ON INFRA GENERALIZED $\#\alpha$-CLOSED SETS IN INFRA TOPOLOGICAL SPACES

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Abstract: In this paper, the relatively new notions of Infra generalized $\#\alpha$-closed set, Infra generalized $\#\alpha$-continuous functions, Infra generalized $\#\alpha$-irresolute mappings are introduced and explored some of its characteristics.

Keywords: infra generalized $\#\alpha$-closed sets; infra generalized $\#\alpha$-continuous functions; infra generalized $\#\alpha$-irresolute mapping.

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1. INTRODUCTION


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2. PRELIMINARIES

Throughout this paper, $(X, \tau_{iX})$ (or $X$) represent a Infra topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $X$, $icp(A)$ and $iip(A)$ denote the Infra closure point of $A$ and the Infra interior point of $A$ and also $icp_\alpha(A)$, $icp_b(A)$ denote $i\alpha cp(A)$, $ibcp(A)$ respectively.

The following recalls requisite definitions in Infra topological spaces that will be necessitated in the sequel of our work.

**Definition 2.1.** [1] Let $X$ be any arbitrary set. An Infra topological space on $X$ is a collection $\tau_{iX}$ subsets of $X$ such that the following axioms are satisfying:

(1) $\phi, X \in \tau_{iX}$.

(2) The intersection of the elements of any sub collection of $\tau_{iX}$ in $X$. Terminology, the ordered pair $(X, \tau_{iX})$ is called Infra-topological space. We simply say $X$ is an Infra space.

**Definition 2.2.** [1] Let $(X, \tau_{iX})$ be an infra-topological space and $A \subset X$. $A$ is called an infra open set (ios) if $A \subset \tau_{iX}$.

**Definition 2.3.** [1] Let $(X, \tau_{iX})$ be an infra topological space. A subset $B \subset X$ is called infra-closed set (ics) in $X$ if $X-B$ is infra-open set in $X$.

**Definition 2.4.** [1] Let $(X, \tau_{iX})$ be an infra topological space and $A \subset X$. The Infra Closure Point (ICP) of $A$ is a set denoted by $icp(A)$ and given by : $icp(A)= \cap_{i} B_i : A \subset B_i, X - B_i \in \tau_{iX}$. (i.e) $icp(A)$ is the intersection of all infra closed set containing the set $A$. 
Definition 2.5. [1] Let $(X, \tau_X)$ be an infra topological space and $A \subseteq X$. The Infra Interior Point (IIP) of $A$ is a set denoted by $\text{iip}(A)$ and given by: $\text{iip}(A) = \cup \{ O_i : O_i \subseteq A, O_i \in \tau_{iX} \}$ (i.e) $\text{iip}(A)$ is the union of all infra open set contained in the set $A$.

Definition 2.6. [9] Let $(X, \tau_X)$ be an infra topological space. $A$ is called infra semi-open if $A \subseteq \text{icp}(\text{iip}(A))$ and infra semi-closed set if $\text{iip}(\text{icp}(A)) \subseteq A$.

Definition 2.7. [9] Let $(X, \tau_X)$ be an infra topological space. $A$ is called infra pre-open if $A \subseteq \text{iip}(\text{icp}(A))$ and infra pre-closed set if $\text{icp}(\text{iip}(A)) \subseteq A$.

Definition 2.8. [9] Let $(X, \tau_X)$ be an infra topological space. $A$ is called infra $\alpha$-open if $A \subseteq \text{iip}(\text{icp}(\text{iip}(A)))$ and infra $\alpha$-closed set if $\text{icp}(\text{iip}(\text{icp}(A))) \subseteq A$.

Definition 2.9. [9] Let $(X, \tau_X)$ be an infra topological space. $A$ is called infra $\beta$-open if $A \subseteq \text{iip}(\text{icp}(\text{iip}(A)))$ and infra $\beta$-closed set if $\text{icp}(\text{iip}(\text{icp}(A))) \subseteq A$.

Definition 2.10. [10] Let $(X, \tau_X)$ be an infra topological space. $A$ is called infra $b$-open if $A \subseteq \text{iip}(\text{icp}(A)) \cup \text{icp}(\text{iip}(A))$ and infra $b$-closed set if $\text{iip}(\text{icp}(A)) \cup \text{icp}(\text{iip}(A)) \subseteq A$.

Definition 2.11. A subset $A$ of a space $(X, \tau)$ is called

1. a infra generalized- closed set (briefly ig-closed) [10] if $\text{icp}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.
2. a infra $\alpha$ generalized- closed set (briefly i$\alpha$g-closed) if $\text{icp}_{\alpha}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra semi-open.
3. a infra generalized semi- closed set (briefly igs-closed) [10] if $\text{iscp}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.
4. an infra $\alpha$ generalized- closed set (briefly i$\alpha$g-closed) [10] if $\text{i}$ $\alpha$ $\text{cp}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.
5. an infra generalized $\alpha$- closed set (briefly ig$\alpha$-closed) [10] if $\text{i}$ $\alpha$ $\text{cp}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra $\alpha$- open.
6. a infra generalized pre- closed set (briefly igp-closed) [10] if $\text{ipcp}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.
(7) a infra generalized $\beta$- closed set (briefly ig$\beta$- closed) [10] if $ii\beta cp(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.

(8) a infra generalized $b$- closed set (briefly ig$b$- closed) [10] if $icp_b(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.

(9) a infra generalized sp- closed set (briefly igsp- closed) [10] if ispcp$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra open.

(10) a infra generalized $^*b$- closed set (briefly ig$^*b$- closed) [10] if icp$_b(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is infra g- open.

**Definition 2.12.** A subset $A$ of a space $(X, \tau_{iX})$ is called

(1) **Infra generalized- continuous**[11] if $f^{-1}(V)$ is Infra generalized- closed in $X$, for every Infra closed set $V$ of $Y$.

(2) **Infra $\alpha$-generalized- continuous**[11] if $f^{-1}(V)$ is Infra $\alpha$-generalized- closed in $X$, for every Infra closed set $V$ of $Y$.

(3) **Infra generalized $b$- continuous**[11] if $f^{-1}(V)$ is Infra generalized $b$- closed in $X$, for every Infra closed set $V$ of $Y$.

(4) **Infra generalized $p$- continuous**[11] if $f^{-1}(V)$ is Infra generalized $p$- closed in $X$, for every Infra closed set $V$ of $Y$.

(5) **Infra generalized $s$- continuous**[11] if $f^{-1}(V)$ is Infra generalized $s$- closed in $X$, for every Infra closed set $V$ of $Y$.

(6) **Infra generalized $\beta$- continuous**[11] if $f^{-1}(V)$ is Infra generalized $\beta$- closed in $X$, for every Infra closed set $V$ of $Y$.

(7) **Infra generalized sp- continuous**[11] if $f^{-1}(V)$ is Infra generalized sp- closed in $X$, for every Infra closed set $V$ of $Y$.

(8) **Infra generalized $^*b$- continuous**[11] if $f^{-1}(V)$ is Infra generalized $^*b$- closed in $X$, for every Infra closed set $V$ of $Y$.

**Definition 2.13.** A subset $A$ of a space $(X, \tau_{iX})$ is called

(1) **Infra generalized- irresolute**[11] if $f^{-1}(V)$ is Infra generalized- closed in $X$, for every Infra generalized- closed set $V$ of $Y$. 
3. Characteristics of Infra Generalized \( \# \alpha \)-Closed Sets in Infra Topological Spaces

In this section, we introduce the notion of Infra \( \# \alpha \)-closed sets and study some of its basic properties.

**Definition 3.1.** Let \((X, \tau_X)\) be a Infra topological space. A subset \(A\) of \(X\) is called an Infra generalized \( \# \alpha \)-closed set (briefly ig\( \# \alpha \)-closed) if \(\text{icp}_\alpha(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is Infra g-open.

**Theorem 3.2.** Every Infra-closed set is Infra g-closed set.

**Proof:** Let \(A\) be a Infra-closed set in \(X\). Let \(U\) be Infra open set, such that \(A \subseteq U\). Since \(A\) is Infra closed, \(\text{icp}(A) = A \subseteq U\). Therefore \(\text{icp}(A) \subseteq U\). Hence \(A\) is Infra g-closed set in \(X\).

**Remark 3.3.** The converse of the above theorem need not be true as seen from the following example.
Example 3.4. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{d\}\}$. Let $A = \{b\}$. Here $A$ is Infra $g$-closed set but not Infra-closed set of $(X, \tau_X)$.

Theorem 3.5. Every Infra-closed set is Infra $g^\#\alpha$-closed set.
Proof: Let $A$ be a Infra-closed set in $X$. Let $U$ be Infra $g$-open set, such that $A \subseteq U$. Since $A$ is Infra closed, $icp_\alpha(A) \subseteq icp(A) \subseteq U$. Therefore $icp_\alpha(A) \subseteq U$. Hence $A$ is Infra $g^\#\alpha$-closed set in $X$.

Remark 3.6. The converse of the above theorem need not be true as seen from the following example.

Example 3.7. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $A = \{b\}$. Here $A$ is Infra $g^\#\alpha$-closed set but not a Infra-closed set of $(X, \tau_X)$.

Theorem 3.8. Every Infra $\alpha$-closed set is Infra $g^\#\alpha$-closed set.
Proof: Let $A$ be a Infra $\alpha$-closed set in $X$. Let $U$ be Infra $g$-open set, such that $A \subseteq U$. Since $A$ is Infra $\alpha$-closed set. We have, $icp_\alpha(A) = A \subseteq U$. Therefore $icp_\alpha(A) \subseteq U$. Hence $A$ is Infra $g^\#\alpha$-closed set in $X$.

Remark 3.9. The converse of the above theorem need not be true as seen from the following example.

Example 3.10. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b, d\}\}$. Let $A = \{a, b, c\}$. Here $A$ is Infra $g^\#\alpha$-closed set but not a Infra $\alpha$-closed set of $(X, \tau_X)$.

Theorem 3.11. Every Infra $g^\#\alpha$-closed set is Infra $gs$-closed set.
Proof: Let $A$ be a Infra $g^\#\alpha$-closed set in $X$. Let $U$ be Infra open set, such that $A \subseteq U$. Since every Infra open set is Infra $g$-open and $A$ is Infra $g^\#\alpha$-closed, we have, $iscp_\alpha(A) \subseteq icp_\alpha(A) \subseteq U$. Then $iscp_\alpha(A) \subseteq U$. Hence $A$ is Infra $gs$-closed set in $X$.

Remark 3.12. The converse of the above theorem need not be true as seen from the following example.

Example 3.13. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, d\}\}$. Let $A = \{a, b, d\}$. Here $A$ is Infra $gs$-closed set but not a Infra $g^\#\alpha$-closed set of $(X, \tau_X)$. 
Theorem 3.14. Every Infra g\(^\#\)\(\alpha\)-closed set is Infra gp-closed set.

Proof: Let A be a Infra g\(^\#\)\(\alpha\)-closed set in X. Let U be Infra open set, such that A \(\subseteq\) U. Since every Infra open set is Infra g-open and A is Infra g\(^\#\)\(\alpha\)-closed, we have, picp(A) \(\subseteq\) icp\(_\alpha\)(A) \(\subseteq\) U. Then picp(A) \(\subseteq\) U. Hence A is Infra gp-closed set in X.

Remark 3.15. The converse of the above theorem need not be true as seen from the following example.

Example 3.16. Let X = \{a, b, c, d\} with the topology \(\tau = \{X, \phi, \{a\}, \{d\}\}\). Let A = \{a, c, d\}. Here A is Infra gp-closed set but not a Infra g\(^\#\)\(\alpha\)-closed set of \((X, \tau_\text{IX})\).

Theorem 3.17. Every Infra g\(^\#\)\(\alpha\)-closed set is Infra \(\alpha\)g-closed set.

Proof: Let A be a Infra g\(^\#\)\(\alpha\)-closed set in X. Let U be Infra open set, such that A \(\subseteq\) U. Since every Infra open set is Infra g-open and A is Infra g\(^\#\)\(\alpha\)-closed, we have, icp\(_\alpha\)(A) = A \(\subseteq\) U. Therefore, icp\(_\alpha\)(A) \(\subseteq\) U. Hence A is Infra \(\alpha\)g-closed set in X.

Remark 3.18. The converse of the above theorem need not be true as seen from the following example.

Example 3.19. Let X = \{a, b, c, d\} with the topology \(\tau = \{X, \phi, \{b\}, \{a, b\}, \{b, d\}\}\). Let A = \{a, b, d\}. Here A is Infra \(\alpha\)g-closed set but not a Infra g\(^\#\)\(\alpha\)-closed set of \((X, \tau_\text{IX})\).

Theorem 3.20. Every Infra g\(^\#\)\(\alpha\)-closed set is Infra g\(\beta\)-closed set.

Proof: Let A be a Infra g\(^\#\)\(\alpha\)-closed set in X. Let U be Infra open set, such that A \(\subseteq\) U. Since every Infra open set is Infra g-open and A is Infra g\(^\#\)\(\alpha\)-closed, we have, \(\beta\)icp(A) \(\subseteq\) icp\(_\alpha\)(A) \(\subseteq\) U. Then \(\beta\)icp(A) \(\subseteq\) U. Hence A is Infra g\(\beta\)-closed set in X.

Remark 3.21. The converse of the above theorem need not be true as seen from the following example.

Example 3.22. Let X = \{a, b, c\} with the topology \(\tau = \{X, \phi, \{a\}, \{b\}\}\). Let A = \{a, b\}. Here A is Infra g\(\beta\)-closed set but not a Infra g\(^\#\)\(\alpha\)-closed set of \((X, \tau_\text{IX})\).

Theorem 3.23. Every Infra g\(^\#\)\(\alpha\)-closed set is Infra gb-closed set.

Proof: Let A be a Infra g\(^\#\)\(\alpha\)-closed set in X. Let U be Infra open set, such that A \(\subseteq\) U. Since
every Infra open set is Infra $g^\#\alpha$-closed, we have, $icp_b(A) \subseteq icp_\alpha \subseteq U$. Then $icp_b(A) \subseteq U$. Hence $A$ is Infra $gb$-closed set in $X$.

**Remark 3.24.** The converse of the above theorem need not be true as seen from the following example.

**Example 3.25.** Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b\}, \{c\}, \{a, b\}\}$. Let $A = \{b, c\}$. Here $A$ is Infra $gb$-closed set but not a Infra $g^\#\alpha$-closed set of $(X, \tau_X)$.

**Theorem 3.26.** Every Infra $g^\#\alpha$-closed set is Infra $g^*b$-closed set.

**Proof:** Let $A$ be a Infra $g^\#\alpha$-closed set in $X$. Let $U$ be Infra open set, such that $A \subseteq U$. Since $A$ is Infra $g^\#\alpha$-closed, we have, $icp_b(A) \subseteq icp_\alpha \subseteq U$. Then $icp_b(A) \subseteq U$. Hence $A$ is Infra $g^*b$-closed set in $X$.

**Remark 3.27.** The converse of the above theorem need not be true as seen from the following example.

**Example 3.28.** Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{b\}, \{c\}, \{a, c\}\}$. Let $A = \{b\}$. Here $A$ is Infra $g^*b$-closed set but not a Infra $g^\#\alpha$-closed set of $(X, \tau_X)$.

**Theorem 3.29.** Every Infra $g^\#\alpha$-closed set is Infra gsp-closed set.

**Proof:** Let $A$ be a Infra $g^\#\alpha$-closed set in $X$. Let $U$ be Infra open set, such that $A \subseteq U$. Since every Infra open set is Infra $g$-open and $A$ is Infra $g^\#\alpha$-closed, we have, $\beta icp(A) \subseteq icp_\alpha(A) \subseteq U$. Then $\beta icp(A) \subseteq U$. Hence $A$ is Infra gsp-closed set in $X$.

**Remark 3.30.** The converse of the above theorem need not be true as seen from the following example.

**Example 3.31.** Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{b, c\}\}$. Let $A = \{a, b\}$. Here $A$ is Infra gsp-closed set but not a Infra $g^\#\alpha$-closed set of $(X, \tau_X)$.

**Theorem 3.32.** Let $A \subseteq X$. If $A$ is Infra $g^\#\alpha$-closed in $(X, \tau_X)$, then $icp_\alpha(A)$ - $A$ contains no non-empty Infra $g$-closed set.

**Proof:** Let $F$ be any Infra $g$-closed set such that $F \subseteq icp_\alpha(A) - A$. Then $A \subseteq X - F$ and $X - F$
is Infra g-open in \((X, \tau)\). Since \(A\) is Infra \(g^\#\alpha\)-closed in \(X\), \(\text{icp}_\alpha(A) \subseteq X - F\), therefore \(F \subseteq X - \text{icp}_\alpha(A)\). Thus \(F \subseteq (\text{icp}_\alpha(A) - A) \cap (X - \text{icp}_\alpha(A)) = \phi\).

**Theorem 3.33.** Let \(A\) be any Infra \(g^\#\alpha\)-closed set in \((X, \tau_X)\). If \(A \subseteq B \subseteq \text{icp}_\alpha(A)\), then \(B\) is also an Infra \(g^\#\alpha\)-closed set.

**Proof:** Let \(B \subseteq U\) where \(U\) is Infra \(g^\#\alpha\)-open \((X, \tau)\). Then \(A \subseteq U\). Also since \(A\) is Infra \(g^\#\alpha\)-closed, \(\text{icp}_\alpha(A) \subseteq U\). Since \(B \subseteq \text{icp}_\alpha(A)\), \(\text{icp}_\alpha(B) \subseteq \text{icp}_\alpha(A) \subseteq U\). This implies, \(\text{icp}_\alpha(B) \subseteq U\). Thus \(B\) is an Infra \(g^\#\alpha\)-closed set.

**Theorem 3.34.** If \(A\) and \(B\) are Infra \(g^\#\alpha\)-closed, then \(A \cap B\) is Infra \(g^\#\alpha\)-closed set.

**Proof:** Given that \(A\) and \(B\) are Infra \(g^\#\alpha\)-closed sets in \(X\). Let \(A \cap B \subseteq U\), \(U\) is Infra \(g^\#\alpha\)-open in \(X\). Since \(A\) is Infra \(g^\#\alpha\)-closed, \(\text{icp}_\alpha(A) \subseteq U\), whenever \(A \subseteq U\), \(U\) is Infra \(g^\#\alpha\)-open in \(X\). Since \(B\) is Infra \(g^\#\alpha\)-closed, \(\text{icp}_\alpha(B) \subseteq U\), whenever \(B \subseteq U\), \(U\) is Infra \(g^\#\alpha\)-open in \(X\). By the fact[9], \(\text{icp}_\alpha(A \cap B) = \text{icp}_\alpha(A) \cap \text{icp}_\alpha(B)\). It follows that \(\text{icp}_\alpha(A \cap B) \subseteq U\), whenever \(A \cap B \subseteq U\), \(U\) is Infra \(g^\#\alpha\)-open in \(X\). Hence \(A \cap B\) is Infra \(g^\#\alpha\)-closed.

**Example 3.35.** Let \(X = \{a, b, c, d\}\) with the topology \(\tau = \{X, \phi, \{b\}, \{a, b\}, \{b, d\}\}\). Let \(A = \{a, d\}, B = \{c, d\}\) are Infra \(g^\#\alpha\)-closed set. Then \(A \cap B = \{d\}\) is also an Infra \(g^\#\alpha\)-closed set.

**Theorem 3.36.** If \(A \subseteq Y \subseteq X\) and \(A\) is Infra \(g^\#\alpha\)-closed in \(X\) then \(A\) is Infra \(g^\#\alpha\)-closed relative to \(Y\).

**Proof:** Given that \(A \subseteq Y \subseteq X\) and \(A\) is a Infra \(g^\#\alpha\)-closed set in \(X\). We have to prove that \(A\) is Infra \(g^\#\alpha\)-closed set relative to \(Y\). Let us assume that \(A \subseteq Y \cap U\), where \(U\) is Infra \(g\)-open in \(X\). Since, \(A\) is Infra \(g^\#\alpha\)-closed set, \(A \subseteq U\), which implies \(\text{icp}_\alpha(A) \subseteq U\). From this, we get \(Y \cap \text{icp}_\alpha(A) \subseteq Y \cap U\). Hence, \(A\) is Infra \(g^\#\alpha\)-closed set relative to \(Y\).

### 4. Properties of Infra \(g^\#\alpha\)-Continuous Functions

In this section we set forth the concept of Infra \(g^\#\alpha\)-continuous function. The relationship between Infra \(g^\#\alpha\)-continuous function and other defined Infra continuous functions are explored.
Definition 4.1. Let $f: (X, \tau_{iX}) \to (Y, \tau_{iX})$ be a Infra topological space $X$ into a Infra topological space $Y$ is called $g^#\alpha$-continuous, if the inverse image of every Infra closed set in $Y$ is Infra $g^#\alpha$-closed set in $X$.

Theorem 4.2. If a map $f: (X, \tau_{iX}) \to (Y, \tau_{iX})$ from a Infra topological space $X$ into a Infra topological space $Y$ is Infra continuous, then it is Infra $g^#\alpha$-continuous.

Proof: Let $f: (X, \tau_{iX}) \to (Y, \tau_{iX})$ be Infra continuous. Let $F$ be any Infra closed set in $Y$. Then the inverse image $f^{-1}(F)$ is Infra closed in $X$. Since, every Infra closed set is Infra $g^#\alpha$-closed set, thus $f^{-1}(F)$ is Infra $g^#\alpha$-closed in $X$. Hence $f$ is Infra $g^#\alpha$-continuous.

Remark 4.3. The converse of the above theorem need not be true as seen from the following example.

Example 4.4. Let $X = Y = \{a, b, c, d\}$ with the Infra topologies $\tau = \{X, \emptyset, \{a\}, \{a,b\}, \{a,d\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b,d\}\}$, with the identity mapping. Then for the closed set $F = \{c\}$ in $Y$, $f^{-1}(\{c\}) = \{c\}$ implies $f$ is not Infra continuous, since $f^{-1}(\{c\})$ is not Infra closed in $X$.

Theorem 4.5. If a map $f: (X, \tau_{iX}) \to (Y, \tau_{iX})$ from a Infra topological space $X$ into a Infra topological space $Y$ is Infra continuous, then it is Infra $g$-continuous.

Proof: Let $f: (X, \tau_{iX}) \to (Y, \tau_{iX})$ be Infra continuous. Let $F$ be any Infra closed set in $Y$. Then the inverse image $f^{-1}(F)$ is Infra closed in $X$. Since, every Infra closed set is Infra $g$-closed set, thus $f^{-1}(F)$ is Infra $g$-closed in $X$. Hence $f$ is Infra $g$-continuous.

Remark 4.6. The converse of the above theorem need not be true as seen from the following example.

Example 4.7. Let $X = Y = \{a, b, c, d\}$ with the Infra topologies $\tau = \{X, \emptyset, \{a\}, \{a,b\}, \{a,d\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{d\}\}$, with the identity mapping. Then for the closed set $F = \{a,b,c\}$ in $Y$, $f^{-1}(\{a,b,c\}) = \{a,b,c\}$ implies $f$ is not Infra continuous, since $f^{-1}(\{a,b,c\})$ is not Infra closed in $X$.

Theorem 4.8. If a map $f: (X, \tau_{iX}) \to (Y, \tau_{iX})$ from a Infra topological space $X$ into a Infra topological space $Y$ is Infra $\alpha$-continuous, then it is Infra $g^#\alpha$-continuous.
Proof: Let \( f: (X, \tau_X) \to (Y, \tau_Y) \) be Infra \( \alpha \)-continuous. Let \( F \) be any Infra \( \alpha \)-closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is Infra \( \alpha \)-closed in \( X \). Since, every Infra \( \alpha \)-closed set is Infra \( \gamma \# \alpha \)-closed, thus \( f^{-1}(F) \) is Infra \( \gamma \# \alpha \)-closed in \( X \). Hence \( f \) is Infra \( \gamma \# \alpha \)-continuous.

Remark 4.9. The converse of the above theorem need not be true as seen from the following example.

Example 4.10. Let \( X = Y = \{a, b, c, d\} \) with the Infra topologies \( \tau = \{X, \phi, \{a\}, \{a,b\}, \{a,d\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{d\}\} \), with the mapping defined by \( f(a) = a, f(b) = b, f(c) = c, f(d) = d \).

For the closed set \( F = \{a,b,c\} \) in \( Y \), \( f^{-1}(\{a,b,c\}) = \{a,b,c\} \) implies \( f \) is not Infra \( \alpha \)-continuous, since \( f^{-1}(\{a,b,c\}) \) is not Infra \( \alpha \)-closed in \( X \).

Theorem 4.11. If a map \( f:(X, \tau_X) \to (Y, \tau_Y) \) from a Infra topological space \( X \) into a Infra topological space \( Y \) is Infra \( \gamma \# \alpha \)-continuous, then it is Infra \( \gamma \)-gs-continuous.

Proof: Let \( f: (X, \tau_X) \to (Y, \tau_Y) \) be Infra \( \gamma \# \alpha \)-continuous. Let \( F \) be any Infra \( \gamma \# \alpha \)-closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is Infra \( \gamma \# \alpha \)-closed in \( X \). Since, every Infra \( \gamma \# \alpha \)-closed set is Infra \( \gamma \)-gs-closed, thus \( f^{-1}(F) \) is Infra \( \gamma \)-gs-closed in \( X \). Hence \( f \) is Infra \( \gamma \)-gs-continuous.

Remark 4.12. The converse of the above theorem need not be true as seen from the following example.

Example 4.13. Let \( X = Y = \{a, b, c, d\} \) with the Infra topologies \( \tau = \{X, \phi, \{a\}, \{d\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,d\}\} \), with the identity mapping. For the closed set \( F = \{c,d\} \) in \( Y \), \( f^{-1}(\{a,b,c\}) = \{c,d\} \) implies \( f \) is not Infra \( \gamma \# \alpha \)-continuous, since \( f^{-1}(\{c,d\}) \) is not Infra \( \gamma \# \alpha \)-closed in \( X \).

Theorem 4.14. If a map \( f:(X, \tau_X) \to (Y, \tau_Y) \) from a Infra topological space \( X \) into a Infra topological space \( Y \) is Infra \( \gamma \# \alpha \)-continuous, then it is Infra \( \gamma \)-gs-continuous.

Proof: Let \( f: (X, \tau_X) \to (Y, \tau_Y) \) be Infra \( \gamma \# \alpha \)-continuous. Let \( F \) be any Infra \( \gamma \# \alpha \)-closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is Infra \( \gamma \# \alpha \)-closed in \( X \). Since, every Infra \( \gamma \# \alpha \)-closed set is Infra \( \gamma \)-gs-closed, thus \( f^{-1}(F) \) is Infra \( \gamma \)-gs-closed in \( X \). Hence \( f \) is Infra \( \gamma \)-gs-continuous.

Remark 4.15. The converse of the above theorem need not be true as seen from the following example.
Example 4.16. Let $X = Y = \{a, b, c\}$ with the Infra topologies $\tau = \{X, \emptyset, \{a\}, \{b\}, \{b,c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}\}$, with the mapping defined by $f(a) = a, f(b) = b, f(c) = c$. For the Infra closed set $F = \{a,b\}$ in $Y$, $f^{-1}(\{a,b\}) = \{a,b\}$ implies $f$ is not Infra $g^\#\alpha$-continuous, since $f^{-1}(\{a,b\})$ is not Infra $g^\#\alpha$-closed in $X$.

Theorem 4.17. If a map $f:(X, \tau_{\alpha}) \to (Y, \tau_{\alpha})$ from a Infra topological space $X$ into a Infra topological space $Y$ is Infra $g^\#\alpha$-continuous, then it is Infra $\alpha g$-continuous.

Proof: Let $f: (X, \tau_{\alpha}) \to (Y, \tau_{\alpha})$ be Infra $g^\#\alpha$-continuous. Let $F$ be any Infra $g^\#\alpha$-closed set in $Y$. Then the inverse image $f^{-1}(F)$ is Infra $g^\#\alpha$-closed in $X$. Since, every Infra $g^\#\alpha$-closed set is Infra $\alpha g$-closed, thus $f^{-1}(F)$ is Infra $\alpha g$-closed in $X$. Hence $f$ is Infra $\alpha g$-continuous.

Remark 4.18. The converse of the above theorem need not be true as seen from the following example.

Example 4.19. Let $X = Y = \{a, b, c\}$ with the Infra topologies $\tau = \{X, \emptyset, \{a\}, \{c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}\}$, with the mapping defined by $f(a) = a, f(b) = b, f(c) = c$. Then for the closed set $F = \{b,c\}$ in $Y$, $f^{-1}(\{b,c\}) = \{b,c\}$ implies $f$ is not Infra $g^\#\alpha$-continuous, since $f^{-1}(\{b,c\})$ is not Infra $g^\#\alpha$-closed in $X$.

Theorem 4.20. If a map $f:(X, \tau_{\alpha}) \to (Y, \tau_{\alpha})$ from a Infra topological space $X$ into a Infra topological space $Y$ is Infra $g^\#\alpha$-continuous, then it is Infra $g^\#\beta$-continuous.

Proof: Let $f: (X, \tau_{\alpha}) \to (Y, \tau_{\alpha})$ be Infra $g^\#\alpha$-continuous. Let $F$ be any Infra $g^\#\alpha$-closed set in $Y$. Then the inverse image $f^{-1}(F)$ is Infra $g^\#\alpha$-closed in $X$. Since, every Infra $g^\#\alpha$-closed set is Infra $g^\#\beta$-closed, thus $f^{-1}(F)$ is Infra $g^\#\beta$-closed in $X$. Hence $f$ is Infra $g^\#\beta$-continuous.

Remark 4.21. The converse of the above theorem need not be true as seen from the following example.

Example 4.22. Let $X = Y = \{a, b, c,d\}$ with the Infra topologies $\tau = \{X, \emptyset, \{b\}, \{a,b\}, \{b,d\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b\}, \{a,d\}\}$, with the mapping defined by $f(a) = a, f(b) = b, f(c) = c, f(d) = d$. Then for the closed set $F = \{b,c\}$ in $Y$, $f^{-1}(\{b,c\}) = \{b,c\}$ implies $f$ is not Infra $g^\#\alpha$-continuous, since $f^{-1}(\{b,c\})$ is not Infra $g^\#\alpha$-closed in $X$. 
Theorem 4.23. If a map \( f: (X, \tau_{iX}) \to (Y, \tau_{iX}) \) from a Infra topological space \( X \) into a Infra topological space \( Y \) is Infra \( g^\# \alpha \)-continuous, then it is Infra \( gb \)-continuous.

Proof: Let \( f: (X, \tau_{iX}) \to (Y, \tau_{iX}) \) be Infra \( g^\# \alpha \)-continuous. Let \( F \) be any Infra \( g^\# \alpha \)-closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is Infra \( g^\# \alpha \)-closed in \( X \). Since, every Infra \( g^\# \alpha \)-closed set is Infra \( gb \)-closed, thus \( f^{-1}(F) \) is Infra \( gb \)-closed in \( X \). Hence \( f \) is Infra \( gb \)-continuous.

Remark 4.24. The converse of the above theorem need not be true as seen from the following example.

Example 4.25. Let \( X = Y = \{a, b, c, d\} \) with the Infra topologies \( \tau = \{X, \phi, \{b\}, \{a,b\}, \{b,d\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,d\}\} \), with the mapping defined by \( f(a) = a, f(b) = b, f(c) = c, f(d) = d \). Then for the closed set \( F = \{b,c\} \) in \( Y \), \( f^{-1}(\{b,c\}) = \{b,c\} \) implies \( f \) is not Infra \( g^\# \alpha \)-continuous, since \( f^{-1}(\{b,c\}) \) is not Infra \( g^\# \alpha \)-closed in \( X \).

Theorem 4.26. If a map \( f: (X, \tau_{iX}) \to (Y, \tau_{iX}) \) from a Infra topological space \( X \) into a Infra topological space \( Y \) is Infra \( g^\# \alpha \)-continuous, then it is Infra \( g^*b \)-continuous.

Proof: Let \( f: (X, \tau_{iX}) \to (Y, \tau_{iX}) \) be Infra \( g^\# \alpha \)-continuous. Let \( F \) be any Infra \( g^\# \alpha \)-closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is Infra \( g^\# \alpha \)-closed in \( X \). Since, every Infra \( g^\# \alpha \)-closed set is Infra \( g^*b \)-closed, thus \( f^{-1}(F) \) is Infra \( g^*b \)-closed in \( X \). Hence \( f \) is Infra \( g^*b \)-continuous.

Remark 4.27. The converse of the above theorem need not be true as seen from the following example.

Example 4.28. Let \( X = Y = \{a, b, c, d\} \) with the Infra topologies \( \tau = \{X, \phi, \{b\}, \{a\}, \{d\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,d\}\} \), with the mapping defined by \( f(a) = a, f(b) = b, f(c) = c, f(d) = d \). Then for the closed set \( F = \{c,d\} \) in \( Y \), \( f^{-1}(\{c,d\}) = \{c,d\} \) implies \( f \) is not Infra \( g^\# \alpha \)-continuous, since \( f^{-1}(\{c,d\}) \) is not Infra \( g^\# \alpha \)-closed in \( X \).

Theorem 4.29. If a map \( f: (X, \tau_{iX}) \to (Y, \tau_{iX}) \) from a Infra topological space \( X \) into a Infra topological space \( Y \) is Infra \( g^\# \alpha \)-continuous, then it is Infra \( gsp \)-continuous.

Proof: Let \( f: (X, \tau_{iX}) \to (Y, \tau_{iX}) \) be Infra \( g^\# \alpha \)-continuous. Let \( F \) be any Infra \( g^\# \alpha \)-closed set in \( Y \). Then the inverse image \( f^{-1}(F) \) is Infra \( g^\# \alpha \)-closed in \( X \). Since, every Infra \( g^\# g\alpha b \)-closed set is Infra \( gsp \)-closed, thus \( f^{-1}(F) \) is Infra \( gsp \)-closed in \( X \). Hence \( f \) is Infra \( gsp \)-continuous.
Remark 4.30. The converse of the above theorem need not be true as seen from the following example.

Example 4.31. Let \( X = Y = \{a, b, c\} \) with the Infra topologies \( \tau = \{X, \emptyset, \{a\}, \{c\}\} \) and \( \sigma = \{Y, \emptyset, \{a\}, \{b\}\} \), with the mapping defined by \( f(a) = a, f(b) = b, f(c) = c \). Then for the closed set \( F = \{a, c\} \) in \( Y \), \( f^{-1}(\{a, c\}) = \{a, c\} \) implies \( f \) is not Infra \( g^\#\alpha \)-continuous, since \( f^{-1}(\{a, c\}) \) is not Infra \( g^\#\alpha \)-closed in \( X \).

Theorem 4.32. If a map \( f: (X, \tau_iX) \rightarrow (Y, \tau_iX) \) from a Infra topological space \( X \) into a Infra topological space \( Y \), then the following statements are equivalent.

(1) \( f \) is Infra \( g^\#\alpha \)-continuous.
(2) The inverse image of each Infra open set in \( Y \) is Infra \( g^\#\alpha \)-open in \( X \).

Proof: Assume that \( f: (X, \tau_iX) \rightarrow (Y, \tau_iX) \) be Infra \( g^\#\alpha \)-continuous. Let \( G \) be Infra open in \( Y \). Then \( G^c \) is Infra closed in \( Y \). Since \( f \) is Infra \( g^\#\alpha \)-continuous, \( f^{-1}(G^c) \) is Infra \( g^\#\alpha \)-closed in \( X \). But \( f^{-1}(G^c) = X - f^{-1}(G) \). Thus \( X - f^{-1}(G) \) is Infra \( g^\#\alpha \)-closed in \( X \) and so \( f^{-1}(G) \) is Infra \( g^\#\alpha \)-open in \( X \). Therefore (i) implies (ii).

Conversely assume that the inverse image of each Infra open set in \( Y \) is Infra \( g^\#\alpha \)-open in \( X \). Let \( F \) be any Infra closed set in \( Y \). The \( F^c \) is Infra open in \( Y \). By assumption, \( f^{-1}(F^c) \) is Infra \( g^\#\alpha \)-open in \( X \). But \( f^{-1}(F^c) = X - f^{-1}(F) \). Thus \( X - f^{-1}(F) \) is Infra \( g^\#\alpha \)-open in \( X \) and so \( f^{-1}(F) \) is Infra \( g^\#\alpha \)-closed in \( X \). Therefore \( f \) is Infra \( g^\#\alpha \)-continuous. Hence (ii) implies (i).

Thus (i) and (ii) are equivalent.

Theorem 4.33. If \( f: (X, \tau_iX) \rightarrow (Y, \tau_iX) \) and \( g: (Y, \tau_iX) \rightarrow (Z, \tau_iX) \) be any two functions, then \( g \circ f: (X, \tau_iX) \rightarrow (Z, \tau_iX) \) is Infra \( g^\#\alpha \)-continuous and \( f \) is Infra \( g^\#\alpha \)-continuous.

Proof: Let \( V \) be any Infra closed set in \( Z \). Since \( g \) is Infra continuous, \( g^{-1}(V) \) is Infra closed in \( Y \) and since \( f \) is Infra \( g^\#\alpha \)-continuous, \( f^{-1}(g^{-1}(V)) \) is Infra \( g^\#\alpha \)-closed in \( X \). Hence \( (g \circ f)^{-1}(V) \) is Infra \( g^\#\alpha \)-closed in \( X \). Thus \( g \circ f \) is Infra \( g^\#\alpha \)-continuous.

5. Properties of Infra \( g^\#\alpha \)-Irresolutive Maps

In this section we set forth the concept of \( g^\#\alpha \)-irresolutive function. The relationship between Infra \( g^\#\alpha \)-irresolutive function and other defined Infra irresolute functions are explored.
Definition 5.1. Let \( f: (X, \tau_X) \to (Y, \tau_Y) \) be a Infra topological space \( X \) into a Infra topological space \( Y \) is called Infra \( g^\# \alpha \)-irresolute, if the inverse image of every Infra \( g^\# \alpha \)-closed set in \( Y \) is Infra \( g^\# \alpha \)-closed set in \( X \).

Theorem 5.2. A map \( f: (X, \tau_X) \to (Y, \tau_Y) \) is Infra \( g^\# \alpha \)-irresolute if and only if the inverse image of every Infra \( g^\# \alpha \)-open set in \( Y \) is Infra \( g^\# \alpha \)-open in \( X \).

Proof: Assume that \( f \) is Infra \( g^\# \alpha \)-irresolute. Let \( A \) be any Infra \( g^\# \alpha \)-open set in \( Y \). Then \( A^c \) is Infra \( g^\# \alpha \)-closed set in \( Y \). Since \( f \) is Infra \( g^\# \alpha \)-irresolute, \( f^{-1}(A^c) \) is Infra \( g^\# \alpha \)-closed in \( X \). But \( f^{-1}(A^c) = X - f^{-1}(A) \) and so \( f^{-1}(A) \) is Infra \( g^\# \alpha \)-open in \( X \). Hence the inverse image of every Infra \( g^\# \alpha \)-open set in \( Y \) is Infra \( g^\# \alpha \)-open set in \( X \).

Conversely, assume that the inverse image of every Infra \( g^\# \alpha \)-open set in \( Y \) is Infra \( g^\# \alpha \)-open in \( X \). Let \( A \) be any Infra \( g^\# \alpha \)-closed set in \( Y \). Then \( A^c \) is Infra \( g^\# \alpha \)-open in \( Y \). By assumption, \( f^{-1}(A^c) \) is Infra \( g^\# \alpha \)-open in \( X \). But \( f^{-1}(A^c) = X - f^{-1}(A) \) and so \( f^{-1}(A) \) is Infra \( g^\# \alpha \)-closed in \( X \). Therefore \( f \) is Infra \( g^\# \alpha \)-irresolute.

Theorem 5.3. If a map \( f: (X, \tau_X) \to (Y, \tau_Y) \) is Infra \( g^\# \alpha \)-irresolute, then it is Infra \( g^\# \alpha \)-continuous.

Proof: Assume that \( f \) is Infra \( g^\# \alpha \)-irresolute. Let \( F \) be any Infra closed set in \( Y \). Since every Infra closed set is Infra \( g^\# \alpha \)-closed, \( F \) is Infra \( g^\# \alpha \)-closed in \( Y \). Since \( f \) is Infra \( g^\# \alpha \)-irresolute, \( f^{-1}(F) \) is Infra \( g^\# \alpha \)-closed in \( X \). Therefore \( f \) is Infra \( g^\# \alpha \)-continuous.

Remark 5.4. The converse of the above theorem need not be true as seen from the following example.

Example 5.5. Let \( X = Y = \{a, b, c, d\} \) with the Infra topologies \( \tau = \{X, \phi, \{a\}, \{a,b,d\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a,b\}, \{a,d\}\} \), with the identity mapping. Here \( f \) is Infra \( g^\# \alpha \)-continuous. But \( f \) is not Infra \( g^\# \alpha \)-irresolute, since for the closed set \( F = \{a, b\} \) in \( Y \) implies, \( f^{-1}(\{a,b\}) = \{a,b\} \) is not Infra \( g^\# \alpha \)-closed in \( X \).

Theorem 5.6. Let \( X, Y \) and \( Z \) be any Infra topological spaces. For any Infra \( g^\# \alpha \)-irresolute map \( f: (X, \tau_X) \to (Y, \tau_Y) \) and any Infra \( g^\# \alpha \)-continuous map \( g: (Y, \tau_Y) \to (Z, \tau_Z) \) the composition \( gof: (X, \tau_X) \to (Z, \tau_Z) \) is Infra \( g^\# \alpha \)-continuous.
Proof: Let $F$ be any Infra closed set in $Z$. Since $g$ is Infra $g^\#\alpha$-continuous, $g^{-1}(F)$ is Infra $g^\#\alpha$-closed in $Y$. Since $f$ is Infra $g^\#\alpha$-irresolute, $f^{-1}(g^{-1}(F))$ is Infra $g^\#\alpha$-closed in $X$. But $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$. Therefore $gof: (X,\tau_X) \to (Z,\tau_Z)$ is Infra $g^\#\alpha$-continuous.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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