PERIODIC SOLUTIONS FOR AN IMPULSIVE PREDATOR-PREY MODEL WITH HOLLING TYPE FUNCTIONAL RESPONSE AND TIME DELAYS

S. MAHALAKSHMI\textsuperscript{1,*}, V. PIRAMANANTHAM\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, BIT-Campus Anna University, Tiruchirappalli 620024, India
\textsuperscript{2}Department of Mathematics, Bharathidasan University, Tiruchirappalli 620024, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper we propose an impulsive predator-prey model with time delays. By applying the continuation theorem of coincidence degree theory, we establish a better estimation on the difference between the supremum and infimum of a differentiable piecewise continuous periodic function.

Keywords: impulsive Predator-prey; Holling-type functional response; time delay.

2010 AMS Subject Classification: 34C25, 34K13, 92D25.

1. INTRODUCTION

One of the powerful and effective methods on the existence of periodic solutions to periodic systems is the continuation method, which gives easily verifiable sufficient conditions. In [5] Bazykin proposed the following Predator-prey system.

\begin{equation}
\begin{aligned}
   &u'(t) = u(t) \left( a - \epsilon u(t) - \frac{bv(t)}{1 + \alpha u(t)} \right) \\
   &v'(t) = v(t) \left( -c + \frac{du(t)}{1 + \alpha u(t)} - \eta v(t) \right)
\end{aligned}
\end{equation}

*Corresponding author
E-mail address: mahasenthil2@gmail.com
Received June 17, 2021
where \( u(t), v(t) \) represents the densities of prey and predator population respectively where \( a, b, c, d, \alpha, \eta, \varepsilon \) are positive parameters. System (1) is called Holling Type II predator system model. It is investigated in [5] for the stability of equilibrium and condimension two bifurcations. The global behavior of system (1) has been discussed by many authors, for example [3], [16]. In [2] the effects of the periodicity of ecological and environmental parameters and time delays due to gestation and negative feedbacks on the global dynamics of predator-prey systems with Holling-type-II functional response.

\[
x_1'(t) = x_1(t) \left[ r_1(t) - a_{11}(t)x_1(t - \tau_1(t)) - \frac{a_{12}(t)x_2(t)}{1 + mx_1(t)} \right] \\
x_2'(t) = x_2(t) \left[ -r_2(t) + \frac{a_{21}(t)x_1(t - \tau_2(t))}{1 + mx_1(t - \tau_2(t))} - a_{22}(t)x_2(t - \tau_3(t)) \right]
\]

(2)

where \( a_{11}, a_{12}, a_{21}, a_{22}, \tau \) are continuous \( \omega \)-periodic functions \( \tau_1, \tau_3 \geq 0 \) denote the time delays due to negative feedbacks of the prey and the predator population \( \tau_2 \) is a time delay due to gestation that is mature adult predators can only contribute to the reproduction of predator biomass \( a_{11}(t), a_{22}(t) \) are the intra-specific rates of the prey and the predator respectively, \( a_1 \) is the capturing rate of predator \( a_{22}(t)/a_{12}(t) \) is the conversion rate of nutrients into the reproduction of the predator. Time delays due to gestation is a common example, the consumption of prey by the predator throughtout its past history governs the present birth rate of predator.[1, 2, 5, 7, 13]. It is well known from the fundamental theory of impulsive differential equations [4, 6, 9, 10, 11, 12] that the system (2) has a unique solution.

In this paper we shall consider (2) with impulsive effects. Precisely, we consider the following delayed impulsive system

\[
x_1'(t) = x_1(t) \left[ r_1(t) - a_{11}(t)x_1(t - \tau_1(t)) - \frac{a_{12}(t)x_2(t)}{1 + mx_1(t)} \right] \\
x_2'(t) = x_2(t) \left[ -r_2(t) + \frac{a_{21}(t)x_1(t - \tau_2(t))}{1 + mx_1(t - \tau_2(t))} - a_{22}(t)x_2(t - \tau_3(t)) \right] \\
\Delta x_1(t) = x_1(t^+) - x_1(t) = d_{1k}x_1(t); t = t_k \\
\Delta x_2(t) = x_2(t^+) - x_2(t) = d_{2k}x_1(t); t = t_k
\]

(3)

where the assumptions are the same as in (2), \( d_{1k}, d_{2k} \in (-1, 0)(k \in \mathbb{N}), t_k \) is a strictly increasing sequence with \( t_1 > 0 \) and \( \lim_{k \to \infty} \) and assume that \( d_{1(k+q)} = d_{1k}, d_{2(k+q)} = d_{2k}, t_{k+q} = t_k + \omega \) for \( k \in \mathbb{N} \).
In the next section, by using the continuation theorem of coincidence degree theory, we discuss the existence of positive \( \omega \)-periodic solutions of system (3) and in section [3] the uniqueness and global stability of the positive \( \omega \)-periodic solutions of system (3).

2. Existence of Positive Periodic Solutions

In this section, we prove the existence of solutions of periodic solution. For the reader’s convenience, we provide some notations and definitions and also we first prepare the functional analytic settings:

Let \( PC_\omega \) be the space of all functions \( \phi \) such that \( \phi \) left continuous at all points, \( \phi \) is right continuous at \( t \neq t_k \), \( \lim_{t \to t_k^-} \phi(t) \) exists and \( \phi(t + \omega) = \phi(t) \), \( PC'_\omega \) the space of all functions \( \phi \in PC_\omega \) which are continuously differentiable at \( t \neq t_k \), \( \lim_{t \to t_k^-} \phi(t) \) exists and \( \lim_{t \to t_k^+} \phi'(t) \) and \( \lim_{t \to t_k^-} \phi(t) \) exist, \( k \in \mathbb{Z}^+ \).

Let \( X, Z \) be normed linear spaces, \( L : \text{Dom} \ L \subset X \to Z \) be a linear transformation, and \( N : X \to Z \) be a continuous function. The map \( L \) is known as a Fredholm map of index zero if \( \dim \text{Ker} \ L = \text{codim} \text{Im} \ L < +\infty \) and \( \text{Im} \ L \) is closed in \( Z \). If \( L \) is a Fredholm mapping of index zero there exist continuous projectors \( P : X \to X \), \( Q : Y \to Y \) such that \( \text{Im} \ L = \text{Im} (I - Q) \). This implies that the restriction \( L|_p \) of \( L \) to \( \text{Dom} \ L \cap \text{Ker} \ P : (I - P)X \to \text{Im} \ L \) is invertible. The inverse of \( L_p \) is denoted by \( K_p \). If \( \Omega \) is an open bounded subset of \( X \), the mapping \( N \) will be called \( L \)-compact on \( \Omega \) if \( QN(\Omega) \) is bounded and \( K_p(I - Q)N : \Omega \to X \) is compact. Since \( \text{Im} \ Q \) is isomorphic to \( \text{Ker} \ L \), there exists an isomorphism \( J : \text{Im} \ A \to \text{Ker} \ L \).

We will make some notations and definitions which will be used in the proof of the main theorem

\[
\begin{align*}
d_1 &= \sum_{k=1}^{n} \log(1 + d_{1k}) \\
d_2 &= \sum_{k=1}^{n} \log(1 + d_{2k}) \\
\bar{f} &= \frac{1}{\omega} \int_{0}^{\omega} f(t)dt \\
f^L &= \sup_{t \in [0, \omega]} |f(t)| \\
f^M &= \inf_{t \in [0, \omega]} |f(t)|
\end{align*}
\]

**Definition 2.1.** The set \( \mathcal{F} \subset PC_\omega \) is said to be equicontinuous if for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( u \in \mathcal{F}, k \in \mathbb{Z}^+ \) and \( t' and t'' \in [t_{k-1}, t_k] \cap [0, \omega] \) and \( |t' - t''| < \delta \), then \( |u'(t') - u(t'')| < \epsilon \)
Lemma 2.2. [15] The set $\mathcal{F} \subset PC_\omega$ is relatively compact if and only if

(i) $\mathcal{F}$ is bounded, that is, $\|u\| = \sup_{t \in [0, \omega]} \|u(t)\| \leq M$ for each $u \in \mathcal{F}$;

(ii) $\mathcal{F}$ is quasi-equicontinuous.

Our existence theorem for periodic solution of the equation (2) is proved with the help of the following theorem of Gaines and Mawhin [8]

**Theorem 2.3.** Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\overline{\Omega}$. Suppose that

(i) for each $\lambda \in (0, 1)$, every solution $x$ of $Lx \neq \lambda Nx$ is such that $x \not\in \Omega$;

(ii) for each $\lambda \in \partial \Omega \cap \text{Ker}L$, $QNx \neq 0$;

(iii) $\deg\{JQN, \Omega \cap \text{Ker}L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution lying in $\text{Dom} L \cap \overline{\Omega}$.

**Theorem 2.4.** The system (2) has at least one positive $\omega$-periodic solution provided that

$$
(A_1) \quad (\overline{r}_1 + d_1)(\overline{a}_{21} - m(\overline{r}_2 + d_2)) - (\overline{a}_{11})(\overline{r}_2 + d_2)e^{(2(\overline{r}_\omega + d_1))} > 0
$$

**Proof.** Let $x_1(t) = e^{u_1(t)}, x_2(t) = e^{u_2(t)}$. Then we obtain the following equivalent system:

$$
\begin{align*}
    u_1'(t) &= \begin{bmatrix} r_1(t) - a_{11}(t)e^{u_1(t-\tau_1(t))} - \frac{a_{12}(t)e^{u_2(t)}}{1 + mu_1(t)} \end{bmatrix} \\
    u_2'(t) &= \begin{bmatrix} -r_2(t) + \frac{a_{21}(t)e^{u_1(t-\tau_2(t))}}{1 + me^{u_1(t-\tau_2(t))}} - \frac{a_{22}(t)e^{u_2(t-\tau_3(t))}}{1 + me^{u_1(t-\tau_3(t))}} \end{bmatrix} \\
    \Delta u_1(t) &= \log(1 + d_{1k}) \\
    \Delta u_2(t) &= \log(1 + d_{2k})
\end{align*}
$$

(4)

It is easy to see that if system (4) has one $\omega$-Periodic solution $(u_1^*, u_2^*)$ then the corresponding $x^*(t) = (x_1^*(t), x_2^*(t))^T$ is a periodic solution of (3). Therefore, to complete proof, it suffices to show that the system (4) has at least one $\omega$ periodic solution.
Let

\[ X = \{ u = (u_1, u_2)^T \in PC_\omega([0, \omega], \mathbb{R}^2) : u_i(t + \omega) = u_i(t), i = 1, 2 \}, Z = X \times \mathbb{R}^{2(q+1)}. \]

Let us define

\[ \|u\| = \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_2(t)|, \]

and for any \((u, \eta) \in Z\)

\[ \|(u, \eta)\| = \|u\| + \sum_{j=1}^{2(q+1)} |\eta_j|. \]

Then \(X\) and \(Z\) are Banach spaces. Set \(L : \text{Dom } L \cap X \to Z, L(u) = (u'(t), \Delta u(t_k)_{k=1}^q), \)

where

\[ \text{Dom } L = \{ u = (u_1, u_2)^T \in PC_\omega([\mathbb{R}, \mathbb{R}^2) : u_i \in PC_\omega, i = 1, 2 \}. \]

and \(N : X \to Z,\)

\[ N\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} r_1(t) - a_{11}(t)e^{a_1(t-\tau_1(t))} - \frac{a_{12}(t)e^{a_2(t)}}{1 + me^{a_1(t)}} \\ -r_2(t) + \frac{a_{21}(t)e^{a_1(t-\tau_2(t))}}{1 + me^{a_1(t-\tau_1(t))}} - a_{22}e^{a_2(t-\tau_2(t))} \end{pmatrix} \cdot \begin{pmatrix} \log(1 + d_{1k}) \\ \log(1 + d_{2k}) \end{pmatrix}_{k=1}^q. \]

\[ P : X \to X, P((u_1, u_2)^T) = (\overline{u_1}, \overline{u_2})^T \text{ and } Q : Z \to Z,\]

\[ Q\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \left\{ \begin{pmatrix} m_k \\ n_k \end{pmatrix}_{k=1}^q \right\} = \begin{pmatrix} \frac{1}{\omega} \int_0^\omega u_1(t) \, dt + \frac{1}{\omega} \sum_{k=1}^q m_k \\ \frac{1}{\omega} \int_0^\omega u_2(t) \, dt + \frac{1}{\omega} \sum_{k=1}^q n_k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{k=1}^q. \]

It is easy to see that

\[ \text{Ker } L = \left\{ u = (u_1, u_2)^T \in X : \exists c \in \mathbb{R}^2, (u_1(t), u_2(t)) = c, \text{ for } t \in \mathbb{R} \right\}. \]

\[ \text{Im } L = \left\{ y = (u, \eta_1, \eta_2, \ldots, \eta_{2q}) \in Y : \exists u \in \text{Dom } L, \int_0^\omega u(s) \, ds + \sum_{k=1}^{2q} \eta_k = 0 \right\}. \]

Since \(\text{Im } L\) is closed in \(Y\) and \(\text{dimker } L = \text{codimIm } L = 2, L\) is a Fredholm mapping of index zero. Moreover, the generalized inverse (to \(L\)) \(K_p : \text{Im } L \to \text{Ker } P \cap \text{Dom } L\) is

\[ K_p(u) = \int_0^t u(s) \, ds + \sum_{0 < k < t} \eta_k - \frac{1}{\omega} \int_0^\omega \int_0^t u(s) \, ds \, dr - \sum_{k=1}^{2q} \eta_k. \]
Then direct computation gives us

\[
QN \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = \left( \begin{array}{c} \frac{1}{\omega} \int_0^\omega \left[ (r_1(t) - a_{11}(t)) e^{u_1(t - \tau_1(t))} - \frac{a_{12}(t)e^{u_2(t)}}{1 + m e^{u_1(t)}} \right] \, dt + \frac{1}{\omega} \sum_{k=1}^n \log(1 + d_{1k}) \\
\frac{1}{\omega} \int_0^\omega \left[ -r_2(t) e^{u_1(t - \tau_2(t))} - a_{22}(t) e^{u_2(t - \tau_3(t))} \right] \, dt + \frac{1}{\omega} \sum_{k=1}^n \log(1 + d_{2k}) \end{array} \right), \quad \left\{ \begin{array}{c} 0 \\
0 \end{array} \right\}_{k=1}^q
\]

and

\[
K_p(I - Q)N \left( \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = \left( \begin{array}{c} \frac{1}{\omega} \int_0^\omega \left[ r_1(s) - a_{11}(s) e^{u_1(s - \tau_1(s))} - \frac{a_{12}(s)e^{u_2(s)}}{1 + m e^{u_1(s)}} \right] \, ds + \sum_{\alpha < k < t} \log(1 + d_{1k}) \\
\frac{1}{\omega} \int_0^\omega \left[ -r_2(s) + a_{21}(s) e^{u_1(s - \tau_2(s))} - a_{22}(s) e^{u_2(s - \tau_3(s))} \right] \, ds + \sum_{k=1}^n \log(1 + d_{2k}) \end{array} \right)
\]

Clearly, \(QN\) and \(K_p(I - Q)N\) are continuous. Furthermore, it follows from Lemma 2.2 that \(QN(\overline{\Omega})\) and \(K_p(I - Q)N(\overline{\Omega})\) are relatively compact for any open bounded set \(\Omega \subset X\). Therefore, \(N\) is \(L\)-compact on \(\overline{\Omega}\) for any open bounded set \(\Omega \subset X\). In the following, we consider the operator equation \(Lu = \lambda Nu\), \(\lambda \in (0, 1)\), that is,

\[
\begin{align*}
u_1^1(t) &= \lambda \left[ r_1(t) - a_{11}(t) e^{u_1(t - \tau_1(t))} - \frac{a_{12}(t)e^{u_2(t)}}{1 + m e^{u_1(t)}} \right], \quad t \neq t_k, \\
u_1^2(t) &= \lambda \left[ -r_2(t) + a_{21}(t) e^{u_1(t - \tau_2(t))} - a_{22}(t) e^{u_2(t - \tau_3(t))} \right], \quad t \neq t_k, \\
\Delta u_1(t) &= \lambda \left[ \log(1 + d_{1k}) \right], \\
\Delta u_2(t) &= \lambda \left[ \log(1 + d_{2k}) \right].
\end{align*}
\]
Integration of both sides of the system (5) from 0 to \( \omega \) gives

\[
\omega (\bar{T}_1) + d_1 = \int_0^\omega \left[ a_{11} e^{u_1(t-\tau_1(t))} + \frac{a_{12}(t)e^{u_2(t)}}{1+pe^{u_2(t)}} \right] dt.
\]

(6)

\[
\omega (\bar{T}_2) + d_2 = \int_0^\omega \left[ a_{21} e^{u_1(t-\tau_2(t))} - \frac{a_{22}(t)e^{u_2(t-\tau_2(t))}}{1+pe^{u_2(t-\tau_2(t))}} \right] dt.
\]

It follows from the first equation of (5) that

\[
\int_0^\omega |u'_1(t)| dt < \int_0^\omega \left[ r_1(t) + a_{11}(t)e^{u_1(t-\tau_1(t))} + \frac{a_{12}(t)e^{u_2(t)}}{1+pe^{u_2(t)}} \right] dt
\]

\[= 2(\bar{T}_1 \omega + d_1)\]

\[
\int_0^\omega |u'_2(t)| dt < \int_0^\omega \left[ r_2(t) + \frac{a_{21}(t)e^{u_1(t)}}{1+pe^{u_2(t-\tau_2(t))}} - a_{22}(t)e^{u_2(t-\tau_2(t))} \right] dt
\]

Since \((u_1, u_2)^T \in X\), there exists \( \varepsilon_i, \eta_i \in [0, \omega], \ i = 1, 2 \) such that

\[
u(\xi_1) = \min_{t \in [0, \omega]} u_1(t) \quad u(\eta_1) = \max_{t \in [0, \omega]} u_1(t)
\]

(7)

\[
u(\xi_2) = \min_{t \in [0, \omega]} u_2(t) \quad u(\eta_2) = \max_{t \in [0, \omega]} u_2(t)
\]

It follows from (5) and (7) that

(8)

\[
\int_0^\omega a_{11}(t)e^{u_1(t-\tau_1(t))} dt < \bar{T}_1 \omega + d_1
\]

which gives

\[
u_1(\varepsilon_1) < \ln \left( \frac{\bar{T}_1}{a_{11}} + d_1 \right)
\]

(9)

\[
u_1(t) \leq \nu_1(\varepsilon_1) + \int_0^\omega |u'_1(t)| dt
\]

\[< \ln \left( \frac{\bar{T}_1 + d_1}{a_{11}} \right) + 2(\bar{T}_1 \omega + d_1)
\]

It follows from the second equation of (5) and (7) that

\[
\int_0^\omega a_{22}(t)e^{u_2(t-\tau_2(t))} dt \leq \int_0^\omega a_{21}(t) e^{u_2(t-\tau_2(t))} dt \leq \frac{a_{21}\omega + d_2}{m}
\]

which implies that

\[
u_2(\varepsilon_2) + \int_0^\omega |u'_2(t)| dt < \ln \left[ \frac{\tilde{a}_{22}}{m\tilde{a}_{23}} + d_2 \right] + 2 \left[ \frac{\tilde{a}_{21} \omega}{d_2} \right]
\]
(10) \[ u_2(\epsilon_2) \leq \ln \left( \frac{\bar{a}_{21}}{m\bar{a}_{22}} \right) + d_2. \]

On the other hand, (5) yields that

(11) \[ \int_0^\omega a_{21}(t)e^{u_1(t-\tau_2(t))}dt \geq \bar{r}_2 \omega + d_2, \]

implying that

(12) \[ u_1(\eta_1) \geq \ln \frac{\bar{r}_2 \omega + d_2}{a_{21}}. \]

We derive from (11) and (12) that

(13) \[ u_1(t) \geq u_1(\eta_1) - \int_0^\omega |u_1'(t)|dt \geq \ln \frac{\bar{r}_1 + d_1}{a_{11}} - 2(\bar{r}_1 \omega + d_2) \]

which together with (9) gives

(14) \[ \max_{t \in [0, \omega]} |u_1(t)| < \max \left| \ln \frac{\bar{r}_1 + d_1}{a_{11}} + 2(\bar{r}_1 \omega + d_1), \ln \frac{\bar{r}_2 + d_1}{a_{11}} + 2(\bar{r}_1 \omega + d_2) \right| : \]

\[ = H_1 \]

Once again from the second equation of (5)

(15) \[ \int_0^\omega e^{a_{22}(t-\tau_3(t))}dt = \int_0^\omega \frac{a_{22}(t)e^{u_1(t-\tau_2(t))}}{1 + me^{u_1(t-\tau_2(t))}}dt - \int_0^\omega r_2(t)dt - d_2 \]

\[ \geq \bar{a}_{22} \omega e^{u_1(\epsilon_1)} \frac{1}{1 + me^{u_1(\epsilon_1)}} - \bar{r}_2 \omega - d_2 \]

which implies

(16) \[ \bar{a}_{22}(1 + me^{u_1(\epsilon_1)})e^{u_2(\eta_2)} \geq (\bar{a}_{21}) - m(\bar{r}_2 - d_2)e^{u_1(\epsilon_1)} - \bar{r}_2 - d_2. \]

Therefore

(17) \[ u_1(\eta_1) \leq u_1(\epsilon_1) + \int_0^\omega |u_1'(t)|dt \]

\[ < u_1(\epsilon_1) + 2(\bar{r}_1 \omega + d_1) \]

\[ \bar{r}_1 + \frac{d_1}{\omega} \leq \bar{a}_{11}e^{u_1(\eta_1)} + \bar{a}_{12}e^{u_2(\eta_2)}. \]

We derive from (16) that

(18) \[ \bar{a}_{22}(1 + me^{u_1(\epsilon_1)})e^{u_2(\eta_2)} \geq \frac{(\bar{a}_{21} - m(\bar{r}_2 - d_2))(\bar{r}_1 + d_1 - \bar{a}_{12}e^{u_2(\eta_2)})}{\bar{a}_{11}e^{2(\bar{r}_1 \omega + d_1)}} \]

which together with (2.9) implies
Clearly if it exits and the following satisfies taken sufficiently large such that each solution $(\alpha^*, \beta^*)$ of the following algebraic equations
\begin{align*}
\frac{\bar{a}_1}{\bar{0}} + \bar{r}_1 - \bar{a}_{11} e^\alpha - \frac{\bar{a}_{12} e^\beta}{1 + m e^\alpha} &= 0 \\
-\bar{r}_2 + \frac{\bar{d}_2}{\bar{o}} + \frac{\bar{a}_{21} e^\alpha}{1 + m e^\alpha} - \bar{a}_{22} e^\beta &= 0
\end{align*}

satisfies
\begin{equation}
|| (\alpha^*, \beta^*)^T || = |\alpha^*| + |\beta^*| < H
\end{equation}

if it exits and the following
\begin{align*}
\max \left[ \left| \ln \frac{\bar{r}_1 + d_1}{\bar{a}_{11}} \right|, \left| \ln \frac{\bar{r}_2 + d_2}{\bar{a}_{21}} \right| \right] \\
+ \max \left[ \left| \ln \frac{\bar{a}_{21}}{m \bar{a}_{22}} \right|, \left| \ln \frac{(\bar{r}_1 + d_1)(\bar{a}_{21} - m(\bar{r}_2 + d_2)) - \bar{a}_{11}(\bar{r}_2 + d_2)}{\bar{a}_{11} \bar{a}_{22} (1 + m(\bar{r}_1 + d_1))(1 + m(\bar{r}_1 + d_1))} \right| \right]
\end{align*}

< H.
We now take $\Omega = (u_1(t), u_2(t))^T \in X : \|(u_1, u_2)^T\| < H$. This satisfies the condition (a) in Theorem 2.3. When $(u_1(t), u_2(t))^T \in \partial \Omega \cap \text{Ker}L = \partial \Omega \cap \mathbb{R}^2$, $(u_1, u_2)^T$ is a constant vector in $\mathbb{R}^2$ with $|u_1| + |u_2| = H$. If (22) has at least one solution, then

$$QN\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} m_k \\ n_k \end{pmatrix}_{k=1} \right]^{n} = \begin{pmatrix} \frac{\bar{a}_{11}}{\bar{m}} + r_1 - \bar{a}_{11}e^{u_1} - \frac{\bar{a}_{12}e^{u_2}}{1 + m e^{u_1}} \\ -\bar{r}_2 + \frac{\bar{d}_2}{\bar{d}} + \frac{\bar{a}_{21}e^{u_1}}{1 + m e^{u_1}} - \bar{a}_{22}e^{u_2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{k=1} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If (22) does not have a solution, we can directly derive

$$QN\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This proves that condition (ii) in Theorem 2.3 is satisfied. Finally we prove that condition (iii) in Theorem 2.3 holds. Now we define $\phi : \text{Dom } L \times [0,1] \to X$ by

$$\phi(u_1, u_2, \mu) = \begin{pmatrix} \bar{r}_1 + d_1 - \bar{a}_{11}e^{u_1} \\ \frac{\bar{a}_{21}e^{u_1}}{1 + m e^{u_1}} - \bar{a}_{22}e^{u_2} \end{pmatrix} + \mu \begin{pmatrix} \bar{a}_{12}e^{u_1} \\ \bar{r}_2 - d_2 \end{pmatrix}$$

where $\mu \in [0,1]$ is a parameter. When $(u_1(t), u_2)^T \in \partial \Omega \cap \text{Ker}L \phi(u_1, u_2, \mu) \neq 0$ otherwise, there is a constant vector $(u_1, u_2)^T$ with $|u_1 + u_2| = H$ satisfying $\phi(u_1, u_2, \mu) = 0$

$$\bar{r}_1 + d_1 - \bar{a}_{11}e^{u_1} - \mu \frac{\bar{a}_{21}e^{u_1}}{1 + m e^{u_1}} = 0$$

$$\frac{\bar{a}_{21}e^{u_1}}{1 + m e^{u_1}} - \bar{a}_{22}e^{u_2} - \mu (\bar{r}_2 + d_2) = 0$$

A similar argument in (14) and (21) shows that

$$|u_1| < \max \left\{ |\ln \frac{\bar{r}_1 + d_1}{\bar{a}_{11}}, |\ln \frac{\bar{r}_1 + d_1}{\bar{a}_{21}}| \right\}$$

$$|u_2| < \max \left\{ |\ln \frac{\bar{a}_{21}}{\bar{m}} \bar{a}_{22}, |\ln \frac{(\bar{r}_1 + d_1)(\bar{a}_{21} - m(\bar{r}_2 + d_2)) - \bar{a}_{11}(\bar{r}_2 + d_2)}{} \right\}$$

It follows from (24) that $|u_1| + |u_2| < H$ which leads to a contradiction. Using the property of topological degree and taking $J = I : \text{Im}Q \to \text{Ker}L$, we have $(u_1, u_2)^T \to (u_1, u_2)^T$

$$\deg(QN((u_1, u_2)^T, \Omega \cap \text{Ker}L, (0,0)^T)) = \deg(\phi(u_1, u_2, 1), \Omega \cap \text{Ker}L, (0,0)^T)$$

$$= \deg(\phi(u_1, u_2, 0), \Omega \cap \text{Ker}L, (0,0)^T)$$

$$= \deg\left((\bar{r}_1 + d_1) - \bar{a}_{11}e^{u_1}, \frac{\bar{a}_{21}e^{u_1}}{1 + m e^{u_1}} - \bar{a}_{22}e^{u_2} \right), \partial \cap \text{Ker}L, (0,0)^T)$$
The system of algebraic equations
\[ \bar{r}_1 + \frac{d_1}{\omega} - \bar{a}_{11}e^{u_1} = 0 \]
\[ \frac{\bar{a}_{21}e^{u_1}}{1 + me^{u_1}} - \bar{a}_{22}e^{u_2} = 0 \]
which has a unique solution \((u_1^*, u_2^*)\) which satisfies
\[ u_1^* = \ln \frac{\bar{r}_1 + d_1}{\bar{a}_{11}}, \]
\[ u_2^* = \ln \frac{\bar{a}_{21}(\bar{r}_1 + d_1)}{\bar{a}_{22}(\bar{a}_{11} + m(\bar{r}_1 + d_1))} \]
Thus a direct calculation shows that
\[
\text{deg}(JQN(u_1, u_2)^T, \Omega \cap KerL, (0, 0)^T) = \begin{vmatrix}
\frac{-\bar{a}_{11}e^{u_1}}{} & 0 \\
\frac{\bar{a}_{21}e^{u_1}}{1 + me^{u_1}} & -\bar{a}_{22}e^{u_2} \\
\end{vmatrix} = \text{sgn}\{\bar{a}_{11}\bar{a}_{22}e^{u_1^*+u_2^*}\} = 1
\]
Finally, it is easy to show that the set \(k_p(I - Q)Nx| x \in \bar{\Omega}\) is equicotinous and uniformly bounded. By using the Arzela-Ascoli theorem, we see that \(k_p(I - Q)N : \bar{\Omega} \rightarrow X\) is compact. Consequently \(L\) is compact. By now we have proved that \(\Omega\) satisfies the Lemma(1.1). Hence (4) has at least one \(\omega\) periodic solution. As a consequence, system (2) has at least one positive \(\omega\)–periodic solution. \(\Box\)

3. Stability of Positive Periodic Solutions

We consider the nonimpulsive delay differential equation
\[
\begin{align*}
y_1'(t) &= y_1(t)\left(r_1(t) - a_{11}(t)y_1(t - \tau_1(t)) - \frac{a_{12}(t)y_2(t)}{D_1(t) + my_1(t)}\right) \\
y_2'(t) &= y_2(t)\left(-r_2(t) + a_{21}(t)\frac{y_1(t - \tau_2(t))}{D_2(t) + my_1(t - \tau_2(t))} - a_{22}(t)y_2(t - \tau_3(t))\right)
\end{align*}
\]
with the initial conditions
\[
y_i(s) = f_i(s), f_i(0) > 0, f_i \in \mathbb{R}, [-\tau, 0], R_+, i = 1, 2
\]
\[
\tau = \max_{t \in [0, \omega]} \tau_1(t), \tau_2(t), \tau_3(t)
\]
where

\[ a_{11}(t) = \Pi_{0 < k, t} a_{11}(1 + d_1 k) \]
\[ D_1(t) = \Pi_{0 < k, t} (1 + d_1 k)(1 + d_2 k)^{-1} \]
\[ D_2(t) = \Pi_{0 < k, t} (1 + d_1 k)^{-1}(1 + d_2 k) \]

**Lemma 3.1.** Let \( x(t) = (x_1(t), x_2(t))^T \) is a solution of (4) with initial conditions. Then there exists a \( T_1 > 0 \) such that \( 0 < x_i(t) \leq H_i, i = 1, 2 \) for \( t \geq T_1 \), where

\[
H_1 = H_2 > H = \max \left\{ \frac{r_1^U}{a_{11}}, \frac{r_2^L}{a_{21}} \right\}
\]

**Lemma 3.2.** Let \( x(t) = (x_1(t), x_2(t))^T \) is a solution of (4) with initial conditions. Then there exists a \( T_2 > 0 \) such that \( 0 < x_i(t) \geq h_i, i = 1, 2 \) for \( t \geq T_2 \), where

\[
h_1 = h_2 < H = \max \left\{ \frac{r_1^L - a_{11}^U H_1}{a_{11}}, e^{-(r_2^U)} \right\}
\]

**Theorem 3.3.** Assume that the conditions of Theorem(2.3) hold. In addition, assume

\[
\int_0^\omega A(t) \, dt > 0
\]

where

\[
A(t) = \min \{ \min a_{11}(t) + a_{11}(t)D_1(t)h - mHa_{22}(t)D_2(t) \}, \quad \min \{ a_{11}(t)mh + a_{11}(t)D_1(t)h + mHa_{22}(t)D_2(t) \}
\]

**Proof.** Let \( x^*(t) = (x_1^*(t), x_2^*(t))^T \) be a positive \( \omega \)-periodic solution (2) then \( y^*(t) = (y_1^*(t), y_2^*(t))^T, (y^*(t) = \Pi_{0 < k, t} (1 + d_1 k)^{-1} x_i^*(t), i = 1, 2) \) is a positive \( \omega \)-periodic solution (26) and let \( (y_1(t), y_2(t))^T \) be any positive solution of system (26) with the initial conditions (27). It follows from Lemma 3.1, and Lemma 3.2 that there exists \( T, H_i \) and \( h_i \) such that \( \forall t \geq T \)

\[
h_i \leq y_i^*(t) \leq H_i, h_i \leq y_i(t) \leq H_i, i = 1, 2.
\]

Choose Lyapunov function as follows

\[
V(t) = \sum_{i=1}^2 |\log y_i^*(t) - \log y_i(t)|.
\]
Since for any impulsive time \( t_k \) we have

\[
V(t^+_k) = \sum_{i=1}^{2} |log d_i y_i^+(t_k) - log d_i y_i^-(t_k)| = V(t_k).
\]

\( V(t) \) is continuous for all \( t \geq 0 \)

On the other hand, from (2.3) we can obtain that for any \( t \in \mathbb{R}^k \) and \( t \neq t_k \)

\[
\frac{1}{h} |y_i^+(t) - y_i(t)| \leq |log y_i^+(t) - y_i(t)| \leq \frac{1}{h} |y_i^+(t) - y_i(t)|.
\]

_calculated the upper-right derivative of \( V(t) \) along the solutions of (3.1) it follows that

\[
D^+V(t) = \sum_{i=1}^{2} \left( \frac{y_i^+(t)}{y_i(t)} - \frac{y_i^-(t)}{y_i^+(t)} \right) \text{sgn}(y_i^+(t) - y_i(t))
\]

\[
\leq \text{sgn}(y_i^+(t) - y_i(t)) \left[ -a_{11}[y_i^+(t - \tau_1(t)) - y_i(t - \tau_1(t))] 
\right. \\
- a_{12}(t) \left[ \frac{y_2(t)}{D_1(t) + my_1^*(t)} - \frac{y_2(t)}{D_1(t) + my_1(t)} \right]
\]

\[
+ \text{sgn}(y_2^+(t) - y_2(t)) \left[ a_{21}(t) \left[ \frac{y_2(t - \tau_2(t))}{D_2(t) + my_1^*(t - \tau_2(t))} - \frac{y_2(t - \tau_2(t))}{D_2(t) + my_1(t - \tau_2(t))} \right]
\]

\[
- a_{22}(t)[y_2^+(t - \tau_3(t)) - y_2(t - \tau_3(t))]
\]

\[
D^+V(t) \leq -\text{sgn}(y_i^+(t) - y_i(t))[a_{11}(y_i^+(t - \tau_1(t)) - y_i(t - \tau_1(t)))]
\]

\[
-\text{sgn}(y_2^+(t) - y_2(t))[a_{22}(t)[y_2^+(t - \tau_3(t)) - y_2(t - \tau_3(t))]] + \Delta_1 + \Delta_2
\]

where

\[
\Delta_1 = -\text{sgn}(y_i^+(t) - y_i(t))a_{12}(t) \left[ \frac{y_2(t)}{D_1(t) + my_1^*(t)} - \frac{y_2(t)}{D_1(t) + my_1(t)} \right]
\]

\[
= -\text{sgn}(y_i^+ - y_i)a_{12}(t) \left[ \frac{y_2^*(t)(D_1(t) + my_1(t)) - y_2(t)(D_1(t) + my_1^*(t))}{(D_1(t) + my_1^*(t))(D_1(t) + my_1(t))} \right]
\]

\[
= -\text{sgn}(y_i^+ - y_i)a_{12}(t) \left[ \frac{y_2^*D_1(t) + m(y_2^*y_1 - y_2^*y_2 - y_2y_1 + y_2y_1) - D_1(t)y_2 + D_1(t)y_2 - y_2D_1(t)}{(D_1(t) + my_1^*(t))(D_1(t) + my_1(t))} \right]
\]

\[
\leq -a_{12}mh|y_2^* - y_2| - a_{12}mh|y_i^* - y_i| - a_{12}D_1(t)|y_2^* - y_2|
\]
\[
\Delta_2 = -\text{sgn}(y_2^*(t) - y_2(t)) a_{21}(t) \left[ \frac{y_1^*(t)}{D_2(t) + my_1^*(t)} - \frac{y_1(t)}{D_2(t) + my_1(t)} \right] \\
= -\text{sgn}(y_2^* - y_2) a_{21} \left[ \frac{D_2(t) y_1^* + my_1^* - D_2(t) y_1 - my_1^* y_1}{(D_2(t) + my_1^*)(D_2(t) + my_1)} \right] \\
\leq a_{21}(t) D_2(t) |y_1^* - y_1|
\]

It follows from \( \Delta_1 \) and \( \Delta_2 \) that

\[
D^+ V(t) \leq -a_{11}(t) |y_1^* - y_1| - a_{22}(t) |y_2^* - y_2| + a_{21}(t) D_2(t) |y_1^* - y_1| \\
- a_{12} m h |y_2^* - y_2| - a_{12}(t) m h |y_1^* - y_1| - a_{12}(t) D_1(t) |y_2^* - y_2| \\
\leq |y_1^* - y_1| [-a_{11}(t) + a_{21}(t) D_2(t) - a_{12} m h] \\
+ |y_2^* - y_2| [-a_{22}(t) - a_{12}(t) m h - a_{12}(t) D_1(t)] \\
\leq -[a_{11}(t) - a_{21}(t) D_2(t) + a_{12} m h] |y_1^* - y_1| \\
- [a_{22}(t) + a_{12}(t) m h - a_{12}(t) D_1(t)] |y_2^* - y_2| \\
\leq -A(t)
\]

From this, we further have any \( t \geq 0; V(t) \leq V(0) e^{(-\int_0^t A(u) du)} \). From (26) we can \( \int_0^t A(u) du \to \infty \) as \( t \to \infty \). Hence, \( V(t) \to 0 \) as \( t \to \infty \). Further from (33) we have

\[
\lim_{t \to \infty} |y_i^* - y_i(t)| = \lim_{t \to \infty} \left[ \Pi_{0 < t_k, t} (1 + d_i k)^{-1} |x_i^*(t) - x_i(t)| \right] = 0, i = 1, 2
\]

Therefore \( \lim_{t \to \infty} |x_i^*(t) - x_i(t)| = 0, i = 1, 2 \)

\[\square\]

4. **Illustrating Example**

**Example 4.1.** To illustrate the result obtained, we consider the system

\[
x'_1(t) = x_1(t) \left[ (1 + 0.1 \sin t) - 0.1 x_1(t - 0.1) - \frac{x_2(t)}{1 + x_1(t)} \right] \\
x'_2(t) = x_2(t) \left[ -\frac{1}{10^5} (2 + \sin t) + \frac{9 x_1(t - 0.3)}{1 + x_1(t - 0.3)} - x_2(t - 0.1) \right] \\
\Delta x_1(t) = (\frac{1}{e} - 1)x_1(t), t = t_k \\
\Delta x_2(t) = (\frac{1}{e} - 1)x_2(t), t = t_k
\]
It is very simple to verify the conditions of the Theorem 2.4 and the system has at least one positive periodic solutions.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES
