Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7805-7820
https://doi.org/10.28919/jmcs/6610
ISSN: 1927-5307

NUMERICAL SOLUTION OF STOCHASTIC TIME FRACTIONAL HEAT TRANSFER EQUATION WITH ADDITIVE NOISE

ARCHANA MOURYA[1,*], KALYANRAO TAKALE[2], SHRIKISAN GAIKWAD[3]

[1]Department of Applied Science and Maths, K. K. Wagh Institute of Engg. Edu. and Research Nashik, India

[2]Department of Mathematics, RNC Arts, JDB Commerce and NSC Science College, Nashik-Road, Nashik, India

[3]Department of Mathematics, New Arts, Commerce and Science College, Ahmednagar, India

Abstract. In the present paper, we developed an explicit finite difference scheme for stochastic time fractional heat transfer equation. Furthermore, we proved the conditional stability of the solution of the scheme and analyze the effect of the multiplier of the random noise in mean square stability. Also, we discuss the convergence of the explicit scheme in the mean square sense. The approximate solution of the practical problem is obtain by developed scheme and it is represented graphically by Mathematica software.

Keywords: fractional derivative; time fractional; stochastic heat equation; explicit scheme; stability; mean square.

2010 AMS Subject Classification: 34K37, 65M30, 35R60.

1. Introduction

Stochastic partial differential equations are very useful in describing random effects occuruing in the fields of science, engineering and mathematical finance. Several numerical methods have been developed to solve stochasstic partial differential equations (SPDEs), Allen [2] has constructed finite element and difference approximation of some linear SPDEs. Kamrani and

*Corresponding author

E-mail address: archana1704@gmail.com

Received August 07, 2021

Hosseini [9] discussed explicite and implicit finite difference methods and their stability convergence for general SPDEs. Furthermore, Soheili et al. [29] presented higher order finite difference scheme and Saul'yev scheme for solving linear parabolic SPDEs.

Now a days, fractional calculus is a developing branch of mathematics which deals with the derivative and integrals of non-integer order and attracted lots of attention in several fields such as physics, chemistry, engineering, hydrology and finance. Recently, fractional diffusion equations are becoming more popular in many areas of applications [11, 14, 23, 27, 37] and stochastic fractional partial differential equations[1, 36] etc. Stochastic partial differential equations with fractional time derivative can be used to describe random effects on transport of particles in medium with thermal memory or particles subjected to sticking and trapping [4]. The theoretical analysis of fractional stochastic partial differential equations have been intensively investigated in the literature [4, 15, 16], also, Sweilam et al. [31] presented compact finite difference method to solve stochastic fractional advection diffusion equation. Amaneh et al [3] introduced stochastic generalized fractional diffusion equation and used finite difference method for finding numerical solution. Chen et al [4] introduced a class of SPDEs with time fractional derivatives and proved existence and uniqueness of their solutions.

The classical heat equation deals with heat propagation in homogeneous medium and the time fractional diffusion equation has been widely used to model the anomalous diffusion exhibiting sub diffusive behavior due to particle sticking and trapping phenomena [13]. When noise is introduced in partial differential equations, turns it to SPDEs. According to the nature of the noise term there are two main classes of SPDEs like SPDEs with additive noise and SPDEs with multiplicative noise. In additive SPDEs, noise term does not depends on the state of the system, while the noise term depends on the state of the system in multiplicative SPDEs. Additive noise occurs due to temporal fluctuations of internal degrees of freedom, while multiplicative noise arrives due to random variations of some external control parameters [8]. Fractional stochastic partial differential equation considers both memory and environmental noise effect. Now a days, Liu et al [36] studied some properties of a class of fractional stochastic heat equations. Babaei et al [1] applied a spectral collection method to solve a class of time fractional heat equation driven by Brownian motion. Guang-an Zou [6] presented a Galerkin finite element method for time fractional stochastic heat equation driven by multiplicative noise. Stochastic fractional heat equations driven by additive noise are yet to be investigated. At the same time, it is very difficult to find exact solution of stochastic fractional heat equations by analytical methods but finite difference methods that gives large algebraic system to solve in place of differential equation. In this connection, the theme of this paper is to develop a fractional order explicit finite difference scheme for stochastic time fractional heat transfer equation (STFHTE) driven by additive noise with initial and boundary conditions, given as follows,

$$\frac{\partial^\alpha S(x,t)}{\partial t^\alpha} = D\frac{\partial^2 S(x,t)}{\partial x^2} + \sigma \dot{W}(t);\ (x,t) \in \Omega : [0,L] \times [0,T]$$

$$Initial\ condition:\ S(x,0) = S_0(x),\ 0 \le x \le L$$

$$Boundary\ conditions:\ S(0,t) = S(L,t) = 0,\ x \to \infty,\ t \ge 0$$

where, $D > 0$ is the diffusivity constant, $\sigma > 0$ is the constant noise intensity, i.e.additive noise. Now, $W(t)$ is one dimensional standard Winner process, where $\dot{W}(t) = \frac{\partial W(t)}{\partial t}$, such that white noise $\Delta W(t)$ is a random variable generated from Gaussian distribution with zero mean and standard deviation Δt and S(x,t) is temperature at the position x and at time t. Here, $\frac{\partial^\alpha S(x,t)}{\partial t^\alpha}$ is Caputo time fractional derivative of order α. We consider the following definitions for further developments.

Definition 1.1. *The Caputo time fractional derivative of order* α, $(0 < \alpha \le 1)$ *is defined as follows* [22]

$$\frac{\partial^\alpha S(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma 1-\alpha} \int_0^t \frac{\partial S(x,t)}{\partial \xi} \frac{d\xi}{(t-\xi)^\alpha}, & 0 < \alpha < 1 \\ \frac{\partial S(x,t)}{\partial t}, & \alpha = 1 \end{cases}$$

where $\Gamma(.)$ is a Gamma function.

Definition 1.2. ***Euler-Maruyama Scheme:***

For the d-dimensional Stochastic differential equation of Itó type [25]

$$\begin{cases} dS(t) = f(t,S(t))dt + g(t,S(t))dW(t), t \ge 0 \\ S(0) = S_0 \in R^d, \end{cases}$$

where $f,g:R^d \to R^d$, *the Euler-Maruyama Scheme for computing approximations* S_i^k *takes the form*

$$S_i^{k+1} = S_i^k + f(S_i^k)\Delta t + g(S_i^k)\Delta W_k,$$

where

$$\Delta W^k = W(t_{k+1} - W(t_k) = W((k+1)\Delta t) - W(k\Delta t)$$

and $\Delta W^k \sim \mathscr{N}(0,\Delta t)$.

Definition 1.3. *A sequence of random variables* $\{X_{n_k}\}$, $(n, k > 0)$ *converges in mean square to random variable* X *if* $\lim\limits_{n_k \to \infty} ||X_{n_k} - X|| = 0$ [30].

Definition 1.4. *A Stochastic difference scheme is stable in mean square if there are positive constants* ε, δ *and constants* k, b *such that* [30]

$$E|S_i^{k+1}|^2 \leq ke^{bt}|S^0|^2$$

for all $0 \leq t = (k+1)\Delta t$, $0 \leq \Delta x \leq \varepsilon$ *and* $0 \leq \Delta t \leq \delta$.

Definition 1.5. *A Stochastic difference scheme* $L_i^k S_i^k = G_i^k$ *approximating Stochastic partial differential equation* $L_v = G$ *is convergent in mean square at time* $t = (k+1)\Delta t$ *if*

$$E|S_i^k - S|^2 \to 0,$$

as $i \to \infty$, $n \to \infty$, $(\Delta x, \Delta t) \to (0,0)$ *and* $(i\Delta x, k\Delta t) \to (x,t)$ [30].

Definition 1.6. *The symmetric second order difference quotient in space at time level* $t = t_k$ *is given as follows* [28]

$$\frac{\partial^2 S(x,t)}{\partial x^2} = \frac{S(x_{i-1},t_k) - 2S(x_i,t_k) + S(x_{i+1},t_k)}{h^2}$$

We organize the paper as follows: In section 2 , we develop the explicit fractional order finite difference scheme for Stochastic time fractional heat transfer equation. The stability of the solution is proved in section 3 and section 4 deals with convergence of the scheme. Finally, in the last section, the numerical solutions of Stochastic time fractional heat transfer equation is obtained and simulated graphically by Mathematica.

2. Finite Difference Scheme

We consider the following Stochastic time fractional heat transfer equation driven by additive noise (STFHTE) with initial and boundary conditions

$$\frac{\partial^\alpha S(x,t)}{\partial t^\alpha} = D\frac{\partial^2 S(x,t)}{\partial x^2} + \sigma \dot{W}(t);\ (x,t) \in \Omega : [0,L]\times[0,T] \tag{1}$$

$$Initial\ condition:\ S(x,0) = S_0(x),\ 0 \le x \le L \tag{2}$$

$$Boundary\ conditions:\ S(0,t) = S(L,t) = 0,\ t \ge 0 \tag{3}$$

where, $D > 0$ is the diffusivity constant, $\sigma > 0$ is the constant noise intensity, i.e.additive noise. Now, $W(t)$ is one dimensional standard Winner process, where $\dot{W}(t) = \frac{\partial W(t)}{\partial t}$, such that white noise $\Delta W(t)$ is a random variable generated from Gaussian distribution with zero mean and standard deviation Δt. Note that for $\alpha = 1$, we recover in the limit the well known Stochastic heat transfer equation of Markovian process

$$\frac{\partial S(x,t)}{\partial t} = D\frac{\partial^2 S(x,t)}{\partial x^2} + \sigma \dot{W}(t),\ x \in R;\ t \ge 0.$$

For the numerical approximation of the explicit scheme, we define $h = \frac{L}{N}$ and $\tau = \frac{T}{N}$ the space and time steps respectively, such that $t_k = k\tau$; $k = 0,1,\cdots,N$ be the integration time $0 \le t_k \le T$ and $x_i = ih$ for $i = 0,1,\cdots,N$. Define $S_i^k = S(x_i,t_k)$ and let S_i^k denote the numerical approximation at the mesh point (x_i,t_k) to the exact solution $S(x_i,t_k)$.

The time fractional derivative $\frac{\partial^\alpha S(x,t)}{\partial t^\alpha}$ at the mesh point (x_i,t_{k+1}), $i = 1,2,\cdots N-1$, $k = 0,1,2,\cdots$. is approximated as follows

$$\begin{aligned}
\frac{\partial^\alpha S(x_i,t_{k+1})}{\partial t^\alpha} &\approx \frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k+1}} \frac{1}{(t_{k+1}-\xi)^\alpha}\frac{\partial S(x_i,\xi)}{\partial \xi}d\xi \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{S(x_i,t_{j+1})-S(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{S(x_i,t_{j+1})-S(x_i,t_j)}{\tau}\int_{(k-j)\tau}^{(k+1-j)\tau}\frac{d\eta}{\eta^\alpha}
\end{aligned}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{S(x_i,t_{k+1-j}) - S(x_i,t_{k-j})}{\tau} \int_{(j)\tau}^{(j+1)\tau} \frac{d\eta}{\eta^{\alpha}}$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} \frac{S(x_i,t_{k+1-j}) - S(x_i,t_{k-j})}{\tau} \times [(j+1)^{1-\alpha} - j^{1-\alpha}]$$

(4)
$$\frac{\partial^{\alpha} S(x_i,t_{k+1})}{\partial t^{\alpha}} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [S(x_i,t_{k+1}) - S(x_i,t_k)] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j [S(x_i,t_{k+1-j}) - S(x_i,t_{k-j})]$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha},\ j = 0,1,2,\cdots,k.$

We adopt a symmetric second order difference quotient in space at time level $t = t_k$ for approximating the second order space derivative,

$$\frac{\partial^2 S(x,t)}{\partial x^2} = \frac{S(x_{i-1},t_k) - 2S(x_i,t_k) + S(x_{i+1},t_k)}{h^2}. \tag{5}$$

Therefore, from equations , (4) and (5), we approximated the Stochastic fractional heat transfer equation (1) as follows

$$\begin{aligned} &\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [S_i^{k+1} - S_i^{k}] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j [S_i^{k-j+1} - S_i^{k-j})] \\ &= D \frac{[S(x_{i-1},t_k) - 2S(x_i,t_k) + S(x_{i+1},t_k)]}{h^2} + \sigma \frac{W((k+1)\tau) - W(k\tau)}{\tau} \end{aligned} \tag{6}$$

After simplification, we get

$$\begin{aligned} S_i^{k+1} = rS_{i-1}^{k} + (1-2r-b_1)S_i^{k} + rS_{i+1}^{k} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) S_i^{k-j} \\ + b_k S_i^0 + a\left[\frac{W((k+1)\tau) - W(k\tau)}{\tau}\right] \end{aligned} \tag{7}$$

where $i = 1,2,\cdots,N-1,\ k = 0,1,2,\cdots.,\ r = \frac{D\,\tau^{\alpha}\Gamma(2-\alpha)}{h^2},\ a = \sigma\tau^{\alpha}\Gamma(2-\alpha)$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha},\ j = 1,2,\cdots k.$

The initial condition is approximated as $S_i^0 = S_0,\ i = 1,2,\cdots,N-1.$

The boundary conditions are approximated as $S_0^k = 0,\ S_N^k = 0,\ k = 0,1,2,\cdots,N-1.$

$$S_i^{k+1} = rS_{i-1}^{k} + (1-2r-b_1)S_i^{k} + rS_{i+1}^{k} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) S_i^{k-j} + b_k S_i^0 + aW^k \tag{8}$$

where, $W^k \sim \sqrt{k\tau}\,\mathcal{N}(0,1)$, $k = 0,1,2,\cdots N-1$.

Therefore, the complete discretization of STFHTE with initial and boundary conditions is

$$S_i^1 = rS_{i-1}^0 + (1-2r)S_i^0 + rS_{i+1}^0 \; for\; k = 0 \tag{9}$$

$$S_i^{k+1} = rS_{i-1}^k + (1-2r-b_1)S_i^k + rS_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})S_i^{k-j} + b_k S_i^0 + aW^k,\; for\; k \geq 1 \tag{10}$$

$$\text{Initial condition: } S_i^0 = S_0, i = 1,2,\cdots,N-1. \tag{11}$$

$$\text{Boundary conditions: } S_0^k = 0 \; and \; S_N^k = 0. \tag{12}$$

where $r = \frac{D\,\tau^\alpha \Gamma(2-\alpha)}{h^2}$, $a = \sigma\tau^\alpha\Gamma(2-\alpha)$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0,1,2,\cdots,k$.

Therefore, the fractional approximated IBVP $(9)-(12)$ can be written in the following matrix equation form

$$S^1 = BS^0,\; for\; k = 0 \tag{13}$$

$$S^{k+1} = AS^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})S^{k-j} + b_k S^0 + +aW^k,\; for\; k \geq 1 \tag{14}$$

where $S^k = (S_1^k, S_2^k, ..., S_{N-1}^k)^T$, $k = 0,1,2...,N-1$, A and B are tri-diagonal matrices of order $N-1$ given by

$$A = \begin{pmatrix} (1-2r-b_1) & r & 0 & 0\cdots & 0 & 0 & 0 \\ r & (1-2r-b_1) & r & 0\cdots & 0 & 0 & 0 \\ 0 & r & (1-2r-b_1) & r\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0\cdots & r & (1-2r-b_1) & r \\ 0 & 0 & \cdots & 0\cdots & 0 & r & (1-2r-b_1) \end{pmatrix}$$

$$B = \begin{pmatrix} (1-2r) & r & 0 & 0\cdots & 0 & 0 & 0 \\ r & (1-2r) & r & 0\cdots & 0 & 0 & 0 \\ 0 & r & (1-2r) & r\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0\cdots & r & (1-2r) & r \\ 0 & 0 & \cdots & 0\cdots & 0 & r & (1-2r) \end{pmatrix}.$$

3. STABILITY

In the present section, we discuss the stability of the solution of fractional order explicit finite difference scheme $(9)-(12)$ for the Stochastic time fractional heat transfer equation $(1)-(3)$.

Theorem 3.1. *The solution of stochastic time fractional heat transfer equation* $(1)-(3)$ *obtained by developed fractional order explicit finite difference scheme* $(9)-(12)$ *is conditionally stable in mean square sense, when* $r \leq \min\left\{ \frac{1}{2}, \frac{1-b_1}{2} \right\}, 0 \leq b_1 \leq 1$.

Proof: Proof: We define $E|S^k|^2 = \sup_{1\leq i\leq N-1} E|S_i^k|^2$, where $S^k = (S_1^k,\, S_2^k,\, S_3^k \cdots,\, S_{N-1}^k)^T$.
Then equation (9) leads to

$$\begin{aligned} E|S_i^k| &= E|rS_{i-1}^0 + (1-2r)S_i^0 + rS_{i+1}^0|^2 \\ &\leq |r+(1-2r)+r|^2 \sup_{1\leq i\leq N-1} E|S_i^0|^2, \text{(when} \, 1-2r \geq 0 \Rightarrow r \leq \frac{1}{2}) \end{aligned}$$

$$\therefore E|S_i^1|^2 \leq K\,E|S^0|^2, \text{ when } r \leq \frac{1}{2}.$$

We assume that

$$E|S_i^k|^2 \leq KE|S^0|^2,\ \forall n \leq k.$$

Then we prove

$$E|S_i^{k+1}|^2 \leq KE|S^0|^2 \text{ for} \, n = k+1.$$

From equation (10), we obtain

$$E\left|S_i^{k+1}\right|^2 = E\left|rS_{i-1}^k + (1-2r-b_1)S_i^k + rS_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})S_i^{k-j} + b_k S_i^0 + aW^k\right|^2$$

$$E\left|S_i^{k+1}\right|^2 \le E\left|rS_{i-1}^k + (1-2r-b_1)S_i^k + rS_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})S_i^{k-j} + b_k S_i^0 + a\sqrt{\Delta t}V_n\right|^2$$

when $(1-2r-b_1) \ge 0 \Rightarrow r \le \frac{1-b_1}{2}$, where $V_n \sim \mathscr{N}(0,1)$ and V_n is Normally distributed with mean 0 and variance 1 i.e. $\mathscr{N}(0,1)$ random variable.

We know that $E[V_n] = 0$ and $E[V_n^2] = 1$, therefore,

$$\begin{aligned}
E|S_i^{k+1}|^2 &\le |r + (1-2r-b_1) + r + (b_1 - b_2 + b_2 - b_3 + \cdots + (b_{k-1} - b_k) + b_k|^2 \sup_{1\le i\le N-1} E|S_i^k|^2 \\
&\quad + (a^2\Delta t) \sup_{1\le i\le N-1} E|S_i^0|^2 \\
&\le \sup_{1\le i\le N-1} E|S_i^k|^2 + a^2\Delta t \sup_{1\le i\le N-1} E|S_i^0|^2 \\
&\le (1 + a^2\Delta t) \sup_{1\le i\le N-1} E|S_i^0|^2 \\
&\le K \sup_{1\le i\le N-1} E|S_i^0|^2, \text{ where } K = (1 + a^2\Delta t). \\
&\le K E|S^0|^2, \text{ where } K = (1 + a^2\Delta t).
\end{aligned}$$

Hence, by induction, we prove

$$E|S_i^k|^2 \le KE|S^0|^2,\ \forall n,\ K = (1 + a^2\Delta t),\ \text{when } r \le \min\left\{\ \tfrac{1}{2}, \tfrac{1-b_1}{2}\ \right\}, 0 \le b_1 \le 1.$$

Therefore, we prove that the solution of the scheme is conditionally stable in mean square sense.

4. CONVERGENCE

In the present section, we discuss convergence of the fractional order explicit finite difference scheme $(9)-(12)$ developed for the STFHTE $(1)-(3)$.

Theorem 4.1. *The fractional order explicit finite difference scheme* $(9)-(12)$ *for stochastic time fractional heat transfer equation* $(1)-(3)$ *is convergent in mean square sense with* $r \le \min\left\{\frac{1}{2}, \frac{1-b_1}{2}\right\}$.

Proof: Let $\bar{S}^k = \left(\bar{S}^k_1, \bar{S}^k_2, \bar{S}^k_3, \cdots, \bar{S}^k_{N-1}\right)^T$ and $S^k = \left(S^k_1, S^k_2, S^k_3, \cdots, S^k_{N-1}\right)^T$ be the vectors of exact solution and approximate solution of the STFHET $(1)-(3)$ respectively. Then the fractional order explicit finite difference scheme $(9)-(12)$ becomes

$$\bar{S}^{k+1}_i = r\bar{S}^k_{i-1} + (1-2r-b_1)\bar{S}^k_i + r\bar{S}^k_{i+1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})\bar{S}^{k-j}_i + b_k\bar{S}^0_i + aW^k + T^k \tag{15}$$

where T^k is vector of truncation error at time level t_k. Suppose, $|e^k_i| = |\bar{S}^k_i - S^k_i|$ is the error vector. Let us assume that

$$E|e^k_l|^2 = \max_{1\le i\le N-1} E|e^k_i|^2 = ||E^{*^k}||^2_\infty \text{ and } T^k_l = \max_{1\le i\le N-1} |T^k_i| = h^2 O\big(\tau + h^2\big) \text{for } l = 1,2,3,\cdots$$

For k=0, from equation (9), we obtain

$$\begin{aligned}
E|e^1_l|^2 &= E|re^0_{i-1} + (1-2r)e^0_i + re^0_{i+1}|^2 + r|\tau^1_i| \\
&\le |r + (1-2r) + r|^2 E|e^0_l|^2 + r|\tau^1_l| \ \text{(when } 1-2r \ge 0 \Rightarrow r \le \frac{1}{2}) \\
&\le E|e^0_l|^2 + r|\tau^1_l| \\
&\le E|e^0_l|^2 + r|\tau^1_l| \\
&\le E|e^0_l|^2 + rh^2 O(\tau + h^2) \\
\therefore E\|E^{*^1}\|^2_\infty &\le E\|E^{*^0}\|^2_\infty + \tau^\alpha \Gamma(2-\alpha) O(\tau + h^2)
\end{aligned}$$

For $n = k$, we assume that

$$E\|E^{*^k}\|^2_\infty \le E\|E^{*^0}\|^2_\infty + k\tau^\alpha \Gamma(2-\alpha) O(\tau + h^2).$$

For $n = k+1$, we prove that

$$E\|E^{*^{k+1}}\|^2_\infty \le E\|E^{*^0}\|^2_\infty + (k+1)\tau^\alpha \Gamma(2-\alpha) O(\tau + h^2).$$

Now, from (10), we have

$$
\begin{aligned}
E\|e_l^{k+1}\|^2 &= E|re_{i-1}^k+(1-2r-b_1)e_i^k+re_{i+1}^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})e_i^{k-j}+b_ke_i^0|^2+r|\tau_l^k| \\
&\le |r+(1-2r-b_1)+r+(b_1-b_2+b_2-b_3+\cdots+b_{k-1}+b_k) \\
&\qquad +b_k|^2E|e_l^k|^2+r|\tau_l^k| \qquad (when\ 1-2r-b_1\ge 0\Rightarrow r\le\frac{1-b_1}{2}) \\
&\le E|e_l^k|^2+r|\tau_l^k| \\
&\le E\|E^{*^k}\|_\infty^2+r|\tau_l^k| \\
&\le \left(E\|E^{*^0}\|_\infty^2+k\tau^2\Gamma(2-\alpha)O(\tau+h^1)\right)+\tau^\alpha\Gamma(2-\alpha)O(\tau+h^2) \\
&\le E\|E^{*^0}\|_\infty^2+(k+1)\tau^\alpha\Gamma(2-\alpha)O(\tau+h^2)
\end{aligned}
$$

$$\therefore E\|E^{*^{k+1}}\|_\infty^2\le E\|E^{*^0}\|_\infty^2+(k+1)\tau^\alpha\Gamma(2-\alpha)O(\tau+h^2).$$

Therefore, we conclude that $E|\bar{S}_i^k-S_i^k|^2\to 0$ as $(h,\tau)\to(0,0)$ when $r\le\min\left\{\frac{1}{2},\frac{1-b_1}{2}\right\}$. Hence, we prove that the scheme is conditionally convergent in mean square sense. This completes the proof.

5. Numerical Solutions

We consider the following stochastic time fractional heat transfer equation with initial and boundary conditions

$$\frac{\partial^\alpha S}{\partial t^\alpha}=\frac{\partial^2 S}{\partial x^2}+\sigma\dot{W}(t), 0<x<1,\ 0<t<1 \tag{16}$$

$$\text{initial condition:}\, S(x,0)=\sin\pi x,\ 0\le x\le 1 \tag{17}$$

$$\text{boundary conditions:}\, S(0,t)=S(1,t)=0, t\ge 0. \tag{18}$$

where $0<\alpha\le 1$, $\sigma>0$ is the constant noise intensity. Now, $W(t)$ is one dimensional standard Winner process, where $\dot{W}(t)=\frac{\partial W(t)}{\partial t}$, such that white noise $\Delta W(t)$ is a random variable generated from Gaussian distribution with zero mean and standard deviation Δt. In Figure 1, we plot the exact solution of stochastic heat transfer equation in absence of noise term.

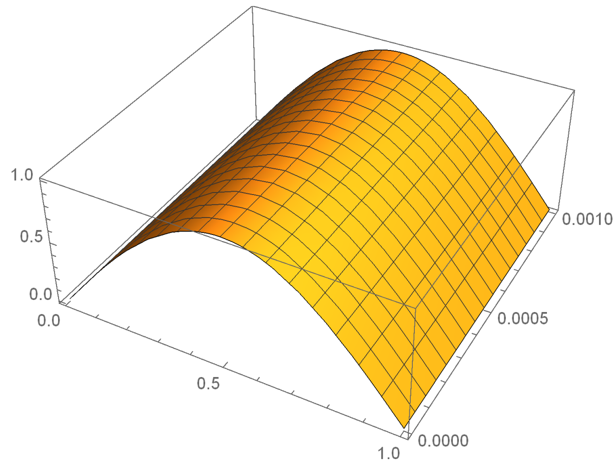
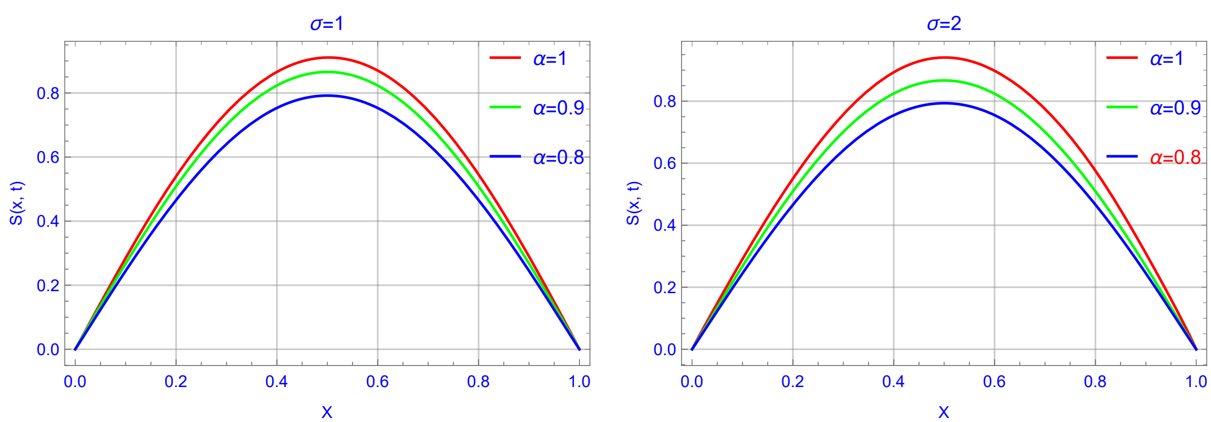


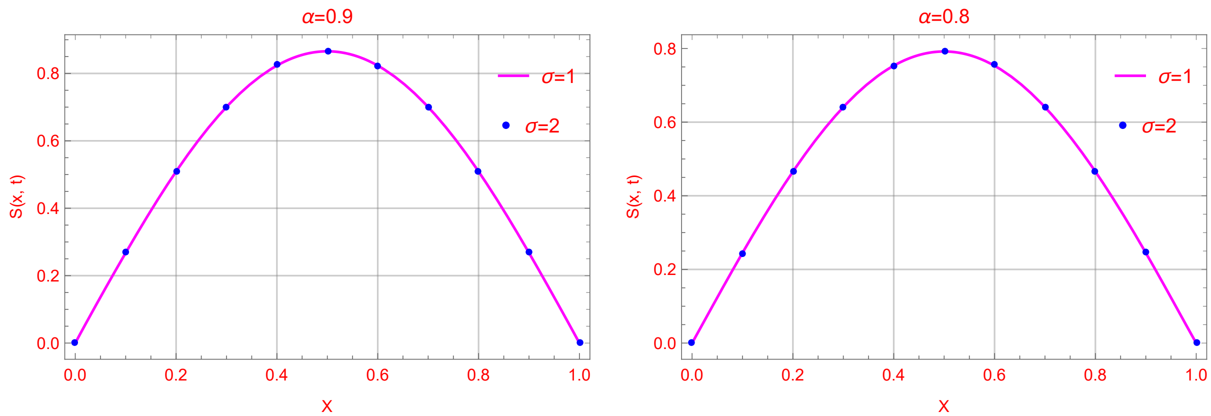
FIGURE 1. The exact solution of Stochastic heat transfer equation $(5.1)-(5.3)$ without noise.



(A) σ=1

(B) σ=2

FIGURE 2. Approximation solution of the STFHTE with constant noise intensity $\sigma = 1,2$ and the parameters $\alpha = 1, 0.9, 0.8$ respectively.



(A) α=0.9 (B) α=0.8

FIGURE 3. Comparison of approximation solution of the STFHTE with constant noise intensity $\sigma = 1, 2$ and the parameters $\alpha = 0.9, 0.8$ respectively.

In Figure 2, we obtain the numerical solution of the STFHTE (5.1)-(5.3) for $\alpha = 1, 0.9, 0.8$ with additive noise $\sigma = 1, 2$ respectively and observed that these solutions are good approximation to the exact solutions. Therefore the developed fractional order explicit finite difference scheme provides a effective approximate solution. We observed that as α increased the amplitude of the solution behavior is increased. From Figure 3, we observed that the approximate solutions of STFHTE (5.1)-(5.3) for $\alpha = 0.8$ and $\alpha = 0.9$ with $\sigma = 1, 2$ are approximately same. Therefore we conclude that if the additive noise intensity changes the numerical solutions are approximately same.

6. CONCLUSIONS

(1) In present paper, we successfully developed the fractional order explicit finite difference scheme for Stochastic time fractional heat transfer equation.
(2) The stability and consistency of the scheme are investigated for Stochastic time fractional heat transfer equation in mean square sense and the bound of stability is $r \leq \min\left\{\frac{1}{2}, \frac{1-b_1}{2}\right\}$, where $0 \leq b_1 \leq 1$.
(3) The numerical examples with plots of the results are given to demonstrate the efficiency of the scheme. It has been observed that the numerical scheme is very powerful and provide approximate solution which is very near to the exact solution.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Babaei, H. Jafari, S. Banihashemi, A Collocation Approach for Solving Time-Fractional Stochastic Heat Equation Driven by an Additive Noise, Symmetry. 12 (2020), 904.

[2] E.J. Allen, S.J. Novosel, Z. Zhang, Finite element and difference approximation of some linear stochastic partial differential equations, Stoch. Stoch. Rep. 64 (1998), 117–142.

[3] A. Sepahvandzadeh, B. Ghazanfari, N. Asadian, Numerical Solution of Stochastic Generalized Fractional Diffusion Equation by Finite Difference Method, Math. Comput. Appl. 23 (2018), 53.

[4] Z.-Q. Chen, K.-H. Kim, P. Kim, Fractional time stochastic partial differential equations, Stoch. Proc. Appl. 125 (2015), 1470–1499.

[5] D. Dhaigude, K. Takale, A weighted average finite difference schemefor one dimensional pennes bioheat equation, Proc. Neural Parallel Sci. Comput. 4 (2010), 124-128.

[6] G. Zou, A Galerkin finite element method for time-fractional stochastic heat equation, Computers Math. Appl. 75 (2018), 4135–4150.

[7] M.G. Hahn, K. Kobayashi, S. Umarov, Fokker-Planck-Kolmogorov equations associated with time-changed fractional Brownian motion, Proc. Amer. Math. Soc. 139 (2011), 691–691.

[8] T. Kamppeter, F.G. Mertens, E. Moro, A. Sánchez, A.R. Bishop, Stochastic vortex dynamics in two-dimensional easy-plane ferromagnets: Multiplicative versus additive noise, Phys. Rev. B. 59 (1999), 11349–11357.

[9] M. Kamrani, S. Mohammad Hosseini, The role of coefficients of a general SPDE on the stability and convergence of a finite difference method, J. Comput. Appl. Math. 234 (2010), 1426–1434.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trijillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol 24, (2006).

[11] J. Klafter, I.M. Sokolov, Anomalous diffusion spreads its wings, Phys. World. 18 (2005), 29–32.

[12] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comput. Phys. 225 (2007), 1533–1552.

[13] M.M. Meerschaert, A. Sikorskii, Stochastic models for fractional calculus, De Gruyter Studies in Mathematics, 43, (2012).

[14] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000), 1–77.

[15] J.B. Mijena, E. Nane, Intermittence and space-time fractional stochastic partial differential equations, Potential Anal. 44 (2016), 295-312.

[16] J. B. Mijena, E. Nane, Space - time fractional stochastic fractional differential equations, Stoch. Process. Appl. 125 (2015), 3301-3326.

[17] K. S. Miller, B. Ross, An introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, (1993).

[18] W. W. Mohammed et al. Mean square convergent finite difference scheme for stochastic parabolic PDEs, Amer. J. Comput. Math. 4 (2014), 280-288.

[19] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Solitons Fractals. 31 (2007), 1248–1255.

[20] M. Namjoo, A. Mohebbian, Approximation of stochastic advection-diffusion equations with finite difference scheme, J. Math. Model. 4 (2016), 1-18.

[21] M. Namjoo, A. Mohebbian, Analysis of stability and convergence of a finite difference approximation for stochastic partial differential equations, Comput. Meth. Differ. Equ. 7 (2019), 334-358.

[22] I. Podlubny, Fractional Differential equations, Academic Press, San Diago (1999).

[23] J. Pruss, Evolutionary integral equations and applications, Monogr. Math. 87, (1983).

[24] M. Rehman, R. A. Khan, A numerical method for solving boundary value problems for fractional differential equations, Appl. Math. Model. 36(3) (2012), 894-907.

[25] C. Roth, Difference methods for stochastic partial differential equations, Z. Angew. Math. 82 (2002), 821-830.

[26] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordan and Breach, Newark, N. J., (1993).

[27] R. Schumer, D.A. Benson, M.M. Meerschaert, S.W. Wheatcraft, Eulerian derivation of the fractional advection-dispersion equation, J. Contaminant Hydrol. 48 (2001), 69-88.

[28] G.D. Smith, Numerical Solution of Partial Differential Equations, 2nd edition, Clarendon Press, Oxford, (1978).

[29] A.R. Soheili, M.B. Niasar, M. Arezoomanadan, Approximation of stochastic parabolic differential equations with two different finite difference scheme, Bull. Iran. Math. Soc. 61-83, (2011).

[30] T.T. Soong, Random Differential equations in science and engineering, Academic Press, New York, (1973).

[31] N.H.Sweilam, D.M. El-Sakout and M.M.Muttardi, Compact finite difference method to numerically solving a stochastic fractional advection diffusion equation, Adv. Differ. Equ, 2020 (2020), 189.

[32] K. Takale, Fractional order finite difference scheme for space fractional diffusion equation, Eng. Autom. Probl. 120-124, (2014).

[33] K.C. Takale, V.R. Nikam, S. Kulkarni, Fractional order finite difference scheme for space fractional diffusion equation, Eng. Autom. Probl. 120-124, (2014).

[34] K. Takale, M. Datar, S. Kulkarni, Approximate solution of space fractional partial differential equations and applications, Int. J. Eng. Computer Sci. Eng. 5 (2018), 216-220.

[35] J.B. Walsh, An introduction to stochastic partial differential equations, lecture notes in mathematics, Springer-Verlag, 1180 (1986), 265-439.

[36] W. Liu, K. Tian, M. Foodun, On some properties of a class of fractional stochastic heat equations, J. Theor. Probab. 30 (2017), 1310-1333.

[37] L. von Wolfersdorf, An identification of memory kernels in linear theory of heat equation, Math. Methods Appl. Sci. 17 (1994), 919-932.