COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS IN CONNECTION WITH \( (p,q) \) CHEBYSHEV POLYNOMIAL

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Abstract. In this present work, authors are introduced a new subclass of bivalent functions \( \mathcal{S}_\Sigma (\alpha, x, p, q) \) with respect to symmetric conjugate points in the open unit disc \( U \) related to \( (p,q) \) polynomials. Further the initial bounds of the subclass and the well known Fekete-Szegő inequality are determined.

Keywords: \( (p, q) \)-Chebyshev polynomials; bi-univalent functions; subordination.

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1. INTRODUCTION

Let \( \mathbb{R}=(-\infty, \infty) \) be the set of real numbers, \( \mathbb{C} \) be the set of complex numbers and

\[ N := 1, 2, 3... = \mathbb{N}_0 \setminus \{0\} \]

be the set of positive integers.

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Let \( \mathcal{A} \) denote the family of normalized analytic functions \( f \) of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U})
\]

in the open disc \( \mathbb{U} = \{ z : z \in \mathbb{C} : |z| < 1 \} \). Further, let \( \mathcal{S} \) denote the class of functions in \( \mathcal{A} \) which are also univalent in \( \mathbb{U} \).

The well-known Koebe one-quarter theorem [2] ensures that the image of \( \mathbb{U} \) under every univalent function \( f \in \mathcal{A} \) contains a disc of radius 1/4. Hence every univalent function \( f \) has an inverse \( f^{-1} \) satisfying

\[
f^{-1}(f(z)) = z, \quad (z \in \mathbb{U})
\]

and

\[
f^{-1}(f(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq 1/4),
\]

where

\[
g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1.1). For example, functions in the class \( \Sigma \) are given below [8]:

\[
\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log \left( \frac{1+z}{1-z} \right).
\]

In 1967, Lewin [5] introduced the class \( \Sigma \) of bi-univalent functions and shown that \( |a_2| < 1.51 \). In 1969, Netanyahu [7] showed that \( \max_{f \in \Sigma} |a_2| = 4/3 \) and Suffridge [9] have given an example of \( f \in \Sigma \) for which \( |a_2| = 4/3 \). Later, in 1980, Brannan and Clunie [1] improved the result as \( |a_2| \leq \sqrt{2} \). In 1985, Kedzier-awski [3] proved this conjecture for a special case when the function \( f \) and \( f^{-1} \) are starlike. In 1984, Tan [10] proved that \( |a_2| \leq 1.485 \) which is the best estimate for the function in the class of bi-univalent functions.

For any integer \( n \geq 2 \) and \( 0 < q < p \leq 1 \), the \( (p,q) \)-Chebyshev polynomials of the second kind is defined by the following recurrence relations:

\[
U_n(x,s,p,q) = (p^n + q^n)xU_{n-1}(x,s,p,q) + (pq)^{n-1}sU_{n-2}(x,s,p,q)
\]

with the initial values \( U_0(x,s,p,q) = 1, U_1(x,s,p,q) = (p+q)x \) and ‘s’ is a variable. By Assuming various values of \( x,s,p \) and \( q \) we get some interesting polynomials as follows:
• When \( x = \frac{x}{2} \), \( s = s \), \( p = p \) and \( q = q \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes \((p, q)\)-Fibonacci polynomials.

• When \( x = x \), \( s = -1 \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Second kind of Chebyshev polynomials.

• When \( x = \frac{x}{2} \), \( s = 1 \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Fibonacci polynomials.

• When \( x = \frac{1}{2} \), \( s = 1 \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Fibonacci numbers.

• When \( x = x \), \( s = 1 \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Pell polynomials.

• When \( x = 1 \), \( s = 11 \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Pell numbers.

• When \( x = \frac{1}{2} \), \( s = 2y \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Jacobsthal polynomials.

• When \( x = \frac{1}{2} \), \( s = 2 \), \( p = 1 \) and \( q = 1 \), the \((p, q)\)-Chebyshev polynomials of the second kind becomes Jacobsthal numbers.

Recently Kızılateş et al. [4] defined \((p, q)\)-Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

The generating function of the \((p, q)\)-Chebyshev polynomials of the second kind is as follows:

\[
G_{p,q}(z) = \frac{1}{1 - xpz \tau_p - xqz \tau_q - spqz^2 \tau_{p,q}}
\]

\[
= \sum_{n=0}^{\infty} U_n(x, s, p, q)z^n \quad (z \in \mathbb{U})
\]

where the Fibonacci operator \( \tau_q \) was introduced by Mason [6], \( \tau_q f(z) = f(qz) \). Similarly, \( \tau_{p,q} f(z) = f(pqz) \).
Definition 1. For $0 < \alpha \leq 1$, a function $s \in \sigma$ is belong to the class $\mathcal{S}(\alpha, x, p, q)$ if it satisfies the following conditions

$$
\begin{align*}
\left\{ \frac{2zs'(z)}{s(z) - s(-\bar{z})} + \frac{2(zs'(z))'}{(s(z) - s(-\bar{z}))'} - \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z) - s(-\bar{z}))'} \right\} < G_{p,q}(z)
\end{align*}
$$

(1.3)

and

$$
\begin{align*}
\left\{ \frac{2wr'(w)}{r(w) - r(-\bar{w})} + \frac{2(wr'(w))'}{(r(w) - r(-\bar{w}))'} - \frac{2\alpha w^2 r''(w) + 2wr'(w)}{\alpha w(r(w) - r(-\bar{w}))'} \right\} < G_{p,q}(w)
\end{align*}
$$

(1.4)

where $r = s^{-1}$.

By setting $\alpha = 0$, $\mathcal{S}(\alpha, x, p, q) = \mathcal{S}(0, x, p, q)$ which holds the following conditions

$$
\frac{2(zs'(z))'}{(s(z) - s(-\bar{z}))'} < G_{p,q}(z) \quad \text{and} \quad \frac{2(wr'(w))'}{(r(w) - r(-\bar{w}))'} < G_{p,q}(w),
$$

where $r$ is the extension of $f^{-1}$.

2. Estimation of Initial Coefficients & Fekete-Szegö Inequality

Theorem 1. A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{S}(\alpha, x, p, q)$, then

$$
|a_2| \leq \frac{u_1(x,s,p,q)}{2} \left[ \frac{\sqrt{u_1(x,s,p,q)(m_2 + n_2)}}{\sqrt{(3 - 2\alpha)u_1^2(x,s,p,q) - 2(2 - \alpha)^2}} \right]
$$

(2.1)

and

$$
|a_3| \leq \frac{u_1(x,s,p,q)}{4} \left[ \frac{(m_2 - n_2)}{(3 - 2\alpha) - \frac{u_1(x,s,p,q)(m_1^2 + n_1^2)}{2(2 - \alpha)^2}} \right].
$$

(2.2)

Proof. Suppose that $f \in \mathcal{S}(\alpha, x, p, q)$, then from (1.3) and (1.4)

$$
\begin{align*}
\left\{ \frac{2zs'(z)}{s(z) - s(-\bar{z})} + \frac{2(zs'(z))'}{(s(z) - s(-\bar{z}))'} - \frac{2\alpha z^2 s''(z) + 2zs'(z)}{\alpha z(s(z) - s(-\bar{z}))'} \right\} = G_{p,q}(\phi(z))
\end{align*}
$$

(2.3)
and for its inverse map \( g = f^{-1} \), we have

\[
\begin{aligned}
\frac{2wr'(w)}{r(w) - r(-\overline{w})} + \frac{2(wr'(w))'}{r(w) - r(-\overline{w})}'
- \frac{2\alpha w^2r''(w) + 2wr'(w)}{\alpha w(r(w) - r(-\overline{w}))' + (1 - \alpha)(r(w) - r(-\overline{w}))}
\end{aligned}
\]

(2.4)

For some analytic functions \( \phi \) and \( \varphi \) such that \( \phi(0) = \varphi(0) = 0 \) and \( |\phi(z)| = |\varphi(w)| < 1 \) for all \( z, w \in \mathbb{U} \). It is well known that if

\[
|\phi(z)| = |m_1z + m_2z^2 + m_3z^3 + ...| < 1
\]

and

\[
|\varphi(w)| = |n_1w + n_2w^2 + n_3w^3 + ...| < 1
\]

where \( z, w \in \mathbb{U} \), then \( |m_k| = |n_k| < 1 \quad (\forall k \in \mathbb{N}) \).

From (2.3) and (2.4),

\[
\frac{2zs'(z)}{s(z) - s(-\overline{z})} + \frac{2(zs'(z))'}{(s(z) - s(-\overline{z}))}' - \frac{2\alpha z^2s''(z) + 2zs'(z)}{\alpha z(s(z) - s(-\overline{z}))' + (1 - \alpha)(s(z) - s(-\overline{z}))}
\]

\[
= U_0(x, s, p, q) + U_1(x, s, p, q)\phi(z) + U_2(x, s, p, q)\phi^2(z) + \cdots
\]

and

\[
\frac{2wr'(w)}{r(w) - r(-\overline{w})} + \frac{2(wr'(w))'}{(r(w) - r(-\overline{w}))}' - \frac{2\alpha w^2r''(w) + 2wr'(w)}{\alpha w(r(w) - r(-\overline{w}))' + (1 - \alpha)(r(w) - r(-\overline{w}))}
\]

\[
= U_0(x, s, p, q) + U_1(x, s, p, q)\varphi(w) + U_2(x, s, p, q)\varphi^2(w) + \cdots
\]

Thus, we write

\[
\begin{aligned}
\frac{2zs'(z)}{s(z) - s(-\overline{z})} + \frac{2(zs'(z))'}{(s(z) - s(-\overline{z}))}' - \frac{2\alpha z^2s''(z) + 2zs'(z)}{\alpha z(s(z) - s(-\overline{z}))' + (1 - \alpha)(s(z) - s(-\overline{z}))}
\end{aligned}
\]

(2.5)

\[
= 1 + U_0(x, s, p, q) + m_1(z) + [U_1(x, s, p, q)m_2 + U_2(x, s, p, q)m_1^2]z^2 + ...
\]

and

\[
\frac{2wr'(w)}{r(w) - r(-\overline{w})} + \frac{2(wr'(w))'}{(r(w) - r(-\overline{w}))}' - \frac{2\alpha w^2r''(w) + 2wr'(w)}{\alpha w(r(w) - r(-\overline{w}))' + (1 - \alpha)(r(w) - r(-\overline{w}))}
\]

(2.6)

\[
= 1 + U_0(x, s, p, q) + n_1(w) + [U_1(x, s, p, q)n_2 + U_2(x, s, p, q)n_1^2]w^2 + \cdots
\]
By equating the coefficients from (2.5) and (2.6)

(2.7) \[ 2(2 - \alpha)a_2 = u_1(x, s, p, q)m_1 \]

(2.8) \[ 2(3 - 2\alpha)a_3 = u_1(x, s, p, q)m_2 + u_2(x, s, p, q)m_1^2 \]

(2.9) \[ -2(2 - \alpha)a_2 = u_1(x, s, p, q)n_1 \]

(2.10) \[ 2(3 - 2\alpha)(2a_2^2 - a_3) = u_1(x, s, p, q)n_1^2. \]

From (2.7) and (2.9)

(2.11) \[ m_1 = -n_1 \]

and

(2.12) \[ 8(2 - \alpha)^2a_2^2 = u_1^2(x, s, p, q)(m_1^2 + n_1^2). \]

By using (2.8) and (2.10) we obtain,

(2.13) \[ 4(3 - 2\alpha)a_2^2 = u_1(x, s, p, q)(m_2 + n_2) + u_2(x, s, p, q)(m_1^2 + n_1^2). \]

By using (2.12) in (2.13) we get,

(2.14) \[ \left[ 4(3 - 2\alpha) - \frac{8(2 - \alpha)^2u_2(x, s, p, q)}{u_1^2(x, s, p, q)} \right] a_2^2 = u_1(x, s, p, q)(m_2 + n_2). \]

From (2.13) we acquired the result which is desired in (2.1).

By subtracting (2.10) from (2.8)

\[ -4(3 - 2\alpha)(a_2^2 - a_3) = u_1(x, s, p, q)(m_2 - n_2) + u_2(x, s, p, q)(m_1^2 - n_1^2). \]

Using (2.11) and (2.12),

\[ 4(3 - 2\alpha)u_1^2(x, s, p, q)(m_1^2 + n_1^2) \frac{8(2 - \alpha)^2}{8(2 - \alpha)^2} + 4(3 - 2\alpha)a_3 = u_1(x, s, p, q)(m_2 - n_2) \]
\[(2.15) \quad a_3 = \frac{u_1(x,s,p,q)(m_2 - n_2)}{4(3 - 2\alpha)} + \frac{u_1^2(x,s,p,q)(m_1^2 + n_1^2)}{8(2 - \alpha)^2}.\]

By using (2.11), we obtain the desired result in (2.2).

\[\Box\]

**Theorem 2.** A function \(f \in \Sigma\) of the form (1.1) is said to be in the class \(\mathcal{S}_\Sigma(\alpha, x, p, q)\), then

\[|a_3 - \mu a_2^2| \leq \begin{cases} |u_1(x,s,p,q)|, & \phi \leq \frac{1}{4(3 - 2\alpha)}, \\ 2|u_1(x,s,p,q)||p|, & \phi \geq \frac{1}{4(3 - 2\alpha)}. \end{cases}\]

**Proof.** From (2.14) and (2.15),

\[a_3 - \mu a_2^2 = \frac{[u_1(x,s,p,q)]^3(m_2 + n_2)(1 - \mu)}{4(3 - 2\alpha)u_1^2(x,s,p,q) - 8(2 - \alpha)^2u_2(x,s,p,q)} + \frac{u_1(x,s,p,q)(m_2 - n_2)}{4(3 - 2\alpha)} \]

\[= u_1(x,s,p,q) \left[ m_2 + \left( \phi + \frac{1}{4(3 - 2\alpha)} \right) + n_2 \left( \phi - \frac{1}{4(3 - 2\alpha)} \right) \right] \]

where

\[\phi = \frac{u_1^2(x,s,p,q)(1 - \mu)}{4(3 - 2\alpha)u_1^2(x,s,p,q) - 8(2 - \alpha)^2u_2(x,s,p,q)}.\]

\[\Box\]

**Corollary 1.** When \(\alpha = 0\),

\[|a_3 - \mu a_2^2| \leq \begin{cases} |u_1(x,s,p,q)|, & \phi \leq \frac{1}{12}, \\ 2|u_1(x,s,p,q)||p|, & \phi \geq \frac{1}{12}. \end{cases}\]

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**


COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS


