qCl\{y_e\} = \phi\).
Let \( qCl\{z_e\} \). And so, \( qCl\{x_e\} = qCl\{z_e\} \). Similarly, \( qCl\{y_e\} = qCl\{z_e\} \). Consequently, \( qCl\{x_e\} = qCl\{y_e\} \).

\[ \square \]

**Theorem 4.4.** Let \((X, \tau_1, \tau_2, E)\) be a soft quasi \(R_0\)-space and \(x_e, y_e \in X\). Then either \(qKer\{x_e\} = qKer\{y_e\}\) or \(qKer\{x_e\} \cap qKer\{y_e\} = \phi\).

**Proof.** Evident. \( \square \)

**Theorem 4.5.** Let \((X, \tau_1, \tau_2, E)\) be a soft bitopological space and \((Y, (\tau_1)_Y, (\tau_2)_Y, E)\) be a subspace of \((X, \tau_1, \tau_2, E)\). If \(X\) is soft \(R_0\) then so is \(Y\).

**Proof.** Let \((X, \tau_1, \tau_2, E)\) be a soft quasi \(R_0\)-space and let \((Y, (\tau_1)_Y, (\tau_2)_Y, E)\) be a subspace of \(X\). Let \((F, E)_Y\) be a soft quasi closed set in \(Y\) and \(x_e\) be a soft point of \(Y\) such that \(x_e \notin (F, E)_Y\). There exists a \((\tau_1)_Y\)-soft closed set \((A, E)\) and a \((\tau_2)_Y\)-soft closed set \((B, E)\) such that \((F, E)_Y = (A, E) \cap (B, E)\). Now there is a \((\tau_1)_Y\)-soft closed set \((F, E)\) such that \((A, E) = (F, E) \cap \tilde{Y}\) and a \((\tau_2)_Y\)-soft closed set \((H, E)\) such that \((A, E) = (F, E) \cap \tilde{Y}\). Therefore \((F, E)_Y = ((F, E) \cap (H, E)) \cap \tilde{Y}\).

Now, the soft point \(x_e \in Y\) and \(x_e \notin (F, E)_Y\) implies that \(x_e \notin (F, E) \cap (H, E)\). It is clear that \((F, E) \cap (H, E)\) is soft quasi closed in \(X\). By hypothesis the space \(X\) is soft quasi \(R_0\), therefore by Theorem 4.2 (5), there exists a soft quasi open set \((G, E)\) in \(X\) such that \((F, E) \cap (H, E) \subseteq (G, E)\) and \(x_e \notin (G, E)\). Thus by Theorem 3.4, \(\tilde{Y} \cap (G, E) = (G, E)_Y\) is a soft quasi open set in \(Y\) such that \(x_e \notin (G, E)_Y\) and \((F, E)_Y \subseteq (G, E)_Y\). Consequently by Theorem 4.2(5) the subspace \((Y, (\tau_1)_Y, (\tau_2)_Y, E)\) is soft quasi \(R_0\). \( \square \)

**5. Conclusion**

In this paper, soft quasi \(R_0\) spaces have been introduced, it is shown with the help of example that union of soft quasi open sets in a soft bitopological space is soft quasi open but intersection of two soft quasi open sets in a soft bitopological space may not be soft quasi open. Several characterizations and properties of soft quasi \(R_0\) spaces have been studied.

**Conflict of Interest**

The author(s) declare that there is no conflict of interests.
REFERENCES