Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7253-7268
https://doi.org/10.28919/jmcs/6605
ISSN: 1927-5307

SOME TOPOLOGICAL PROPERTIES OF MULTI METRIC SPACES

SUJOY DAS*, RANAJOY ROY

Department of Mathematics, Suri Vidyasagar College, Suri, Birbhum-731101, West Bengal, INDIA

Abstract. In the present paper some topological properties of multi metric space are studied. In multi metric space, notions of multi open ball, multi open set, multi limit point and multi derived set are presented for the first time.

Keywords: multi metric; multi open set; multi closed set; multi limit point; multi closure; multi derived set.

2010 AMS Subject Classification: 54E35, 08A05.

1. Introduction

Multiset (bag) is a well established notion both in mathematics and in computer science ([10], [11], [22]). In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object is allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained ([21], [23], [24]). In various counting arguments it is convenient to distinguish between a set like $\{a,b,c\}$ and a collection like $\{a,a,a,b,c,c\}$. The latter, if viewed as a set, will be identical to the former. However, it has some of its elements purposely listed several times. We formalize it by defining a multiset as a collection of elements, each considered with certain multiplicity. For the sake of convenience a multiset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer.

*Corresponding author
E-mail address: sujoy_math@yahoo.co.in
Received August 05, 2021

From 1989 to 1991, Wayne D. Blizard made a through study of multiset theory, real valued multisets and negative membership of the elements of multisets ([1], [2],[3],[4]). K. P. Girish and S. J. John introduced and studied the concepts of multiset topologies, multiset relations, multiset functions, chains and antichains of partially ordered multisets ([12], [13],[14],[15],[16]). D. Tokat studied the concept of soft multi continuous function [25]. Concepts of multigroups and soft multigroups are found in the studies of Sk. Nazmul and S. K. Samanta ([18], [19]). Many other authors like Chakrabarty et al. ([5], [6], [7], [8]), S. P. Jena et al. ([17]), J. L. Peterson ([20]) also studied various properties and applications of multisets.

Classical set theory states that a given element can appear only once in a set; it assumes that all mathematical objects occur without repetition. Thus there is only one number four, one field of complex numbers, etc. So, the only possible relation between two mathematical objects is either they are equal or they are different. However in the physical world it is observed that there is enormous repetition. For instance, there are many hydrogen atoms, many water molecules, many strands of DNA, etc. Coins of the same denomination and year, electrons or grains of sand appear similar, despite being obviously separate.

Functional analysis is an important branch of Mathematics and it has many applications in Mathematics and Sciences. Metric space is the beginning of functional analysis and it has several applications in many branch of functional analysis. An extension of metric spaces is done by using multi set and multi number instead of crisp real set and crisp real number in ([9]). In this paper we study some topological properties of multi metric spaces.

2. PRELIMINARIES

Definition 2.1. [12] A **multi set** M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$ where N represents the set of non negative integers.
Here, $C_M(x)$ is the number of occurrences of the element x in the mset M. We represent the mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i in the mset M denoted by $x_i \in^{m_i} M, i = 1, 2, ..., n$. However those elements which are not included in the mset M have zero count.

Example 2.2. [12] Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X. Clearly, a set is a special case of an mset.

Definition 2.3. [12] Let M and N be two msets drawn from a set X. Then, the following are defined:

(i) $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.

(ii) $M \subset N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.

(iii) $P = M \cup N$ if $C_P(x) = Max\{C_M(x), C_N(x)\}$ for all $x \in X$.

(iv) $P = M \cap N$ if $C_P(x) = Min\{C_M(x), C_N(x)\}$ for al $x \in X$.

(v) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.

(vi) $P = M \ominus N$ if $C_P(x) = Max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$, where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively.

Let M be an mset drawn from a set X. The **support set** of M, denoted by M^*, is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set. M^* is also called root set.

An mset M is said to be an **empty mset** if for all $x \in X, C_M(x) = 0$. The cardinality of an mset M drawn from a set X is denoted by $Card(M)$ or $|M|$ and is given by $Card(M) = \sum_{x\in X} C_M(x)$.

Definition 2.4. [12] A **domain** X, is defined as a set of elements from which msets are constructed. The **mset space** $[X]^w$ is the set of all msets whose elements are in X such that no element in the mset occurs more than w times. The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1, x_2, ..., x_k\}$ then $[X]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : for\ i = 1, 2, ...k;\ m_i \in \{0, 1, 2, ...w\}\}$.

Definition 2.5. [12] Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the **complement** M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C^c_M(x) = w - C_M(x)$, for all $x \in X$.

Definition 2.6. [12] The **maximum** mset is defined as Z where
$C_Z(x) = Max\{C_M(x) : x \in^k M, M \in [X]^m \text{ and } k \leq m\}$. Thus $C_Z(x) = m\ \forall x \in X$.

Definition 2.7. [12] Let $[X]^w$ be an mset space and $\{M_1, M_2, ...\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection of msets.

(i) The **union** $\bigcup_{i\in I} M_i = \{C_{\cup M_i}(x)/x : C_{\cup M_i}(x) = max\{C_{M_i}(x) : x \in X\}$.

(ii) The **intersection** $\bigcap_{i\in I} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = min\{C_{M_i}(x) : x \in X\}$.

(iii) The **mset addition** $\bigoplus_{i\in I} M_i = \{C_{\bigoplus M_i}(x)/x : C_{\bigoplus M_i}(x) = min\{w, \sum_{i\in I}\{C_{M_i}(x) :\ x \in X\}\}$.

(iv) The **mset complement** $M^c = Z \ominus M = \{C_{M}c(x)/x : C_{M}c(x) = C_Z(x) - C_M(x), x \in X\}$.

Definition 2.8. [12] The **power set** of an mset is denoted by $P^*(M)$ and it is an ordinary set whose members are sub msets of M.

Definition 2.9. [12] Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a **multiset topology** of M if τ satisfies the following properties.

(i) The mset M and the empty mset $\emptyset$ are in τ.

(ii) The mset union of the elements of any sub collection of τ is in τ.

(iii) The mset intersection of the elements of any finite sub collection of τ is in τ.

Mathematically a multiset topological space is an ordered pair (M,τ) consisting of an mset $M \in [X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Note that τ is an ordinary set whose elements are msets. Multiset topology is abbreviated as an M-topology.

Definition 2.10. [9] **Multi point:** Let M be a multi set over a universal set X. Then a multi point of M is defined by a mapping $P_x^k : X \longrightarrow \mathbb{N}$ such that $P_x^k(x) = k$ where $k \leq C_M(x)$.
x and k will be referred to as the **base** and the **multiplicity** of the multi point P_x^k respectively.

Collection of all multi points of an mset M is denoted by M_{pt}.

Definition 2.11. [9] The **mset generated by a collection** B of multi points is denoted by $MS(B)$ and is defined by $C_{MS(B)}(x) = Sup\{k : P_x^k \in B\}$.

An mset can be generated from the collection of its multi points. If M_{pt} denotes the collection of all multi points of M, then obviously $C_M(x) = Sup\{k : P_x^k \in M_{pt}\}$ and hence $M = MS(M_{pt})$.

Definition 2.12. [9] (i) The **elementary union** between two collections of multi points C and D is denoted by $C \sqcup D$ and is defined as $C \sqcup D = \{P_x^k : P_x^l \in C, P_x^m \in D \ \ and \ \ k = max\{l,m\}\}$.

(ii) The **elementary intersection** between two collections of multi points C and D is denoted by $C \sqcap D$ and is defined as $C \sqcap D = \{P_x^k : P_x^l \in C, P_x^m \in D \ \ and \ \ k = min\{l,m\}\}$.

(iii) For two collections of multi points C and D, C is said to be an **elementary subset** of D, denoted by $C \sqsubset D$, iff $P_x^l \in C \Rightarrow \exists m \geq l$ such that $P_x^m \in D$.

The following results can be easily proved:

Theorem 2.13. [9] (i) For two collections of multi points C and D, $C \subset D \Rightarrow C \sqsubset D$, but the converse is not true.

(ii) For two collections of multi points C and D, $C \cup D \supset C \sqcup D$ and the equality does not hold in general.

(iii) For two collections of multi points C and D, $C \cap D \subset C \sqcap D$ and the equality does not hold in general.

(iv) For an mset $M, MS(M_{pt}) = M$.

(v) For a collection B of multi points, $[MS(B)]_{pt} \supset B$.

(vi) For two msets F and G, $F \subset G \Leftrightarrow F_{pt} \subset G_{pt}$.

(vii) For two collections of multi points C and D, $C \subset D \Rightarrow MS(C) \subset MS(D)$.

$(viii)$ For two collections of multi points C and D, $C \sqsubset D \Leftrightarrow MS(C) \subset MS(D)$.

(ix) For two collections of multi points C and D, $MS(C \sqcap D) = MS(C) \cap MS(D)$

(x) For an arbitrary collection $\{B_i : i \in \Delta\}$ of multi points, $MS(\sqcup_{i\in\Delta} B_i) = \cup_{i\in\Delta} MS(B_i)$

(xi) For an arbitrary collection $\{B_i : i \in \Delta\}$ of multi points, $MS(\cup_{i\in\Delta} B_i) = \cup_{i\in\Delta} MS(B_i)$

Definition 2.14. [9] Let $m\mathbb{R}^+$ denotes the multi set over $\mathbb{R}^+$ (set of non-negative real numbers) having multiplicity of each element equal to w, $w \in \mathbb{N}$. The members of $(m\mathbb{R}^+)_{pt}$ will be called **non-negative multi real points**.

Definition 2.15. [9] Let P_a^i and P_b^j be two multi real points of $m\mathbb{R}^+$. We define $P_a^i > P_b^j$ if $a > b$ or $P_a^i > P_b^j$ if $i > j$ when $a = b$.

Definition 2.16. [9] **(Addition of multi real points)** We define $P_a^i + P_b^j = P_{a+b}^k$ where $k = Max\{i, j\}, P_a^i, P_b^j \in (m\mathbb{R}^+)_{pt}$.

Definition 2.17. [9] **(Multiplication of multi real points)** We define multiplication of two multi real points in $m\mathbb{R}^+$ as follows:

$P_a^i \times P_b^j = P_0^1$,if either P_a^i or P_b^j equal to P_0^1

$= P_{ab}^k$, otherwise where $k =$Max $\{i, j\}$

Proposition 2.18. [9] **(Properties of multiplication)** Multiplication of multi real points satisfies the following properties:

(i) Multiplication is Commutative.

(ii) Multiplication is Associative.

(iii) Multiplication is distributive over addition.

Definition 2.19. [9] **Multi Metric:** Let $d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$($M$ being a multi set over a Universal set X having multiplicity of any element atmost equal to w) be a mapping which satisfy the following:

(M1) $d(P_x^l, P_y^m) \geq P_0^1, \forall P_x^l, P_y^m, \in M_{pt}$

(M2) $d(P_x^l, P_y^m) = P_0^1$ iff $P_x^l = P_y^m, \forall P_x^l, P_y^m \in M_{pt}$

(M3) $d(P_x^l, P_y^m) = d(P_y^m, P_x^l), \forall P_x^l, P_y^m \in M_{pt}$

(M4) $d(P_x^l, P_y^m) + d(P_y^m, P_z^n) \geq d(P_x^l, P_z^n), \forall P_x^l, P_y^m, P_z^n \in M_{pt}$.

(M5) For $l \neq m$, $d(P_x^l, P_y^m) = P_0^k, \Leftrightarrow x = y$ and $k = \text{Max}\{l, m\}$.

Then d is said to be a multi metric on M and (M, d) is called a Multi metric (or an M-metric) space.

Example 2.20. [9] Let M be a multi set over X having multiplicity of any element atmost equal to w. We define

$d : M_{pt} \times M_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ such that

$d(P_x^l, P_y^m) = P_0^1$ if $P_x^l = P_y^m$

$= P_0^{Max\{l,m\}}$ if $x = y$ and $l \neq m$

$= P_1^j$ if $x \neq y \; \forall \; P_x^l, P_y^m \in M_{pt}$, $[\; 1 \leq j \leq w \; is \; some \; fixed \; positive \; integer \;]$

Then d is an M-metric on M.

Theorem 2.21. [9] If $d(P_a^i, P_b^j) = P_r^l$ and $d(P_a^p, P_b^q) = P_s^m$, then $r = s$, $P_a^i, P_b^j, P_a^p, P_b^q \in M_{pt}$ and $P_r^l, P_s^m \in (m\mathbb{R}^+)_{pt}$.

Definition 2.22. [9] Let (M,d) be an M-metric space and L be a non null sub mset of M. Then the mapping $d_L : L_{pt} \times L_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ given by $d_L(P_x^a, P_y^b) = d(P_x^a, P_y^b), \; \forall \; P_x^a, P_y^b \in L_{pt}$ is an M-metric on L. The metric is known as the **relative M-metric** induced by d on L. The M-metric space (L, d_L) is called an **M-metric subspace** or simply an **M-subspace** of the M-metric space (M, d).

Definition 2.23. [9] Let (M,d) be an M-metric space and L be a nonempty submset of M. Then the diameter of L, denoted by $\delta(L)$ is defined by:

$\delta(L) = P_a^k$ where $a = Sup\{b : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\}$,

$k = 1$ if $a > b \; \forall P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}$ and

$= Max\{j : P_b^j = d(P_x^l, P_y^m), P_x^l, P_y^m \in L_{pt}\}$ otherwise.

If supremum does not exist finitely, we call L a set of infinite diameter.

Theorem 2.24. [9] For a sub mset L of M in an M-metric space (M, d), $\delta(L) = P_0^1$ iff $L = \{1/a\}$ ie. L consists of a single element of the universal set X with multiplicity 1.

Theorem 2.25. [9] $P \subset Q \Rightarrow \delta(P) \leq \delta(Q)$.

Definition 2.26. [9] Let A and B be two sub msets of M in an M-metric space (M,d). Then the **distance between** A and B, denoted by $\delta(A,B)$, is defined by

$\delta(A,B) = P_a^k$ where $a = \text{Inf }\{b : P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\}$ and

$k = w$ if $a < b \; \forall P_b^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}$,

$k = \text{Min }\{j : P_a^j = d(P_x^l, P_y^m), P_x^l \in A_{pt}, P_y^m \in B_{pt}\}$ otherwise.

3. Multi Open Balls and Multi Open Sets

Definition 3.1. Let (M,d) be an M-metric space, $r > 0$ and $P_a^k \in M_{pt}$. Then the **open ball** with centre P_a^k and radius P_r^1 $[r > 0]$, $i \in \mathbb{N}, 1 \leq i \leq w]$, is denoted by $B(P_a^k, P_r^1)$ and is defined by

$B(P_a^k, P_r^1) = \{P_x^l : d(P_x^l, P_a^k) < P_r^1\}$.

$MS[B(P_a^k, P_r^1)]$ will be called a **multi open ball** with centre P_a^k and radius $P_r^1 > P_0^1$.

Definition 3.2. $B[P_a^k, P_r^1] = \{P_x^l : d(P_x^l, P_a^k) \leq P_r^1\}$ is called the **closed ball** with centre P_a^k and radius P_r^1 $[r > 0]$.

$MS[B[P_a^k, P_r^1]]$ will be called a **multi closed ball** with centre P_a^k and radius P_r^1 $[r > 0]$.

Note 3.3. For $r > s > 0 \Rightarrow P_r^1 > P_s^1 \Rightarrow B(P_a^k, P_r^1) \supset B(P_a^k, P_s^1)$.

Note 3.4. In any M-metric space (M,d), $B(P_a^k, P_r^1) \supset \{P_a^l, 1 \leq l \leq C_M(a)\}$, for any $r > 0$.

Example 3.5. In Example 2.20, for $P_a^k \in M_{pt}$ and $P_r^1 > P_0^1$,

$B(P_a^k, P_r^1) = \{P_a^l, 1 \leq l \leq w\}$, if $0 < r \leq 1$

$= M_{pt}$, if $r > 1$

Theorem 3.6. (Hausdorff Property)

Let (M,d) be an M-metric space and $P_a^k, P_b^l \in M_{pt}$ such that $a \neq b$.Then $\exists\, r > 0$ such that $MS[B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1)] = \emptyset$ which is equivalent to $B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1) = \phi$

Proof. Let (M,d) be an M-metric space and $P_a^k, P_b^l \in M_{pt}$ such that $a \neq b$. If $P_c^m = d(P_a^k, P_b^l)$ then $c > 0$.

Let $0 < r < c/2$ and consider the open balls $B(P_a^k, P_r^1)$ and $B(P_b^l, P_r^1)$.

Then $B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1) = \phi$ since otherwise $\exists$ a multi point $P_x^s \in B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1)$ $\Rightarrow d(P_a^k, P_x^s) < P_r^1$ and $d(P_b^l, P_x^s) < P_r^1 \Rightarrow d(P_a^k, P_b^l) \leq d(P_a^k, P_x^s) + d(P_b^l, P_x^s) < P_r^1 + P_r^1 = P_{2r}^1 < P_c^m$ —- which is a contradiction.

$\therefore B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1) = \phi$ and hence MS$[B(P_a^k, P_r^1) \cap B(P_b^l, P_r^1)] = \emptyset$.

Note 3.7. For two multi points P_a^k and P_a^l with $k \neq l$, any open ball that contains one of them, must contain the other one. (Follows from Note 3.4)

Definition 3.8. Let (M,d) be an M-metric space and $P_a^k \in M_{pt}$. A collection $N(P_a^k)$ of multi points of M is said to be a **nbd** of the multi point P_a^k if $\exists r > 0$ such that $P_a^k \in B(P_a^k, P_r^1) \subset N(P_a^k)$.

MS[$N(P_a^k)$] will be called a **multi nbd** of the multi point P_a^k.

Theorem 3.9. Let N_1 and N_2 are two nbds of a multi point P_a^i in an M-metric space (M,d). Then $N_1 \cap N_2$ is a nbd of P_a^i and hence $MS(N_1 \cap N_2)$ is a multi nbd of P_a^i.

Proof. Since N_1 and N_2 be two nbds of P_a^i, $\exists P_{r_1}^1, P_{r_2}^1$ with $r_1, r_2 > 0$ such that

$P_a^i \in B(P_a^i, P_{r_1}^1) \subset N_1$ and $P_a^i \in B(P_a^i, P_{r_2}^1) \subset N_2$

Let $r =$ Min $\{r_1, r_2\}$. Then $r > 0, r \leq r_1,\ r \leq r_2,. \therefore P_a^i \in B(P_a^i, P_r^1) \in B(P_a^i, P_{r_1}^1) \subset N_1$ and $P_a^i \in B(P_a^i, P_r^1) \in B(P_a^i, P_{r_2}^1) \subset N_2 \ \Rightarrow P_a^i \in B(P_a^i, P_r^1) \subset N_1 \cap N_2 \ \Rightarrow N_1 \cap N_2$ is a nbd of P_a^i in (M,d). Hence MS$(N_1 \cap N_2)$ is a multi nbd of P_a^i in (M,d).

Corrolary: Since $N_1 \cap N_2 \subset N_1 \sqcap N_2$, $N_1 \sqcap N_2$ is a nbd of P_a^i and hence

MS$(N_1 \sqcap N_2)$=MS$(N_1)\cap$ MS(N_2) is a multi nbd of P_a^i.

Definition 3.10. Let B be a collection of multi points of M in an M-metric space (M,d). Then a multi point P_a^k is said to be an **interior point** of B if $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k *and* $r > 0$ such that $B(P_a^k, P_r^1) \subset B$.

Definition 3.11. Let N be a sub multiset of an M-metric space (M,d). Then a multi point P_a^k is said to be an **interior point of** N if it is an interior point of N_{pt}, ie. $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k, *and* $r > 0$ such that $B(P_a^k, P_r^1) \subset N_{pt}$.

Definition 3.12. Let N be a sub mset of an M-metric space (M,d). Then the **interior** of N is defined to be the set consisting of all interior points of N.

The interior of the multi set N is denoted by N^o or Int(N).

MS[Int(N)] is said to be the **multi interior** of N denoted by M-int(N).

Proposition 3.13. Let A and B be two non-null sub msets of an M-metric space (M,d). Then (i) $A_{pt} \cap B_{pt} = (A \cap B)_{pt}$, (ii)$A_{pt} \cup B_{pt} = (A \cup B)_{pt}$.

Proof. **(i)** Clearly $(A \cap B)_{pt} \subset A_{pt} \cap B_{pt}$ Next $P_a^k \in A_{pt} \cap B_{pt} \ \Rightarrow P_a^k \in A_{pt}$ and $P_a^k \in B_{pt} \ \Rightarrow$ $C_A(a) \geq k$ and $C_B(a) \geq k \ \Rightarrow C_{A\cap B}(a) =$ Min $\{C_A(a), C_B(a)\} \geq k \ \Rightarrow P_a^k \in (A \cap B)_{pt}$. $\therefore A_{pt} \cap$

$B_{pt} \subset A \cap B)_{pt}$ and hence $A_{pt} \cap B_{pt} = A \cap B)_{pt}$.

(ii) The proof can be done in a similar way as the above.

Theorem 3.14. Let A and B be two non-null sub msets of an M-metric space (M,d). Then

(i) M-int(A) $\subset$ A

(ii) $A \subset B \Rightarrow Int(A) \subset Int(B)$ and hence M-int(A)$\subset$ M-int(B)

(iii) $Int(A) \cap Int(B) = Int(A \cap B)$

(iv) (a) $Int(A \cap B) \subset Int(A) \sqcap Int(B)$ (b) $Int(A \cap B) \sqsubset Int(A) \sqcap Int(B)$ (c) $Int(A \cap B) \sqsubset Int(A) \cap Int(B)$

(v) $M - int(A \cap B) \subset M - int(A) \cap M - int(B)$

(vi) $Int(A \cup B) \supset Int(A) \cup Int(B)$

Proof. (iii) $A \cap B \subset A \Rightarrow Int(A \cap B) \subset Int(A)$. Similarly $Int(A \cap B) \subset Int(B)$. $\therefore Int(A \cap B) \subset Int(A) \cap Int(B)$. Next let $P_a^k \in Int(A) \cap Int(B)$, $\Rightarrow P_a^k \in Int(A)$ and $P_a^k \in Int(B) \Rightarrow \exists r_1, r_2 > 0$ such that $B(P_a^k, P_{r_1}^1) \subset A_{pt}$ and $B(P_a^k, P_{r_2}^1) \subset B_{pt}$. Let $r =$ Min $\{r_1, r_2\}$. Then $r > 0, i \geq 1$ and $B(P_a^k, P_r^1) \subset B(P_a^k, P_{r_1}^1) \subset A_{pt}$ and $B(P_a^k, P_r^1) \subset B(P_a^k, P_{r_2}^1) \subset B_{pt} \Rightarrow B(P_a^k, P_r^1) \subset A_{pt} \cap B_{pt} = (A \cap B)_{pt} \Rightarrow P_a^k \in Int(A \cap B)$. $\therefore Int(A) \cap Int(B) \subset Int(A \cap B)$. $\therefore Int(A) \cap Int(B) = Int(A \cap B)$.

Definition 3.15. Let (M,d) be an M-metric space. Then a collection B of multi points of M is said to be **open** if every multi point of B is an interior point of B i.e., for each $P_a^k \in B$, $\exists$ an open ball $B(P_a^k, P_r^1)$ with centre at P_a^k, *and* $r > 0$ such that $B(P_a^k, P_r^1) \subset B$.

ϕ is separately considered as an open set.

Definition 3.16. Let (M,d) be an M-metric space. Then $N \subset M$ is said to be **multi open** in (M,d) iff $\exists$ a collection B of multi points of N such that B is open and $MS(B) = N$

The null multiset Φ separately considered as multi open in (M,d).

Definition 3.17. Subtraction of nonnegative multi real points We define substraction of two nonnegative multi real points as follows:

$$P_a^i - P_b^j = P_{a-b}^{Min\{i,j\}} \text{ if } P_a^i > P_b^j$$
$$= P_0^1 \quad \text{ if } P_a^i = P_b^j$$

Proposition 3.18. In an M-metric space every open ball is open.

Proof. Let us consider an open ball $B(P_a^l, P_r^1)$ and let $P_b^m \in B(P_a^l, P_r^1) \Rightarrow d(P_a^l, P_b^m) < P_r^1$ $\Rightarrow P_r^1 - d(P_a^l, P_b^m) = P_c^n > P_0^1 \Rightarrow c > 0$. Let $0 < s < c$. Then $P_r^1 - d(P_a^l, P_b^m) = P_c^n > P_s^1$

and we consider the open ball $B(P_b^m, P_s^1)$.

Now $P_x^u \in B(P_b^m, P_s^1) \Rightarrow d(P_b^m, P_x^u) < P_s^1 \Rightarrow d(P_a^l, P_x^u) \leq d(P_a^l, P_b^m) + d(P_b^m, P_x^u) < d(P_a^l, P_b^m) + P_s^1 < d(P_a^l, P_b^m) + P_r^1 - d(P_a^l, P_b^m) \leq P_r^1$ ($\because$ For any three nonnegative multi real points $P_a^i, P_b^j and P_c^k, P_a^i > P_b^j \Rightarrow P_a^i + P_c^k \geq P_b^j + P_c^k$)

$\Rightarrow P_x^u \in B(P_a^l, P_r^1) \Rightarrow B(P_b^m, P_s^1) \subset B(P_a^l, P_r^1) \Rightarrow B(P_a^l, P_r^1)$ is open.

Consequently it follows that every multi open ball being generated by these open balls are multi open.

Theorem 3.19. In an M-metric space (M,d) a set B of multi points is open iff every multi point of B is an interior point of B ie. iff B is a nbd of each of its multi points.

Theorem 3.20. In an M-metric space (M,d)

(i) Union of arbitrary number of open sets of multi points is open.

(ii) Elementary intersection of two open sets of multi points is open.

(iii) Intersection of two open sets of multi points is open.

Proof. (i) Let $\{B_i : i \in \Delta\}$ be an arbitrary collection of open sets of multi points in (M,d) and $P_x^l \in \bigcup_{i\in\Delta} B_i \Rightarrow P_x^l \in B_i$ for some $i \in \Delta$. Since B_i is open, $\exists\, r > 0$ such that $B(P_x^l, P_r^1) \subset B_i \subset \bigcup_{i\in\Delta} B_i$ and the result follows.

(ii) Let B_1, B_2 be two open sets of multi points in (M,d) and $P_x^l \in B_1 \sqcap B_2 \Rightarrow$ One of B_1 and B_2 contains P_x^l and the other contains P_x^m where $m \geq l$. For definiteness let us assume $P_x^l \in B_1$ and $P_x^m \in B_2$. Since B_1 is open, $\exists\, r,s > 0$ such that $B(P_x^l, P_r^1) \subset B_1$ and $B(P_x^m, P_s^1) \subset B_2$. Now from Note 3.4, it follows that $P_x^l \in B(P_x^m, P_s^1) \subset B_2$ and from the openness of B_2, $\exists\, t > 0$ such that $B(P_x^l, P_t^1) \subset B_2$.

Let $u = \text{Min}\{r,t\}$. Then $u > 0$, $B(P_x^l, P_u^1) \subset B(P_x^l, P_r^1) \subset B_1$ and $B(P_x^l, P_u^1) \subset B(P_x^l, P_t^1) \subset B_2$ $\Rightarrow B(P_x^l, P_u^1) \subset B_1 \cap B_2 \subset B_1 \sqcap B_2$ and the result follows.

(iii) Let B_1, B_2 be two open sets of multi points in (M,d) and $P_x^l \in B_1 \cap B_2 \Rightarrow P_x^l \in B_1$ and $P_x^l \in B_2 \Rightarrow \exists\, r_1, r_2 > 0$ such that $B(P_x^l, P_{r_1}^1) \subset B_1$ and $B(P_x^l, P_{r_2}^1) \subset B_2$.

Let $r = \text{Min}\{r_1, r_2\}$. Then $r > 0$, $B(P_x^l, P_r^1) \subset B(P_x^l, P_{r_1}^1) \subset B_1$ and $B(P_x^l, P_r^1) \subset B(P_x^l, P_{r_2}^1) \subset B_2$ $\Rightarrow B(P_x^l, P_r^1) \subset B_1 \cap B_2 \Rightarrow B_1 \cap B_2$ is open.

Theorem 3.21. In an M-metric space (M,d),

(i) The null sub mset $\emptyset$ is multi open.

(ii) M is multi open.

(iii) Arbitrary union of multi open sets is multi open.

(iv) Intersection of two multi open sets is multi open.

Proof. (i) $\emptyset$ is trivially multi open.

(ii) Since M_{pt} is the collection of all multi points in (M,d), it is obviously open and hence $M = MS(M_{pt})$ is multi open.

(iii) Let $\{M_i : i \in \triangle\}$ be a collection of multi open sets in (M,d).

Then $\exists$ B_i such that M_i =MS(B_i) and B_i is open set of multi points in $(M,d) \forall i \in \triangle$

$\Rightarrow \bigcup_{i\in\triangle} B_i$ is open in (M,d) and since from Theorem [2],

$\bigcup_{i\in\triangle} M_i = \bigcup_{i\in\triangle} MS(B_i) = \text{MS}(\bigcup_{i\in\triangle} B_i)$, it follows that $\bigcup_{i\in\triangle} M_i$ is multi open in (M,d)

(iv)Let $M_i, i = 1,2$ be two multi open sets in (M,d).

Then $\exists$ B_i such that $M_i = \text{MS}$ (B_i) and B_i is open set of multi points in (M,d), $i = 1,2$.

Then $B_1 \sqcap B_2$ is open in (M,d) and since from Theorem 2.13,

$M_1 \cap M_2$ =MS(B_1)$\cap$ MS(B_2) =MS($B_1 \sqcap B_2$) , it follows that $M_1 \cap M_2$ is multi open in (M,d)

Note 3.22. Thus the collection τ of all multi open sets in an M-metric space (M,d) forms a multi set topology on M. The topology is called M-metric topology.

Example 3.23. Arbitrary intersection of multi open sets may not be multi open.

For example consider $\mathbb{R}$ to be a multi set with multiplicity of each element 1.

Define $d : \mathbb{R}_{pt} \times \mathbb{R}_{pt} \longrightarrow (m\mathbb{R}^+)_{pt}$ by $d(P^1_x, P^1_y) = P^1_{|x-y|}$, $\forall P^1_x, P^1_y \in \mathbb{R}_{pt}$.

Consider the collection $\{P_n : n \in \mathbb{N}\}$ of multi sets such that

$P_n = \{1/x : -\frac{1}{n} < x < \frac{1}{n}\}$. Then $P_n, n \in \mathbb{N}$ are multi open sets as $(P_n)_{pt} = \{P^1_x : -\frac{1}{n} < x < \frac{1}{n}\}, n \in \mathbb{N}$ are open sets of multi points in $(\mathbb{R},d)$ and $P_n = \text{MS}((P_n)_{pt})$.

But $\bigcap_{n\in\mathbb{N}} P_n = \{1/0\}$ which is not multi open in $(\mathbb{R},d)$

Definition 3.24. A multi set N in an M-metric space (M,d) is said to be **multi closed** if its complement N^c is multi open in (M,d).

Proposition 3.25. Let $\{N_i : i \in \triangle\}$ be an arbitrary collection of multisets in (M,d). Then $\bigcup_{i\in\triangle}(N_i)^c = (\bigcap_{i\in\triangle} N_i)^c$ and $\bigcap_{i\in\triangle}(N_i)^c = (\bigcup_{i\in\triangle} N_i)^c$

Proof. $\forall x \in X, C_{(\bigcap_{i\in\triangle} N_i)^c}(x) = C_M(x) - C_{\bigcap_{i\in\triangle} N_i}(x) = C_M(x) - \bigwedge_{i\in\triangle} C_{N_i}(x) = \bigvee_{i\in\triangle}[C_M(x) - C_{N_i}(x)] = \bigvee_{i\in\triangle} C_{N_i^c}(x) = C_{\bigcup_{i\in\triangle} N_i^c}(x)$

The other result follows similarly.

Theorem 3.26. In an M-metric space

(i) The null multi set $\emptyset$ is multi closed.

(ii) The absolute multiset M is multi closed.

(iii) Arbitrary intersection of multi closed sets is multi closed.

(iv) Finite union of multi closed sets is multi closed.

Note 3.27. Arbitrary union of multi closed sets may not be multi closed.

In Example 3.23, if we Consider the collection $\{Q_n : n \in \mathbb{N}\}$ of multi sets such that $Q_n = \{1/x : -1 + \frac{1}{n} \leq x \leq 1 - \frac{1}{n}\}$. Then $Q_n, n \in \mathbb{N}$ are multi closed sets.

But $\bigcup_{n\in\mathbb{N}} Q_n = \{1/x : -1 < x < 1\}$ which is not multi closed in $(\mathbb{R}, d)$.

4. Multi Limit Point and Multi Closure

Definition 4.1. Let (M,d) be an M-metric space and B be a collection of multi points of M. Then a multi point P_x^l of M is said to be a **limit point** of B if every open ball $B(P_x^l, P_r^1)$ $(r > 0)$ containing P_x^l in (M,d) contains at least one point of B other than P_x^l.

The set of all limit points of B is said to be the **derived set** of B and is denoted by B^d.

Definition 4.2. Let (M,d) be an M-metric space and $N \subset M$. Then $P_x^l \in M_{pt}$ is said to be a **multi limit point of** N if it is a limit point of N_{pt} ie. if every open ball $B(P_x^l, P_r^1)$ $(r > 0)$ containing P_x^l in (M,d) contains at least one point of N_{pt} other than P_x^l.

A multi limit point of a multi set N may or may not belong to the set N. The multiset generated by the multi limit points of N is called the **multi derived set** of N and is denoted by N^d. Thus $N^d = MS[(N_{pt})^d]$.

Example 4.3. Consider the M-metric space $(\mathbb{R}, d)$ as in Example 3.23.

Let $P = \{1/a : m < a < n\}, m, n \in \mathbb{R}$. Then $P_{pt} = \{P_a^1 : m < a < n\}$. Consider $Q = \{1/b : m \leq b \leq n\}$. Then $Q_{pt} = \{P_b^1 : m \leq b \leq n\}$. $\therefore \forall P_b^1 \in Q_{pt}$ and $\forall r > 0$ $B(P_b^1, P_r^1) \cap [P_{pt} \setminus \{P_b^1\}] = \{P_c^1 : b - r < c < b + r\} \cap [\{P_a^1 : m < a < n\} \setminus \{P_b^1\}] = \{P_e^1 : Max(b - r, m) < e < Min(b + r, n)\} \setminus \{P_b^1\} \neq \emptyset$.

So, $(P_{pt})^d = Q_{pt}$ and hence $P^d = MS[(P_{pt})^d] = MS[Q_{pt}] = Q$.

Note 4.4. If A be a set of multi points and $P_x^l \in A$, then $P_x^m \in A^d\ \forall m \neq l,\ 1 \leq m \leq C_M(x)$ and P_x^l may or may not be a multi limit point of A.

Note 4.5. If each element of a multi set $N \subset M$ have multiplicity at least 2, then each multi point of the multi set is a multi limit point of the multi set and also $(N_{pt})^d \supset \{P_x^l : x \in N^*, 1 \leq l \leq C_M(x)\} \supset N_{pt}$.

Note 4.6. If A be a set of multi points, $P_x^l \notin A$ for any $1 \leq l \leq C_M(x)$ and $P_x^l \in A^d$ for some $1 \leq l \leq C_M(x)$,then $P_x^m \in A^d\ \forall\ 1 \leq m \leq C_M(x)$.

To prove this let us take any $r > 0$. Then as $P_x^l \in A^d$, $[B(P_x^l, P_{\frac{r}{2}}^1) \cap A] - \{P_x^l\} \neq \phi \Rightarrow B(P_x^l, P_{\frac{r}{2}}^1) \cap A \neq \phi$ $(\because P_x^l \notin A)$.

Let $P_a^i \in B(P_x^l, P_{\frac{r}{2}}^1) \cap A$. Then $P_a^i \in A, d(P_x^l, P_a^i) < P_{\frac{r}{2}}^1$ and for any $1 \leq m \leq C_M(x)$, $d(P_x^m, P_a^i) \leq d(P_x^m, P_x^l) + d(P_x^l, P_a^i) < P_0^{Max\{l,m\}} + P_{\frac{r}{2}}^1 = P_{\frac{r}{2}}^{Max\{l,m\}} < P_r^1$

$\Rightarrow P_a^i \in B(P_x^m, P_r^1) \Rightarrow P_a^i \in B(P_x^m, P_r^1) \cap A$ and $P_a^i \neq P_x^m$ as $P_x^m \notin A$, but $P_a^i \in A \Rightarrow [B(P_x^m, P_r^1) \cap A] - \{P_x^m\} \neq \phi \Rightarrow P_x^m \in A^d$.

Theorem 4.7. Let A and B be collections of multi points in (M,d). Then

$(i)\ A^d \cup B^d = (A \cup B)^d \quad (ii)\ (A^d)^d \nsubseteq A^d$ in general.

Proof. (i) Since $A, B \subset A \cup B \Rightarrow A^d, B^d \subset (A \cup B)^d \Rightarrow A^d \cup B^d \subset (A \cup B)^d ------(i)$.

Again $P_x^l \notin A^d \cup B^d \Rightarrow \exists\ r_1, r_2 > 0$, such that $B(P_x^l, P_{r_1}^1)$ and $B(P_x^l, P_{r_2}^1)$ contains no point of A and B respectively other than P_x^l. Let $r = $ Min $\{r_1, r_2\}$. Then $B(P_x^l, P_r^1)$ contains no point of $A \cup B$ other than P_x^l and consequently $P_x^l \notin (A \cup B)^d$ ——-(ii).

Form (i) and (ii) the result follows.

(ii) We show an example in support of this.

Let in an M-metric space (M,d), $A = \{P_a^2\}$ where $a \in^m M, m > 2$.

Then for any $r > 0$, $[B(P_a^2, P_r^1) \cap A] - \{P_a^2\} = \phi$ and hence $P_a^2 \notin A^d ----(1)$.

But $P_a^2 \in B(P_a^3, P_r^1)\ \forall r > 0$ and so $P_a^2 \in [B(P_a^3, P_r^1) \cap A] - \{P_a^3\} \Rightarrow [B(P_a^3, P_r^1) \cap A] - \{P_a^3\} \neq \phi$ ie $P_a^3 \in A^d$. Since $P_a^3 \in A^d$, we can prove in a similar manner as above $P_a^2 \in (A^d)^d -----(2)$.

From (1) and (2) the result follows.

Theorem 4.8. For two sub multi sets P and Q of M, $(P \cup Q)^d = P^d \cup Q^d$.

Proof. $(P \cup Q)^d = MS[\{(P \cup Q)_{pt}\}^d] = MS[(P_{pt} \cup Q_{pt})^d]$

$= MS[(P_{pt})^d \cup (Q_{pt})^d] = MS[(P_{pt})^d] \cup MS[(Q_{pt})^d] = P^d \cup Q^d$.

Definition 4.9. Let (M,d) be an M-metric space and $B \subset M_{pt}$. Then the collection of all points of B together with all limit points of B is said to be the **closure** of B in (M,d) and is denoted by $\overline{B}$. Thus $\overline{B} = B \cup B^d$.

Note 4.10. $P_x^l \in \overline{B}$ iff for any $r > 0,\ B(P_x^l, P_r^1) \cap B \neq \phi$.

Note 4.11. If $B \subset M_{pt}$ and $P_x^l \in \overline{B}$ for some $1 \leq l \leq C_M(x)$, then $P_x^l \in \overline{B}\ \forall\ 1 \leq l \leq C_M(x)$.

Proof. Let $r > 0$ be arbitrary. Since $P_x^l \in \overline{B},\ B(P_x^l, P_{\frac{r}{2}}^1) \cap B \neq \phi$.

Now for any $1 \leq m \leq C_M(x)$, consider the open ball $B(P_x^m, P_r^1)$.

Clearly $B(P_x^l, P_{\frac{r}{2}}^1) \subset B(P_x^m, P_r^1)$, since $P_a^i \in B(P_x^l, P_{\frac{r}{2}}^1)$

$\Rightarrow d(P_x^l, P_a^i) < P_{\frac{r}{2}}^1 \ \Rightarrow d(P_x^m, P_a^i) \leq d(P_x^m, P_x^l) + d(P_x^l, P_a^i) \ < P_0^{Max\{m,l\}} + P_{\frac{r}{2}}^1$

$= P_{\frac{r}{2}}^{Max\{m,l\}} \ < P_r^1 \ \Rightarrow P_a^i \in B(P_x^m, P_r^1)$

So we have as $B(P_x^l, P_{\frac{r}{2}}^1) \cap B \neq \phi$, also $B(P_x^m, P_r^1) \cap B \neq \phi$

$\Rightarrow P_x^m \in \overline{B}\ \forall\ 1 \leq m \leq C_M(x)$.

Theorem 4.12. If $B \subset M_{pt}$ in (M,d), then $\overline{\overline{B}} = \overline{B}$.

Proof. Clearly $\overline{B} \subset \overline{\overline{B}}$. Conversely, $P_x^l \in \overline{\overline{B}} \ \Rightarrow$ for any $r > 0$,

$B(P_x^l, P_r^1) \cap \overline{B} \neq \phi$.

Let $P_y^m \in B(P_x^l, P_r^1) \cap \overline{B}$. Since $P_y^m \in B(P_x^l, P_r^1),\ \exists\ s > 0$ such that

$B(P_y^m, P_s^1) \subset B(P_x^l, P_r^1)$ and as $P_y^m \in \overline{B},\ B(P_y^m, P_s^1) \cap B \neq \phi$

$\Rightarrow B(P_x^l, P_r^1) \cap B \neq \phi \quad \Rightarrow P_x^l \in \overline{B}$ and hence $\overline{\overline{B}} \subset \overline{B}$.

Hence the result follows.

Definition 4.13. Let (M,d) be an M-metric space and $N \subset M$. Then the multi set generated by all multi points and all multi limit points of N is said to be the **multi closure** of N and is denoted by $\overline{N}$.

Thus the multi set generated by all the multi points of $\overline{N_{pt}}$ is the multi closure of N and we have $\overline{N} = MS[\overline{N_{pt}}] = MS[N_{pt} \cup (N_{pt})^d] = MS[N_{pt}] \cup MS[(N_{pt})^d] = N \cup MS[(N_{pt})^d] = N \cup N^d$.

Theorem 4.14. Let (M,d) be an M-metric space and $P \subset M$. Then $\overline{P_{pt}} = (\overline{P})_{pt}$.

Proof. Let $P_x^l \in \overline{P_{pt}} = P_{pt} \cup (P_{pt})^d$. If $P_x^l \in P_{pt} \ \Rightarrow C_P(x) \geq l$. If $P_x^l \in (P_{pt})^d \ \Rightarrow C_{P^d}(x) \geq l$.

$\therefore C_P(x) \ \vee \ C_{P^d}(x) = C_{P \cup P^d}(x) = C_{\overline{P}}(x) \geq l \ \Rightarrow P_x^l \in (\overline{P})_{pt}$.

Next $P_x^l \in (\overline{P})_{pt} \ \Rightarrow C_{\overline{P}}(x) = m \geq l \ \Rightarrow x \in^m \overline{P} = MS(\overline{P_{pt}}) \Rightarrow m = \text{Sup}\ \{l : P_x^l \in \overline{P_{pt}}\} \Rightarrow P_x^m \in \overline{P_{pt}}$

$\Rightarrow P_x^k \in \overline{P_{pt}}\ \forall\ 1 \leq k \leq C_M(x)$ [From Note 4.11] $\Rightarrow P_x^l \in \overline{P_{pt}}$.

Theorem 4.15. Let (M,d) be an M-metric space and $P,Q \subset M$. Then

(i) $\overline{\emptyset} = \emptyset$ and $\overline{M} = M$

(ii) $P \subset \overline{P}$

(iii) $\overline{P} = \overline{\overline{P}}$

(iv) $P \subset Q \Rightarrow \overline{P} \subset \overline{Q}$

(v) $\overline{P} \cup \overline{Q} = \overline{P \cup Q}$

(vi) $\overline{P \cap Q} \subset \overline{P} \cap \overline{Q}$

(vii) $P_x^l \in M_{pt}$ *and* $\delta(P_x^l, Q) = P_0^1 \Rightarrow P_x^l \in \overline{Q}_{pt}$, but the converse is not true in general.

Proof. (i), (ii) and (iv) are obvious.

(iii) We have $(\overline{\overline{P}})_{pt} = \overline{(\overline{P})_{pt}} = \overline{\overline{P_{pt}}} = (\overline{P})_{pt}$, from Theorem 4.12 and Theorem 4.14.

Hence $\mathrm{MS}[(\overline{\overline{P}})_{pt}] = \mathrm{MS}[(\overline{P})_{pt}] \Rightarrow \overline{\overline{P}} = \overline{P}$.

(v) $\overline{P \cup Q} = (P \cup Q) \cup (P \cup Q)^d = (P \cup Q) \cup (P^d \cup Q^d) = (P \cup P^d) \cup (Q \cup Q^d) = \overline{P} \cup \overline{Q}$.

(vi) The proof is obvious.

The equality does not hold in general.

As in Example 3.23, in $(\mathbb{R},d)$ consider $P = \{1/x : 2 \le x < 3\}$ and $Q = \{1/x : 3 \le x < 4\}$. Then $P \cap Q = \emptyset$ which gives $\overline{P \cup Q} = \emptyset$

But $P^d = \{1/x : 2 \le x \le 3\}$ and $Q = \{1/x : 3 \le x \le 4\}$ which gives $P^d \cup Q^d = \{1/3\} \ne \emptyset \Rightarrow \overline{P} \cap \overline{Q} \ne \emptyset$.

(vii) We have, $\delta(P_x^l, Q) = P_0^1 \Rightarrow P_x^l \in Q_{pt} \subset (\overline{Q})_{pt}$.

To show that the converse is not true, we consider the following example.

In an M-metric space (M,d), let $a \in^m M$ where $m > 2$ and $Q = \{2/a\}$. Then $Q_{pt} = \{P_a^1, P_a^2\}$. Clearly P_a^3 is a multi limit point of Q and so $P_a^3 \in (\overline{Q})_{pt}$. Now $d(P_a^3, P_a^1) = P_0^3$ and $d(P_a^3, P_a^2) = P_0^3$ $\Rightarrow \delta(P_a^3, Q) = P_0^3 > P_0^1$. Thus $P_a^3 \in (\overline{Q})_{pt}$, but $\delta(P_a^3, Q) > P_0^1$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] Wayne D. Blizard, Multiset theory, Notre Dame J. Formal Logic 30(1) (1989), 36 – 66.

[2] Wayne D. Blizard, Real-valued multisets and fuzzy sets, Fuzzy Sets Syst. 33(1) (1989), 77 – 97.

[3] Wayne D. Blizard, Negative membership, Notre Dame J. Formal Logic 31(3) (1990), 346 – 368.

[4] Wayne D. Blizard, The development of multiset theory, Modern Logic 1(4) (1991), 319 – 352.

[5] K. Chakrabarty, Bags with interval counts, Found. Comput. Decision Sci. 25(1) (2000), 23– 36.

[6] K. Chakrabarty and I. Despi, n^k-bags, Int. J. Intell. Syst. 22(2) (2007), 223 – 236.

[7] K. Chakrabarty, R. Biswas and S. Nanda, Fuzzy shadows, Fuzzy Sets Systems 101(3) (1999), 413 – 421.

[8] K. Chakrabarty, R. Biswas and S. Nanda, On Yagers theory of bags and fuzzy bags, Computers and Artificial Intelligence (1999).

[9] S. Das, R. Roy, An introduction to multi metric spaces, Adv. Dyn. Syst. Appl. 16(2) (2021), 605-618.

[10] G.F. Clements, On multiset k-families, Discrete Math. 69(2) (1988), 153 – 164.

[11] M. Conder, S. Marshall, A. Slinko, Orders on Multisets and Discrete Cones, Order. 24 (2007), 277–296.

[12] K.P. Girish, S.J. John, Multiset topologies induced by multiset relations, Inform. Sci. 188 (2012), 298 – 313.

[13] K.P. Girish, S.J. John, On multiset topologies, Theory Appl. Math. Computer Sci. 2(1) (2012), 37 – 52.

[14] K.P. Girish, S.J. John, General relations between partially ordered multisets and their chains and antichains, Math. Commun. 14(2) (2009),193 – 206.

[15] K.P. Girish, S.J. John, Relations and functions in multiset context, Inform. Sci. 179(6) (2009), 758 – 768.

[16] K. P. Girish, S. J. John, Rough multisets and information multisystems, Adv. Decision Sci. 2011 (2011), 495392.

[17] S.P. Jena, S.K. Ghosh and B.K. Tripathy, On the theory of bags and lists, Inform. Sci. 132(14) (2001), 241 – 254.

[18] Sk. Nazmul, S.K. Samanta, On soft multigroups, Ann. Fuzzy Math. Inform. 10(2) (2015), 271 – 285.

[19] Sk. Nazmul, P. Majumdar, S. K. Samanta, On multisets and multigroups, Ann. Fuzzy Math. Inform. 6(30) (2013), 643 – 656.

[20] J.L. Peterson, Computation sequence sets, J. Computer Syst. Sci. 13(1) (1976), 1 – 24.

[21] D. Singh, A note on the development of multiset theory, Modern Logic 4(4) (1994), 405 – 406.

[22] D. Singh, A.M. Ibrahim, T. Yohana and J.N. Singh, Complementation in multiset theory, Int. Math. Forum 6(38) (2011), 1877 – 1884.

[23] D. Singh, A.M. Ibrahim, T. Yohana and J.N. Singh, An overview of the applications of multisets, Novi Sad J. Math. 37(2) (2007), 73 – 92.

[24] D. Singh, A.M. Ibrahim, T. Yohana and J.N. Singh, Some combinatorics of multisets, Int. J. Math. Educ. Sci. Technol. 34(4) (2003), 489 – 499.

[25] D. Tokat, I. Osmanoglu, Y. Ucok, On Soft Multi Continuous Functions, J. New Theory, 1 (2015), 50– 58.