NUMERICAL SOLUTION OF CAUCHY SINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND WITH INDEX V = 1

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Abstract: A method based on Gauss-Chebyshev quadrature and barycentric interpolation is used to obtain the numerical solution of Cauchy singular integral equations of the first kind with index equal to 1 at non-Chebyshev nodes. The unknown function in the equation is first expressed as a product of an appropriate weight function and a truncated weighted series of Chebyshev polynomial of the first kind. Some properties of Chebyshev polynomials are then used to reduce the equation to a system of linear equations. On solving the linear system, the numerical solution of the Cauchy singular integral equation is obtained at Chebyshev nodes, after which barycentric interpolation is used to obtain the numerical solution at non-Chebyshev nodes. When the numerical solution obtained is compared with the analytical solution and the absolute error computed, the results are found to be satisfactory.

Keywords: singular integral equation; Chebyshev polynomial; Cauchy; numerical solution.

Subject Classification codes: 45L05, 65R20

1. INTRODUCTION

Solving Cauchy singular integral equations of the first kind with index V =1 using regularization methods is cumbersome and laborious from the standpoint of numerical analysis [9]. Hence, the need for direct computational methods. One of these computational methods uses Gauss-

Chebyshev quadrature, which only gives numerical solutions at Chebyshev nodes. However, the numerical solution at non-Chebyshev nodes may be required in some real-world problems, hence, the need for this study.

The essence of this work is to obtain the numerical solution of Cauchy singular integral equations of the first kind with index V=1 at non-Chebyshev nodes using Gauss-Chebyshev quadrature as a method and barycentric interpolation as a tool.

2. PRELIMINARIES

Many problems in science and engineering can be modeled using Cauchy singular integral equations [9], [10]. For instance, [1] modeled some problems in fracture mechanics using Cauchy singular integral equations, [2] gave an example of Cauchy singular integral equations in neutron transport, [3] listed electrodynamics among the fields where Cauchy singular integral equations are used to model problems, [4] demonstrated the usefulness of Cauchy singular integral equations in hydrodynamics. Hence, the growing need for efficient methods of solving them. Solving these equations both numerically and analytically is difficult due to the presence of strong singularity or Cauchy singularity when $p = s$.

[5] noted that numerical methods for singular integral equations must consider the correct nature of the singularity, else, they will either breakdown or converge slowly. Hence, the need to incorporate the correct singular behavior into the numerical technique for obtaining the solution of the singular integral equation. Many methods for solving Cauchy singular integral equations exist. They include: Muskhelishvili–Vekua regularization method, Carleman-Vekua regularization method, collocation method, Galerkin method, Gauss-Jacobi quadrature method, Gauss-Chebyshev quadrature method, Gauss-Lobatto quadrature method, Lobatto-Chebyshev quadrature method, piecewise polynomial collocation method, among others. Most of these methods are based on the assumption that the solution is either bounded or it has an integrable singularity at the end points.

If the solution has integrable singularities at both endpoints, then the Cauchy singular integral equation has its index as 1. Again, if the solution is bounded at both endpoints, then its index is equal to -1. The equation has its index as 0 if the solution is bounded at one endpoint and has an integrable singularity at the other endpoint.

[6] proposed a method that involves the application of another singular integral of the same form on the singular integral. This procedure converts the Cauchy singular integral equation into a weakly singular Fredholm integral equation of the second kind. There exists a host of numerical algorithms for solving singular Fredholm integral equations with weak singularity and any of these algorithms can be used to solve the regularized equation. In terms of computational time, this procedure is computationally laborious because the original problem has to be converted to a weakly singular Fredholm integral equation before an appropriate numerical technique is used to obtain an approximate solution. Also, in practice, the evaluation of the Fredholm kernel in the regularized equation is often difficult, even for problems in which the Fredholm kernel of the original equation is known in closed form.

Gauss-Chebyshev quadrature formulas and barycentric interpolating polynomials are useful tools in this research work. Gauss-Chebyshev quadrature formula is used to transform the Cauchy-singular integral equation to a linear system. This linear system is then solved numerically using an appropriate algorithm. On solving the linear system, we obtain the numerical solution at Chebyshev nodes of the first kind. Subsequently, barycentric interpolation is used to obtain the numerical solution at non-Chebyshev nodes.

3. FORMULATION OF NUMERICAL SCHEME

Here, we seek to develope a numerical method based on Gauss-Chebyshev quadrature and barycentric interpolation for obtaining the numerical solution of Cauchy singular integral equations of the first kind having an index v = 1 at non-Chebyshev nodes.

Proposition: Let $T_n(s)$ and $U_n(s)$ be Chebyshev polynomials defined as $T_n(s) = \cos n\theta$ and $U_n(s) = \frac{Sin(n+1)\theta}{Sin\theta}$ with $\cos\theta = s$. If $T_n(p_k) = 0$ and $U_{n-1}(s) = 0$, then

$$\begin{cases} \sum_{k=1}^{n} \frac{T_j(p_k)}{n(p_k - s_r)} = 0, \ j = 0 \\ \sum_{k=1}^{n} \frac{T_j(p_k)}{n(p_k - s_r)} = U_{j-1}(s_r), \ 0 < j < n \end{cases} \tag{1}$$

Gauss-Chebyshev Quadrature Formula for singular integrals with Cauchy kernel: Consider the following integral

$$Q(s) = \frac{1}{\pi} \int_{-1}^{1} \frac{g(p)}{p-s} dp \ , \quad -1 < s < 1 \tag{2}$$

Let g(p) = w(p)Ø(p) \tag{3}

Where Ø(p) is bounded in -1 ≤ p ≤ 1.

When the index is 1,

$$w(p) = (1 - p^2)^{-1/2} \tag{4}$$

Which is the weight of Chebyshev polynomial of the first kind $T_m(p)$.

Suppose that in $-1 \leq p \leq 1$, it is possible to approximate $\emptyset(p)$ by the series

$$\emptyset(p) = \sum_{m=0}^{z} B_m T_m(p) \,.$$

(2) becomes

$$Q(s) = \sum_{m=0}^{z} B_m \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s)(1-p^2)^{1/2}} \, dp$$

$$= B_o \frac{1}{\pi} \int_{-1}^{1} \frac{T_o(p)}{(p-s)(1-p^2)^{1/2}} \, dp + \sum_{m=1}^{z} B_m \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s)(1-p^2)^{1/2}} \, dp$$

$$= \sum_{m=1}^{z} B_m U_{m-1}(s), \tag{5}$$

where $U_{m-1}(s)$ is a Chebyshev polynomial of the second kind and according to [7],

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s)(1-p^2)^{1/2}} \, dp = 0, \, for \, m = 0 \quad and$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s)(1-p^2)^{1/2}} \, dp = U_{m-1}(s), \, for \, m > 0$$

$-1 < s < 1$

If we set $s = s_r$ in (2), we obtain

$$Q(s_r) = \frac{1}{\pi} \int_{-1}^{1} \frac{g(p)}{p-s_r} \, dp \tag{6}$$

Hence, we have

$$Q(s_r) = \sum_{m=0}^{z} B_m \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s_r)(1-p^2)^{1/2}} \, dp$$

$$= B_o \frac{1}{\pi} \int_{-1}^{1} \frac{T_o(p)}{(p-s_r)(1-p^2)^{\frac{1}{2}}} \, dp + \sum_{m=1}^{z} B_m \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s_r)(1-p^2)^{\frac{1}{2}}} \, dp$$

$$\Rightarrow Q(s_r) = \sum_{m=1}^{z} B_m \, U_{m-1}(s_r) \tag{7}$$

$$But \, U_{m-1}(s_r) = \sum_{k=1}^{n} \frac{T_m(p_k)}{n(p_k - s_r)} \,, \qquad 0 < m < n$$

Thus (7) becomes

$$Q(s_r) = \sum_{m=1}^{z} \sum_{k=1}^{n} \frac{B_m T_m(p_k)}{n(p_k - s_r)}$$

$$Q(s_r) = \sum_{k=1}^{n} \frac{\emptyset(p_k)}{n(p_k - s_r)} , \qquad (8)$$

where

$$T_n(p_k) = 0, p_k = cos\left(\frac{2k-1}{2n}\pi\right), K = 1, 2, ..., n$$

$$U_{n-1}(s_r) = 0 , s_r = cos\left(\frac{\pi r}{n}\right), r = 1, 2, ..., n - 1$$

Note: p_k's are Chebyshev nodes of the first kind.

If we compare (8) with the Gauss-Chebyshev quadrature formula

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\emptyset(p)}{(1 - p^2)^{\frac{1}{2}}} dp = \sum_{m=1}^{z} \frac{\emptyset(p_k)}{n} , \qquad T_n(p_k) = 0$$

It can be seen that (8) is a Gauss-Chebyshev quadrature formula for the Cauchy singular integral
(2) valid only at the points s = s_r (r = 1, 2... , n-1) and $U_{n-1}(s_r) = 0$.

**Numerical Solution of Cauchy Singular Integral Equation of the First Kind with Index V =
1 at Chebyshev nodes:** Consider the equation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{g(p)}{p-s} dp + \int_{-1}^{1} k(s,p)g(p)dp = y(s) , -1 < s < 1 \qquad (9)$$

Let g(p) = w(p)ϕ(p)

If the index of (9) is 1,

w(p) = $(1 - p^2)^{-1/2}$

Hence,

g(p) = $(1 - p^2)^{-1/2}\phi$(p) $\qquad (10)$

Suppose we express ϕ(p) as a series having the form

$$\phi(p) = \sum_{m=o}^{z} B_m T_m(p) \qquad (11)$$

Substituting (10) into (9), yields

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(p)}{(p-s)(1-p^2)^{1/2}} dp + \int_{-1}^{1} \frac{k(s,p)\phi(p)}{(1-p^2)^{1/2}} dp = y(s) \qquad (12)$$

Also,

When the index is 1, (9) is subject to the compatibility condition

$$\int_{-1}^{1} g(p)dp = C \qquad (13)$$

$$\Rightarrow \int_{-1}^{1} \frac{\phi(p)}{(1-p^2)^{1/2}} dp = C , \tag{14}$$

where c is a constant.

Substituting (11) in (12), we get

$$\sum_{m=o}^{z} B_m \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s)(1-p^2)^{1/2}} dp + \sum_{m=o}^{z} B_m \int_{-1}^{1} \frac{k(s,p)T_m(p)}{(1-p^2)^{1/2}} dp = y(s)$$

$$\Rightarrow B_0 \frac{1}{\pi} \int_{-1}^{1} \frac{T_0(p)}{(p-s)(1-p^2)^{1/2}} dp + \sum_{m=1}^{z} B_m \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(p)}{(p-s)(1-p^2)^{1/2}} dp$$

$$+ \sum_{m=0}^{z} \int_{-1}^{1} \frac{B_m T_m(p) k(s,p)}{(1-p^2)^{1/2}} dp = y(s)$$

$$\Rightarrow \sum_{m=1}^{z} B_m U_{m-1}(s) + \frac{1}{\pi} \int_{-1}^{1} \pi k(s,p) \frac{\phi(p)}{(1-p^2)^{1/2}} dp = y(s) \tag{15}$$

If we set $s = s_r$ in (15), we get

$$\sum_{m=1}^{z} B_m U_{m-1}(s_r) + \frac{1}{\pi} \int_{-1}^{1} \pi k(s_r, p) \frac{\phi(p)}{(1-p^2)^{1/2}} dp = y(s_r) \tag{16}$$

$$But \ \ U_{m-1}(s_r) = \sum_{k=1}^{n} \frac{T_m(p_k)}{n(p_k - s_r)}, 0 < m < n \ \ and$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(p)}{(1-p^2)^{1/2}} dp = \sum_{k=1}^{n} \frac{\phi(p_k)}{n}$$

thus (16) becomes

$$\sum_{m=1}^{z} \sum_{k=1}^{n} \frac{B_m T_m(p_k)}{n(p_k - s_r)} + \pi k(s_r, p_k) \sum_{k=1}^{n} \frac{\phi(p_k)}{n} = y(s_r)$$

$$\Rightarrow \sum_{k=1}^{n} \frac{\phi(p_k)}{n(p_k - s_r)} + \pi k(s_r, p_k) \sum_{k=1}^{n} \frac{\phi(p_k)}{n} = y(s_r)$$

hence,

$$\sum_{k=1}^{n} \frac{1}{n} \left[\frac{1}{p_k - s_r} + \pi k(s_r, p_k) \right] \phi(p_k) = y(s_r) \tag{17}$$

also,

(14) becomes

$$\sum_{k=1}^{n} \frac{\pi}{n} \phi(p_k) = C \tag{18}$$

(17) gives n-1 linear equations in n unknowns and (18) gives 1 equation in n unknowns. However, when they are combined, we obtain a system of n linear equations in n unknowns. Hence, we get,

$$\sum_{k=1}^{n} \frac{1}{n} \left[\frac{1}{p_k - s_r} + \pi k(s_r, p_k) \right] \phi(p_k) = y(s_r)$$

$$\sum_{k=1}^{n} \frac{\pi}{n} \phi(p_k) = C, \tag{19}$$

where

$$T_n(p_k) = 0, p_k = \cos\left(\frac{2k-1}{2n}\pi\right), k = 1, 2, \dots, n$$

$$U_{n-1}(s_r) = 0, s_r = \cos\frac{\pi r}{n}, r = 1, 2, \dots, n-1$$

that is,

$$\frac{1}{n}\left[\frac{1}{p_1 - s_r} + \pi k(s_r, p_1)\right]\phi(p_1) + \frac{1}{n}\left[\frac{1}{p_2 - s_r} + \pi k(s_r, p_2)\right]\phi(p_2) + \cdots$$

$$+\frac{1}{n}\left[\frac{1}{p_{n-1} - s_r} + \pi k(s_r, p_{n-1})\right]\phi(p_{n-1}) + \frac{1}{n}\left[\frac{1}{p_n - s_r} + \pi k(s_r, p_n)\right]\phi(p_n) = y(s_r),$$

$$\frac{\pi}{n}\phi(p_1) + \frac{\pi}{n}\phi(p_2) + \cdots + \frac{\pi}{n}\phi(p_{n-1}) + \frac{\pi}{n}\phi(p_n) = C \tag{20}$$

r = 1, 2, …, n-1.

From (20), we get

$$F\vec{\Phi} = \vec{Y},$$

where $\vec{\Phi}$ and $\vec{Y}$ are the column vectors and F is the coefficient matrix with entries

$$f_{rk} = \frac{1}{n}\left[\frac{1}{p_k - s_r} + \pi k(s_r, p_k)\right] \tag{21}$$

with $p_k = \cos\left(\frac{2k-1}{2n}\pi\right), s_r = \cos\frac{\pi r}{n}$

when k = 1, 2, …, n; r = 1, 2, …, n-1

and

$$f_{rk} = \frac{\pi}{n} \tag{22}$$

when k = 1, 2, 3, …, n; r = n

Thus, we can write (20) as

$$
\begin{bmatrix}
f_{11} & f_{12}\ldots & f_{1n} \\
f_{21} & f_{22}\ldots & f_{2n} \\
\vdots & \vdots & \vdots \\
f_{n\text{-}1\,1} & f_{n\text{-}1\,2}\ldots & f_{n\text{-}1\,n} \\
f_{n1} & f_{n2}\ \ldots & f_{nn}
\end{bmatrix}
\begin{bmatrix}
\phi(p_1) \\
\phi(p_2) \\
\vdots \\
\phi(p_{n-1}) \\
\phi(p_n)
\end{bmatrix}
=
\begin{bmatrix}
y(s_1) \\
y(s_2) \\
\vdots \\
y(s_{n-1}) \\
y(s_n)
\end{bmatrix}
\tag{23}
$$

where $y(s_n) = C$

Thus, the problem has been reduced to a problem of solving a system of n linear equations in n unknowns $\phi(p_1)$, $\phi(p_2)$, …, $\phi(p_n)$.

On solving system (23), we obtain the numerical solution of (12) at the Chebyshev nodes of the first kind p_1, p_2, …, p_n.

Numerical Solution of Cauchy Singular Integral equations of the first kind with Index V = 1 at non-Chebyshev nodes: Suppose the numerical solution of (12) obtained at Chebyshev nodes on solving system (23) are shown in Table 2,

p	$\phi(p)$
p_1	$\phi(p_1)$
p_2	$\phi(p_2)$
p_3	$\phi(p_3)$
$\vdots$	$\vdots$
p_4	$\phi(p_n)$

TABLE 2: Numerical solution at Chebyshev nodes

where p_1, p_2,.., p_n are Chebyshev nodes of the first kind and $\phi(p_1)$, $\phi(p_2)$,.., $\phi(p_n)$ are the numerical solutions at p_1, p_2,..,p_n respectively. The numerical solution of (12) at non-Chebyshev nodes may be needed in some practical problems. One way to meet this need is to construct an interpolating polynomial that fits the data given in Table 2.

Suppose we consider the true form of the barycentric interpolating formula defined as

$$
\varphi(p) = \frac{\displaystyle\sum_{k=0}^{n} \frac{\lambda_k}{p - p_k} \varphi(p_k)}{\displaystyle\sum_{k=0}^{n} \frac{\lambda_k}{p - p_k}},
\tag{24}
$$

where

$$\lambda_k = \frac{1}{\prod_{k \neq j} (p_k - p_j)} \tag{25}$$

and

$$\frac{\lambda_0}{p - p_0} = \lim_{\rho \to \infty} \frac{\lambda_0}{p - p_0} = 0 \tag{26}$$

Using (24), (25) and (26), we obtain the barycentric polynomial interpolant that fits the data given in the Table 2 above. This polynomial can be used to obtain the numerical solution of (12) at non-Chebyshev points.

In solid mechanics, for instance, the numerical solution of (12) at the endpoint +1 is an important quantity called the stress intensity factor of a material at $+1$. To obtain this quantity at +1, we apply (24) and (25) using only the first three nodes in Table 2 above.

3. IMPLEMENTATION OF THE SCHEME

For the sake of completeness, a Gauss-Chebyshev quadrature formula for singular integrals being developed is used to obtain the numerical solution of a Cauchy singular integral equation of the first kind with index equal to 1 at Chebyshev nodes, just the same way it was done by Erdogan and Gupta (1986). After this, the true form of the barycentric interpolation formula is used to obtain the numerical solution of the Cauchy singular integral equation at non-Chebyshev nodes using the numerical solution obtained at the Chebyshev nodes.

Test Problem 1

Consider the following integral equation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\emptyset(p)}{\sqrt{1 - p^2}(p - s)} dp = 4s^2 - 1 \quad , \quad -1 < s < 1$$

subject to the compatibility condition

$$\int_{-1}^{1} \frac{\emptyset(p)}{\sqrt{1-p^2}} dp = 0,$$

where the exact solution is $\emptyset(p) = 4p^3 - 3p$.

(Parihar and Ramachandran, 1999).

When n = 4, we have that

$$p_k = \cos\left(\frac{2k - 1}{8}\pi\right), k = 1, 2, 3, 4$$

$p_1 = 0.92$, $p_2 = 0.38$, $p_3 = -0.38$, $p_4 = -0.92$

also,

$$s_r = \cos\left(\frac{\pi r}{4}\right), r = 1, 2, 3,$$

$s_1 = 0.71$, $s_2 = 0.00$, $s_3 = -0.71$

from (23), we obtain the following matrix

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix} \begin{bmatrix} \emptyset(p_1) \\ \emptyset(p_2) \\ \emptyset(p_3) \\ \emptyset(p_4) \end{bmatrix} = \begin{bmatrix} y(s_1) \\ y(s_2) \\ y(s_3) \\ y(s_4) \end{bmatrix} \tag{27}$$

hence,

using the formulae $f_{rk} = \frac{1}{n}\left[\frac{1}{p_k - s_r} + \pi k(s_r, p_k)\right]$ with $k(s_r, p_k) = 0$, k = 1,2,3,4; r = 1,2,3 and

$f_{rk} = \frac{\pi}{n}$ with k = 1,2,3,4; r = 4, we obtain that $f_{41} = f_{42} = f_{43} = f_{44} = \frac{\pi}{4}$.

Also,

$y(s_r) = 4s_r^2 - 1, \qquad r = 1, 2, 3$

$y(s_1) = 4(0.71)^2 - 1 = 1.0164$

$y(s_2) = 4(0.00)^2 - 1 = 1.0000$

$y(s_3) = 4(-0.71)^2 - 1 = 1.0164$

and

$y(s_4) = 0.0000$

Hence (27) becomes

$$\begin{bmatrix} 1.1905 & -0.7576 & -0.2294 & -0.1534 \\ 0.2717 & 0.6579 & -0.6579 & -0.2717 \\ 0.1534 & 0.2294 & 0.7576 & -1.1905 \\ \frac{\pi}{4} & \frac{\pi}{4} & \frac{\pi}{4} & \frac{\pi}{4} \end{bmatrix} \begin{bmatrix} \emptyset(p_1) \\ \emptyset(p_2) \\ \emptyset(p_3) \\ \emptyset(p_4) \end{bmatrix} = \begin{bmatrix} 1.0164 \\ -1.0000 \\ 1.0164 \\ 0.0000 \end{bmatrix}$$

The above system gives the following results when solved

$\emptyset(p_1) = \emptyset(0.92) = 0.3937$, $\emptyset(p_2) = \emptyset(0.38) = -0.9226$, $\emptyset(p_3) = \emptyset(-0.38) = 0.9226$, $\emptyset(p_4) = \emptyset(-0.92)$

$= -0.3937$.

These results are displayed in Table 3

p	$\emptyset(p)$
0.92	0.3937
0.38	-0.9226
-0.38	0.9226
-0.92	-0.3937

TABLE 3: Numerical solution of test problem 1 at Chebyshev nodes when n = 4

Using (24), (25), (26), the numerical solution of the given problem at non- Chebyshev nodes can be obtained by proceeding thus.

From (24),

when n = 4,

$$\emptyset(p) = \frac{\frac{\lambda_0}{p - p_0}\emptyset(p_0) + \frac{\lambda_1}{p - p_1}\emptyset(p_1) + \frac{\lambda_2}{p - p_2}\emptyset(p_2) + \frac{\lambda_3}{p - p_3}\emptyset(p_3) + \frac{\lambda_4}{p - p_4}\emptyset(p_4)}{\frac{\lambda_0}{p - p_0} + \frac{\lambda_1}{p - p_1} + + \frac{\lambda_2}{p - p_2} + \frac{\lambda_3}{p - p_3} + \frac{\lambda_4}{p - p_4}}$$

Using (26), we obtain

$$\emptyset(p) = \frac{\frac{\lambda_1}{p-0.92}\emptyset(0.92) + \frac{\lambda_2}{p-0.38}\emptyset(0.38) + \frac{\lambda_3}{p+0.38}\emptyset(-0.38) + \frac{\lambda_4}{p+0.92}\emptyset(-0.92)}{\frac{\lambda_1}{p-0.92} + \frac{\lambda_2}{p-0.38} + \frac{\lambda_3}{p+0.38} + \frac{\lambda_4}{p+0.92}}.$$

Again, using (25) yields

$\lambda_1 = 0.7742, \lambda_2 = -1.8743, \lambda_3 = 1.8743, \lambda_4 = -0.7742.$

Hence,

$$\emptyset(p) = \frac{\frac{0.7742}{p-0.92}(0.3937) + \frac{1.8743}{p-0.38}(0.9226) + \frac{1.8743}{p+0.38}(0.9226) + \frac{0.7742}{p+0.92}(0.3937)}{\frac{0.7742}{p-0.92} - \frac{1.8743}{p-0.38} + \frac{1.8743}{p+0.38} - \frac{0.7742}{p+0.92}}. \tag{28}$$

Using (28), we obtain

$\emptyset(0.92) = 0.0000, \emptyset(0.25) = -0.6903, \emptyset(0.5) = -0.9991, \emptyset(0.75) = -0.5452.$

To obtain $\emptyset(1)$, we use only the first three nodes in Table 3

that is,

$$\emptyset(p) = \frac{\frac{\lambda_0}{p-p_0}\emptyset(p_0) + \frac{\lambda_1}{p-p_1}\emptyset(p_1) + \frac{\lambda_2}{p-p_2}\emptyset(p_2) + \frac{\lambda_3}{p-p_3}\emptyset(p_3)}{\frac{\lambda_0}{p-p_0} + \frac{\lambda_1}{p-p_1} + + \frac{\lambda_2}{p-p_2} + \frac{\lambda_3}{p-p_3}},$$

where

$\lambda_1 = 1.4245, \lambda_2 = -2.4366, \lambda_3 = 1.0121.$

Hence,

$$\emptyset(p) = \frac{\frac{1.4245}{p-0.92}(0.3937)+\frac{2.4366}{p-0.38}(-0.9226)+\frac{1.0121}{p+0.38}(0.9226)}{\frac{1.4245}{p-0.92}-\frac{2.4366}{p-0.38}+\frac{1.0121}{p+0.38}} \qquad (29)$$

using (29), we obtain $\emptyset(1) = 0.7743$.

When n = 6, we have that

$$p_k = \cos\left(\frac{2k-1}{12}\pi\right), k = 1, 2, 3, \dots, 6$$

$p_1 = 0.97$, $p_2 = 0.71$, $p_3 = 0.26$, $p_4 = -0.26$, $p_5 = -0.71$, $p_6 = -0.97$.

Also,

$$s_r = \cos\left(\frac{\pi r}{6}\right), r = 1, 2, 3, \dots, 5$$

$s_1 = 0.87$, $s_2 = 0.50$, $s_3 = 0.00$, $s_4 = -0.50$, $s_5 = -0.87$.

From (23), we obtain the following matrix

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix} \begin{bmatrix} \emptyset(p_1) \\ \emptyset(p_2) \\ \emptyset(p_3) \\ \emptyset(p_4) \\ \emptyset(p_5) \\ \emptyset(p_6) \end{bmatrix} = \begin{bmatrix} y(s_1) \\ y(s_2) \\ y(s_3) \\ y(s_4) \\ y(s_5) \\ y(s_6) \end{bmatrix} . \qquad (30)$$

Using the formulae $f_{rk} = \frac{1}{n}\left[\frac{1}{p_k-s_r} + \pi k(s_r, p_k)\right]$ with $k(s_r, p_k) = 0$, k = 1,2,3,…,6; r = 1,2,3,…,5

and

$$f_{rk} = \frac{\pi}{n} \quad \text{with k = 1,2,3,…,6; r = 6,}$$

we obtain that

$$f_{61} = f_{62} = f_{63} = f_{64} = f_{65} = f_{66} = \frac{\pi}{6}$$

Also,

$$y(s_r) = 4S_r^2 - 1, r = 1, 2, 3, 4, 5$$
$$y(s_1) = 4(0.87)^2 - 1 = 2.0276$$
$$y(s_2) = 4(0.50)^2 - 1 = 0.0000$$
$$y(s_3) = 4(0.00)^2 - 1 = -1.0000$$
$$y(s_4) = 4(-0.50)^2 - 1 = 0.0000$$
$$y(s_5) = 4(-0.87)^2 - 1 = 2.0276$$

and

$$y(s_6) = C = 0.0000$$

Hence, (30) becomes

$$\begin{bmatrix} 1.6667 & -1.0417 & -0.2732 & -0.1475 & -0.1055 & -0.0906 \\ 0.3546 & 0.7937 & -0.6944 & -0.2193 & -0.1377 & -0.1134 \\ 0.1718 & 0.2347 & 0.6410 & -0.6410 & -0.2347 & -0.1718 \\ 0.1134 & 0.1377 & 0.2193 & 0.6944 & -0.7937 & -0.3546 \\ 0.0906 & 0.1055 & 0.1475 & 0.2732 & 1.0417 & -1.6667 \\ \dfrac{\pi}{6} & \dfrac{\pi}{6} & \dfrac{\pi}{6} & \dfrac{\pi}{6} & \dfrac{\pi}{6} & \dfrac{\pi}{6} \end{bmatrix} \begin{bmatrix} \emptyset(p_1) \\ \emptyset(p_2) \\ \emptyset(p_3) \\ \emptyset(p_4) \\ \emptyset(p_5) \\ \emptyset(p_6) \end{bmatrix} = \begin{bmatrix} 2.0276 \\ 0.0000 \\ -1.0000 \\ 0.0000 \\ 2.0276 \\ 0.0000 \end{bmatrix}$$

Solving the last system yields

$\emptyset(p_1) = \emptyset(0.97) = 0.7184$

$\emptyset(p_2) = \emptyset(0.71) = -0.7223$

$\emptyset(p_3) = \emptyset(0.26) = -0.7081$

$\emptyset(p_4) = \emptyset(-0.26) = 0.7081$

$\emptyset(p_5) = \emptyset(-0.71) = 0.7223$

$\emptyset(p_6) = \emptyset(-0.97) = -0.7184$

These results are displayed in the Table 4

p	$\emptyset(p)$
0.97	0.7184
0.71	-0.7223
0.26	-0.7081
-0.26	0.7081
-0.71	0.7223
0.97	- 0.7184

TABLE 4: Numerical Solution of test problem 1 at Chebyshev nodes when n = 6

Using (24), (25), (26), the numerical solution of the given problem at non-Chebyshev nodes can be obtained by proceeding thus

from (24), we obtain that

$$\emptyset(p) = \frac{\frac{\lambda_0}{p-p_0}\emptyset(p_0)+\frac{\lambda_1}{p-p_1}\emptyset(p_1)+\frac{\lambda_2}{p-p_2}\emptyset(p_2)+\frac{\lambda_3}{p-p_3}\emptyset(p_3)+\frac{\lambda_4}{p-p_4}\emptyset(p_4)+\frac{\lambda_5}{p-p_5}\emptyset(p_5)+\frac{\lambda_6}{p-p_6}\emptyset(p_6)}{\frac{\lambda_0}{p-p_0}+\frac{\lambda_1}{p-p_1}++\frac{\lambda_2}{p-p_2}+\frac{\lambda_3}{p-p_3}+\frac{\lambda_4}{p-p_4}+\frac{\lambda_5}{p-p_5}+\frac{\lambda_6}{p-p_6}} \tag{31}$$

It follows that

$\lambda_1 = 1.3513, \lambda_2 = -3.6936, \lambda_3 = 5.0449, \lambda_4 = -5.0449, \lambda_5 = 3.6936, \lambda_6 = -1.3513.$

and

$\emptyset(0) = 0.0000$, $\emptyset(0.25) = -0.6858$, $\emptyset(0.5) = -1.0083$, $\emptyset(0.75) = -0.5889$.

To obtain $\emptyset(1)$, only the first three nodes in Table 4 will be used.

That is,

$$\emptyset(p) = \frac{\frac{\lambda_0}{p - p_0}\emptyset(p_0) + \frac{\lambda_1}{p - p_1}\emptyset(p_1) + \frac{\lambda_2}{p - p_2}\emptyset(p_2) + \frac{\lambda_3}{p - p_3}\emptyset(p_3)}{\frac{\lambda_0}{p - p_0} + \frac{\lambda_1}{p - p_1} + + \frac{\lambda_2}{p - p_2} + \frac{\lambda_3}{p - p_3}}$$

where,

$\lambda_1 = 5.4171$, $\lambda_2 = -8.5470$, $\lambda_3 = 3.1299$.

Hence,

$$\emptyset(p) = \frac{\frac{5.4171}{p-0.97}(0.7184) - \frac{8.5470}{p-0.71}(-0.7223) + \frac{3.1299}{p-0.26}(-0.7081)}{\frac{5.4171}{p-0.97} - \frac{8.5470}{p-0.71} + \frac{3.1299}{p-0.26}} \tag{32}$$

and using (32), we obtain $\emptyset(1) = 0.9529$.

When n = 8

$$p_k = \cos\left(\frac{2k - 1}{16}\pi\right), k = 1, 2, \dots, 8$$

$p_1 = 0.98$, $p_2 = 0.83$, $p_3 = 0.56$, $p_4 = 0.20$, $p_5 = -0.20$, $p_6 = -0.56$, $p_7 = -0.83$, $p_8 = -0.98$

also,

$$s_r = \cos\left(\frac{\pi r}{8}\right), r = 1, 2, \dots, 7$$

$s_1 = 0.92$, $s_2 = 0.71$, $s_3 = 0.38$, $s_4 = 0.00$, $s_5 = -0.38$, $s_6 = -0.71$, $s_7 = -0.92$

From (23), we obtain the following matrix

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \\ f_{71} & f_{72} & f_{73} & f_{74} & f_{75} & f_{76} \\ f_{81} & f_{82} & f_{83} & f_{84} & f_{85} & f_{86} \end{bmatrix} \begin{bmatrix} \emptyset(p_1) \\ \emptyset(p_2) \\ \emptyset(p_3) \\ \emptyset(p_4) \\ \emptyset(p_5) \\ \emptyset(p_6) \\ \emptyset(p_7) \\ \emptyset(p_8) \end{bmatrix} = \begin{bmatrix} y(s_1) \\ y(s_2) \\ y(s_3) \\ y(s_4) \\ y(s_5) \\ y(s_6) \\ y(s_7) \\ y(s_8) \end{bmatrix} \tag{33}$$

Using the formulae $f_{rk} = \frac{1}{n}\left[\frac{1}{p_k - s_r} + \pi k(s_r, p_k)\right]$ with $k(s_r, p_k) = 0$, k = 1,2,3,…,8; r = 1,2,3,…,7

and

$f_{rk} = \frac{\pi}{n}$ with k = 1,2,3,…,8; r = 8, we obtain

$f_{81} = f_{82} = f_{83} = f_{84} = f_{85} = f_{86} = f_{87} = f_{88} = \frac{\pi}{8}$.

Also,

$y(s_r)=4s_r^2-1$, r=1, 2, 3, …, 7 yields

$y(s_1)=4(0.92)^2-1=2.3856$, $y(s_2)=4(0.71)^2-1=1.0164$,

$y(s_3)=4(0.38)^2-1=-0.4224$, $y(s_4)=4(0.00)^2-1=-1.0000$,

$y(s_5)=4(-0.38)^2-1=-0.4224$, $y(s_6)=4(-0.71)^2-1=1.0164$, $y(s_7)=4(-0.92)^2-1=2.3856$.

and

$y(s_8) = c = 0.0000$

Hence, (33) becomes

$$
\begin{bmatrix}
2.0834 & -1.3889 & -0.3472 & -0.1736 & -0.1116 & -0.0845 & -0.0714 & -0.0658 \\
0.4630 & 1.0417 & -0.8333 & -0.2451 & -0.1374 & -0.0984 & -0.0812 & -0.0740 \\
0.2083 & 0.2778 & 0.6944 & -0.6944 & -0.2155 & -0.1330 & -0.1033 & -0.0919 \\
0.1276 & 0.1506 & 0.2232 & 0.6250 & -0.6250 & -0.2232 & -0.1506 & -0.1276 \\
0.0919 & 0.1033 & 0.1330 & 0.2155 & 0.6944 & -0.6944 & -0.2778 & -0.2083 \\
0.0740 & 0.0812 & 0.0984 & 0.1374 & 0.2451 & 0.8333 & -1.0417 & -0.4630 \\
0.0658 & 0.0714 & 0.0845 & 0.1116 & 0.1736 & 0.3472 & 1.3889 & -2.0833 \\
\frac{\pi}{8} & \frac{\pi}{8} & \frac{\pi}{8} & \frac{\pi}{8} & \frac{\pi}{8} & \frac{\pi}{8} & \frac{\pi}{8} & \frac{\pi}{8}
\end{bmatrix}
\begin{bmatrix}
\emptyset(p_1) \\ \emptyset(p_2) \\ \emptyset(p_3) \\ \emptyset(p_4) \\ \emptyset(p_5) \\ \emptyset(p_6) \\ \emptyset(p_7) \\ \emptyset(p_8)
\end{bmatrix}
=
\begin{bmatrix}
2.3856 \\ 1.0164 \\ -0.4224 \\ -1.0000 \\ -0.4224 \\ 1.0164 \\ 2.3856 \\ 0.0000
\end{bmatrix}
$$

Solving the system yields

$\emptyset(p_1) = \emptyset(0.98) = 0.8334$, $\emptyset(p_2)= \emptyset(0.83) = -0.2200$, $\emptyset(p_3)= \emptyset(0.56) = -1.0296$,

$\emptyset(p_4)= \emptyset(0.20) = -0.5494$

$\emptyset(p_5)= \emptyset(-0.20) = 0.5494$, $\emptyset(p_6)= \emptyset(-0.56) = 1.0296$, $\emptyset(p_7)= \emptyset(-0.83) = 0.2200$,

$\emptyset(p_8)= \emptyset(-0.98) = -0.8334$

These results are displayed in Table 5

p	$\emptyset(p)$
0.98	0.8334
0.83	-0.2200
0.56	-1.0296
0.20	-0.5494
-0.20	0.5494
-0.56	1.0296
-0.83	0.2200
-0.98	-0.8334

TABLE 5: Numerical solution of test problem 1 at Chebyshev nodes when n = 8

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Using (24), (25), (26), the numerical solution at non-Chebyshev nodes can be obtained.

From (24), we have that

$$\emptyset(p) = \frac{\frac{\lambda_0}{p-p_0}\emptyset(p_0) + \frac{\lambda_1}{p-p_1}\emptyset(p_1) + \frac{\lambda_2}{p-p_2}\emptyset(p_2) + \frac{\lambda_3}{p-p_3}\emptyset(p_3) + \frac{\lambda_4}{p-p_4}\emptyset(p_4) + \frac{\lambda_5}{p-p_5}\emptyset(p_5) + \frac{\lambda_6}{p-p_6}\emptyset(p_6) + \frac{\lambda_7}{p-p_7}\emptyset(p_7) + \frac{\lambda_8}{p-p_8}\emptyset(p_8)}{\frac{\lambda_0}{p-p_0} + \frac{\lambda_1}{p-p_1} + + \frac{\lambda_2}{p-p_2} + \frac{\lambda_3}{p-p_3} + \frac{\lambda_4}{p-p_4} + \frac{\lambda_5}{p-p_5} + \frac{\lambda_6}{p-p_6} + \frac{\lambda_7}{p-p_7} + \frac{\lambda_8}{p-p_8}}$$

$\emptyset(P)$ can be obtained from (26).

Using (25), we get

$\lambda_1 = 3.1567$, $\lambda_2 = -9.1110$, $\lambda_3 = 13.4437$, $\lambda_4 = -15.2992$, $\lambda_5 = 15.2992$, $\lambda_6 = -13.4437, \lambda_7 = 9.1110, \quad \lambda_8 = -3.1567$

hence,

$$\emptyset(P) = \frac{\frac{3.1567}{p-0.98}(0.8334) + \frac{9.1110}{p-0.83}(0.2200) - \frac{13.4437}{p-0.56}(1.0296) + \frac{15.2992}{p-0.20}(0.5494) + \frac{15.2992}{p+0.20}(0.5494) + \frac{13.4437}{p+0.56}(1.0296) + \frac{9.1110}{p+0.83}(0.2200) + \frac{3.1567}{p+0.98}(0.8334)}{\frac{3.1567}{p-0.98} - \frac{9.1110}{p-0.83} + \frac{13.4437}{p-0.56} - \frac{15.2992}{p-0.20} + \frac{15.2992}{p+0.20} - \frac{13.4437}{p+0.56} + \frac{9.1110}{p+0.83} - \frac{3.1567}{p+0.98}} \tag{34}$$

Using (34), we obtain

$\emptyset(0) = 0.0000, \emptyset(0.25) = -0.6714, \emptyset(0.5) = -0.9895, \emptyset(0.75) = -0.6082.$

To obtain $\emptyset(1)$, we use only the first three nodes in Table 5.

That is,

$$\emptyset(p) = \frac{\frac{\lambda_0}{p-p_0}\emptyset(p_0) + \frac{\lambda_1}{p-p_1}\emptyset(p_1) + \frac{\lambda_2}{p-p_2}\emptyset(p_2) + \frac{\lambda_3}{p-p_3}\emptyset(p_3)}{\frac{\lambda_0}{p-p_0} + \frac{\lambda_1}{p-p_1} + + \frac{\lambda_2}{p-p_2} + \frac{\lambda_3}{p-p_3}}$$

where,

$\lambda_1 = 15.8730, \lambda_2 = -24.6914, \lambda_3 = 8.8183.$

Hence,

$$\emptyset(p) = \frac{\frac{15.8730}{p-0.98}(0.8334) + \frac{24.6914}{p-0.83}(0.2200) + \frac{8.8183}{p-0.56}(1.0296)}{\frac{15.8730}{p-0.98} - \frac{24.6914}{p-0.83} + \frac{8.8183}{p-0.56}} \tag{35}$$

Using (35), we obtain

$\emptyset(1) = 1.0064.$

p	Ø(p)		
	n = 4	n = 6	n = 8
0.00	0.0000	0.0000	0.0000
0.25	-0.6903	-0.6858	-0.6714
0.50	-0.9991	-1.0083	-0.9895
0.75	-0.5452	-0.5889	-0.6082
1.00	0.7743	0.9529	1.0064

TABLE 6: Numerical Solution of test problem 1 at some non-Chebyshev nodes

p	Ø(p)		
	Exact Solution $4p^3 - 3p$	Numerical Solution	Absolute Error
0.00	0.0000	0.0000	0.0000
0.25	-0.6875	-0.6714	1.61×10^{-2}
0.50	-1.0000	-0.9895	1.05×10^{-2}
0.75	-0.5625	-0.6082	4.57×10^{-2}
1.00	1.0000	1.0064	6.4×10^{-3}

TABLE 7: Numerical results at some non-Chebyshev nodes when n = 8

Gerasoulis and Srivastav (1994) solved this problem using Piecewise linear functions. The numerical solution they obtained when n = 9 is in Table 8 and a comparison between their solution and the exact solution can also be found in Table 8.

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p	$\emptyset(p)$		
	Exact Solution $4P^3 - 3P$	**Numerical Solution**	**Absolute Error**
0.00	0.0000	0.0000	
0.25	-0.6875	-0.7172	2.97 x 10^{-2}
0.50	-1.0000	-1.0588	5.88 x 10^{-2}
0.75	-0.5625	-0.6504	8.79 x 10^{-2}
1.00	1.0000	0.9147	8.53 x 10^{-2}

TABLE 8: numerical results of Gerasoulis and Srivastav when n = 9

The piecewise linear method was proposed by Gerasoulis and Srivastav (1994) for the purpose of improving on the accuracy of the numerical solution obtained using the Gauss-Chebyshev method. However, a closer look at Table 7 and Table 8 reveals that when barycentric interpolation is implemented on the numerical solution obtained using the Gauss-Chebyshev quadrature method for this problem, a better numerical solution is obtained at non-Chebyshev nodes.

Test Problem 2

Consider the following integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\emptyset(p)dp}{(p-s)\sqrt{1-p^2}} - \frac{1}{3}\int_{-1}^{1}\frac{\emptyset(p)dp}{\sqrt{1-p^2}} = 1, \quad -1 < S < 1$$

Subject to the compatibility condition

$$\int_{-1}^{1}\frac{\emptyset(p)}{\sqrt{1-p^2}}dp = 0$$

The exact solution is $\emptyset(p) = p$

Source: Gori, 1995

Using (3.21), (22), (23), (24), (25) and (26), the numerical results in Table 9 are obtained when n = 6

p	Ø(p)		
	Exact Solution p	Numerical Solution	Absolute Error
0.00	0.0000	0.0000	
0.25	0.2500	0.2472	2.8×10^{-3}
0.50	0.5000	0.5011	1.1×10^{-3}
0.75	0.7500	0.7583	8.3×10^{-3}
1.00	1.0000	0.9982	1.8×10^{-3}

Table 9: Numerical results for test problem 2 when n = 6

4. CONCLUSION

In this research work, a numerical technique based on Gauss-Chebyshev Quadrature and Barycentric interpolation for obtaining the numerical solution of Cauchy Singular integral equations of the first kind with index v = 1 at non-Chebyshev nodes has been developed. We sought an approximate solution to

$$\frac{1}{\pi}\int_{-1}^{1} \frac{g(p)}{p-s}\, dp + \int_{-1}^{1} K(s,p)g(p)dp = y(s), \qquad -1 < s < 1$$

and obtained the numerical solution of it with index v = 1 at non-Chebyshev nodes using Gauss-Chebyshev Quadrature as a method and at the non-Chebyshev nodes using Gauss-Chebyshev Quadrature method and Barycentric Interpolation tool. The results show that the numerical solution obtained by the Gauss-Chebyshev Quadrature-Barycentric Interpolation method were satisfactory when compared with the exact solution.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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