**\( \Lambda_{rs} \)- OPEN SETS AND \( \Lambda_{rs} \)- CLOSED SETS IN TOPOLOGICAL SPACES**

G. AMUTHA*, A. P. DHANA BALAN

Department of Mathematics, Alagappa Govt Arts College, Karaikudi, Tamil Nadu, India

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Abstract: In [16] Maki has introduced the concept of \( \Lambda \)-sets in topological spaces as the sets that coincide with their kernel. The kernel of a set \( A \) is the intersection of all open supersets of \( A \). In this paper we obtain new classes of sets by using regular semi open sets in topological spaces and study their basic properties, and their connections with other kind of topological sets.

Keywords: \( \Lambda_{rs} \)-set; \( V_{rs} \)-set; \( \Lambda_{rs} \)-open set; \( \Lambda_{rs} \)-closed set.

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1. INTRODUCTION

In [16] Maki has introduced the concept of \( \Lambda \)-sets in topological spaces as the sets that coincide with their kernel. The kernel of a set \( A \) is the intersection of all open supersets of \( A \). Caldas and Dontchev [4] built on Maki’s work by introducing \( \Lambda_r \)-sets, \( V_r \)-sets, \( g \Lambda_r \)-sets and \( gV_r \)-sets using semi open sets and semi closed sets. Ganster et al [10] introduced the notion of pre- \( \Lambda \)-sets and pre- \( V \)-sets using pre open sets and pre closed sets. Also M. J. Jeyanthi [13] introduced the concepts of \( \Lambda_r \)-sets and \( V_r \)-sets using regular open sets and regular closed sets.

*Corresponding author

E-mail address: amuthamani8525@gmail.com

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Georgiou [12] defined and investigated $\Lambda_{\delta}$-sets and $(\Lambda, \delta)$ closed sets via $\delta$-open sets and $\delta$-closed sets. Also Caldas [5] introduced $\Lambda_{\alpha}$-sets and $(\Lambda, \alpha)$ closed sets via $\alpha$-open sets and $\alpha$-closed sets. Lellis Thivagar [14] introduced the notions of $\Lambda_{a}$-sets and $\Lambda_{a}$-closed sets via $a$-open set and $a$-closed set.

In this paper we introduce the $\Lambda_{rs}$-closed set and $\Lambda_{rs}$-open set. To define these sets, we are using the set $\Lambda_{rs}$-set. These sets are lies between closed sets and semi closed sets and discussed some relationship between other generalized sets.

Throughout this paper, $(X, \tau)$ (or simply $X$) will always represent a topological space on which no separation axioms are assumed, unless otherwise mentioned. When $A$ is a subset of $X$, $\text{cl}(A)$ and $\text{Int}(A)$ denote the closure and interior of a set $A$, respectively. A subset $A$ of a topological space $X$ is said to be semi-regular [8] if it is both semi-open and semi-closed. In [8], it is pointed out that a set is semi-regular if and only if there exists a regular open set $U$ such that $U \subseteq A \subseteq \text{cl}(U)$. Cameron [6] called semi regular sets regular semi-open.

2. PRELIMINARIES

**Definition 2.1** A subset $A$ of a topological space is called:

1) Semi-open [15] if $A \subseteq \text{cl} (\text{int}(A))$

2) Pre-open [17] if $A \subseteq \text{int} (\text{cl}(A))$

3) $b$-open [2] if $A \subseteq \text{int} (\text{cl}(A)) \cup \text{cl} (\text{int}(A))$

4) Regular open [20] if $A = \text{int} (\text{cl}(A))$

The class of all semi-open (resp. pre-open, $b$-open and regular open) denoted by $\text{SO}(X, \tau)$ (resp. $\text{PO}(X, \tau)$, $\text{BO}(X, \tau)$ and $\text{RO}(X, \tau)$)

The complement of these sets called semi-closed (resp. pre-closed, $b$-closed and regular closed) and the classes of all these sets will be denoted by $\text{SC}(X, \tau)$ (resp. $\text{PC}(X, \tau)$, $\text{BC}(X, \tau)$ and $\text{RC}(X, \tau)$)

**Definition 2.2** A subset $A$ of a topological space is called

1) $\Lambda_{r}$-closed [13] if $A = T \cap C$, where $T$ is $\Lambda_{r}$-set and $C$ is closed set.

2) $(\Lambda, b)$-closed [7] if $A = T \cap C$, where $T$ is $\Lambda_{b}$-set and $C$ is $b$-closed set.
Definition 2.3 A topological space \((X, \tau)\) is said to be locally indiscrete [9] if every open set in it is closed.

Definition 2.4 A subset \(A\) of a space \((X, \tau)\) is called:

1. \(\Lambda\)-set (resp. \(V\)-set) [16] if it is the intersection (resp. union) of open (resp. closed) sets.

2. \(\Lambda_s\)-sets (resp. \(V_s\)-sets) [4] if it is the intersection (resp. union) of semi-open (resp. semi-closed) sets.

3. pre-\(\Lambda\)-sets (resp. pre-\(V\)-set) [10] if it is the intersection (resp. union) of pre-open (resp. pre-closed) sets.

Theorem 2.5 [11] every regular semi open set in \(X\) is semi open but not conversely.

Theorem 2.6 [11] If \(A\) is regular semi open set in \(X\), then \(X/A\) is also regular semi open set.

Theorem 2.7 [11] In a space \(X\), the regular closed sets, regular open sets and clopen sets are regular semi open set.

3. \(\Lambda_{rs}\)-SETS AND \(V_{rs}\)-SETS

Definition 3.1

Let \(A\) be a subset of a topological space \((X, \tau)\). We define the sets as follows \(\Lambda_{rs}(A)\) and \(V_{rs}(A)\) as follows

\[
\Lambda_{rs}(A) = \bigcap \{ G \mid G \in \text{RSO}(X, \tau) \land A \subseteq G \}
\]

\[
V_{rs}(A) = \bigcup \{ F \mid F \in \text{RSC}(X, \tau) \land A \supseteq F \}
\]

Example 3.2

Let \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}\)

\(\text{RSO}(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, d\}, \{b, c, d\}\}\)

\(\Lambda_{rs}(A) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, d\}, \{b, c, d\}, \{d\}\}\)

\(V_{rs}(A) = \{\emptyset, X, \{b, c, d\}, \{a, d\}, \{b, c\}, \{a\}, \{a, b, c\}\}\)

Lemma 3.3

For subsets \(A, B \& A_i, i \in I\) of a topological space \((X, \tau)\), the following properties hold
(i) \( A \subseteq \Lambda_{rs}(A) \)

(ii) \( B \subseteq A \Rightarrow \Lambda_{rs}(B) \subseteq \Lambda_{rs}(A) \)

(iii) \( \Lambda_{rs}(\Lambda_{rs}(A)) = \Lambda_{rs}(A) \)

(iv) If \( A \in \text{RSO}(X, \tau) \) then \( A = \Lambda_{rs}(A) \)

(v) \( \Lambda_{rs}\left(\bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} \Lambda_{rs}(A_i) \)

(vi) \( \Lambda_{rs}\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} \Lambda_{rs}(A_i) \)

(vii) \( \Lambda_{rs}(A^c) = \left(V_{rs}(A)^c\right)^c \)

**Proof**

i) Let \( x \notin \Lambda_{rs}(A) \). Then there exist a regular semi open set \( G \) such that
\[ A \subseteq G \text{ and } x \notin G. \]
Hence \( x \notin A \) and so \( A \subseteq \Lambda_{rs}(A) \).

ii) Let \( x \notin \Lambda_{rs}(A) \). Then there exist a regular semi open set \( G \) such that
\[ A \subseteq G \text{ and } x \notin G. \]
Since, \( B \subseteq A, B \subseteq G \) and hence \( x \notin \Lambda_{rs}(B) \) and so
\[ \Lambda_{rs}(B) \subseteq \Lambda_{rs}(A). \]

iii) From (i) and (ii) we have \( \Lambda_{rs}(A) \subseteq \Lambda_{rs}(\Lambda_{rs}(A)) \)

For another inclusion, if \( x \notin \Lambda_{rs}(A) \). Then there exist a regular semi open set \( G \) such
that \( A \subseteq G \) and \( x \notin G. \) Hence \( \Lambda_{rs}(A) \subseteq G \) and so we have
\[ x \notin \Lambda_{rs}(\Lambda_{rs}(A)). \]
Therefore \( \Lambda_{rs}(A) = \Lambda_{rs}(\Lambda_{rs}(A)) \)

iv) By definition \& \( A \in \text{RSO}(X, \tau) \), we have \( \Lambda_{rs}(A) \subseteq A \).

By (i) we have \( \Lambda_{rs}(A) = A \).
Suppose that there exist a point \( x \in X \) such that \( x \notin \bigcup_{i \in I} A_i \).

Then there exist a regular semi open set \( G \) such that \( \bigcup_{A_i \subseteq G} \bigcup_{i \in I} \Lambda_{rs}(A_i) \) and \( x \notin G \).

Thus for each \( i \in I \), we have \( x \notin \bigcup_{i \in I} \Lambda_{rs}(A_i) \).

Therefore \( \Lambda_{rs} \left( \bigcup_{i \in I} A_i \right) \subseteq \bigcup_{i \in I} \Lambda_{rs}(A_i) \).

Suppose that there exist a point \( x \) such that \( x \notin \bigcap_{i \in I} \Lambda_{rs}(A_i) \).

Then \( \forall i \in I, x \notin \Lambda_{rs}(A_i) \). Hence \( \forall i \in I \) and \( G \in \text{RSO}(X, \tau) \) such that

\[
A_i \subseteq G \quad \text{and} \quad x \notin G. \quad \text{Thus} \quad x \notin \bigcap_{i \in I} \Lambda_{rs}(A_i).
\]

Therefore \( \Lambda_{rs} \left( \bigcap_{i \in I} A_i \right) \subseteq \bigcap_{i \in I} \Lambda_{rs}(A_i) \).

Let \( x \in \Lambda_{rs}(A^C) \). Then there exist a regular semi open set \( G \) such that

\[
A^C \subseteq G \quad \text{and} \quad x \in G. \quad \text{Hence} \quad x \notin G^C, \quad \text{for every regular semi closed set} \ G^C \quad \text{and}
\]

\[
G^C \subseteq A. \quad \text{Therefore} \quad x \notin V_{rs}(A) \quad \text{and} \quad x \in (V_{rs}(A))^C.
\]

Similarly \( (V_{rs}(A))^C \subseteq \Lambda_{rs}(A^C) \). Therefore \( \Lambda_{rs}(A^C) = (V_{rs}(A))^C \).

Hence the proof.

**Lemma 3.4**

For subsets \( A, B \) & \( A_i, i \in I \) of a topological space \( (X, \tau) \), the following properties hold

(i) \( V_{rs}(A) \subseteq A \)

(ii) \( B \subseteq A \Rightarrow V_{rs}(B) \subseteq V_{rs}(A) \)

(iii) \( V_{rs}(V_{rs}(A)) = V_{rs}(A) \)

(iv) If \( A \in \text{RSC}(X, \tau) \) then \( A = V_{rs}(A) \)
(v) \[ V_{rs} \left( \bigcap_{i \in I} A_i \right) \subseteq \bigcap_{i \in I} V_{rs} (A_i) \]

(vi) \[ V_{rs} \left( \bigcup_{i \in I} A_i \right) \supseteq \bigcup_{i \in I} V_{rs} (A_i) \]

**Remark 3.5**

In general we have \( \Lambda_{rs} (S \cap Q) \neq \Lambda_{rs} (S) \cap \Lambda_{rs} (Q) \)
and \( \Lambda_{rs} (S \cup Q) \neq \Lambda_{rs} (S) \cup \Lambda_{rs} (Q) \) as the following example shows.

**Example 3.6**

Let \( X = \{ a, b, c, d \} \), \( \tau = \{ \emptyset, X, \{ c \}, \{ d \}, \{ a, c, d \}, \{ b, c, d \} \} \)
\( RSO(X, \tau) = \{ \emptyset, X, \{ c \}, \{ d \}, \{ a, c, d \}, \{ b, c, d \}, \{ a, b, c \}, \{ a, b, d \} \} \)
If \( S = \{ a, c \}, Q = \{ c, d \} \). Here \( \Lambda_{rs} (S \cap Q) = \Lambda_{rs} (c) = \{ c \} \). But \( \Lambda_{rs} (S) = \Lambda_{rs} (a, c) = \{ a, c \} \)
and \( \Lambda_{rs} (Q) = \Lambda_{rs} (c, d) = X \). Also \( \Lambda_{rs} (S) \cap \Lambda_{rs} (Q) = \{ a, c \} \)

**Example 3.7**

In the previous example \( S = \{ c \}, Q = \{ d \} \)
\( \Lambda_{rs} (S) = \{ c \}, \Lambda_{rs} (Q) = \{ d \} \)
\( \Lambda_{rs} (S) \cup \Lambda_{rs} (Q) = \{ c, d \} \) but \( \Lambda_{rs} (S \cup Q) = X \)

**Definition 3.8**

In a topological space \((X, \tau)\), a subset \( B \) is a semi regular \( \Lambda \)-set (resp. semi regular \( V \)-set)
briefly \( \Lambda_{rs} \)-set (resp. \( V_{rs} \)-set) of \((X, \tau)\) if \( B = \Lambda_{rs} (B) \) (resp. \( B = V_{rs} (B) \))

**Remark 3.9**

By (iv) of lemma 3.3 and 3.4 we have that
a) If \( B \) is a \( \Lambda_{rs} \)-set (or) if \( B \in RSO (X, \tau) \), then \( B \) is a \( \Lambda_{rs} \)-set.
b) If \( B \) is a \( V_{rs} \)-set (or) if \( B \in RSC (X, \tau) \), then \( B \) is a \( V_{rs} \)-set.

**Proposition 3.10**

For a space \((X, \tau)\), the following statements hold
**\(\Lambda_{rs} - \text{OPEN SETS AND } \Lambda_{rs} - \text{CLOSED SETS IN TOPOLOGICAL SPACES}\)**

(i) \(\emptyset, X\) are \(\Lambda_{rs}\) sets and \(V_{rs}\) sets

(ii) Every union of \(V_{rs}\) sets is a \(V_{rs}\) set.

(iii) Every intersection of \(\Lambda_{rs}\) sets is a \(\Lambda_{rs}\) set.

**Proof**

(i) It is obvious

(ii) Let \(\{ A_i / i \in I \}\) be a family of \(V_{rs}\) sets in \((X, \tau)\).

Then \(A_i = V_{rs}(A_i)\) for each \(i \in I\).

Let \(A = \bigcup_{i \in I} A_i\)

Then \(V_{rs}(A) = V_{rs}\left(\bigcup_{i \in I} A_i\right) \supseteq \bigcup_{i \in I} V_{rs}(A_i) = \bigcup_{i \in I} A_i = A\).

\(\text{i.e. } V_{rs}(A) \supseteq A\). By lemma 3.4 (i), we have \(V_{rs}(A) \subseteq A\).

Hence \(A\) is a \(V_{rs}\) set.

(iii) Similarly we can prove the result using the lemma 3.3 [(i) and (v)]

The following example shows that union of \(\Lambda_{rs}\) sets need not be \(\Lambda_{rs}\) set and intersection of \(V_{rs}\) sets need not be a \(V_{rs}\) set.

**Example 3.11**

Let \(X = \{a, b, c, d, e\}\), \(\tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}\}\)

\(\text{RSO}(X, \tau) = \{\{\emptyset, X, \{a\}, \{a, b\}, \{a, e\}, \{c, d\}, \{a, b, e\}, \{b, c, d\}, \{c, d, e\}, \{b, c, d, e\}\}\}

\(\Lambda_{rs}(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, e\}, \{c, d\}, \{a, b, e\}, \{b, c, d\}, \{c, d, e\}, \{b, c, d, e\}, \}

\{b\}, \{e\}, \{b, e\}\}

Here \(\{a, e\}\) and \(\{c, d\}\) are \(\Lambda_{rs}\) sets. But \(\{a, e\} \cup \{c, d\} = \{a, c, b, e\}\) is not set \(\Lambda_{rs}\)-set.

**Example 3.12**

Let \(X\) and \(\tau\) be defined as in example 3.6

We have \(V_{rs}(X, \tau) = \{\emptyset, X, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{c, d\}\),

Here \{a, c\} and \{a, d\} are \(V_{rs}\)-sets. But \{a, c\} \(\cap\) \{a, d\} = \{a\} is not \(V_{rs}\)-set.

**Remark 3.13**

Observe that a subset \(A\) is semi regular \(\Lambda\)-set if \(A^c\) is semi regular \(\vee\)-set.

Also observe that every semi regular open (semi regular closed) set is a semi regular \(\Lambda\)-set (semi regular \(\vee\)-set).

**\(\Lambda_{rs}\)-closed sets and its properties**

**Definition 3.14**

A subset \(A\) of a topological space \((X, \tau)\) is called \(\Lambda_{rs}\)-closed if \(A = T \cap C\), where \(T\) is a \(\Lambda_{rs}\)-set and \(C\) is a closed set.

The collection of all \(\Lambda_{rs}\)-closed sets in a topological space \((X, \tau)\) is denoted by 

\[\Lambda_{rs} C (X, \tau)\]

**Example 3.15**

Let \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\),

\(\Lambda_{rs}\)-closed set = \(\{\emptyset, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{c\}\}\),

**Theorem 3.16**

For a subset \(A\) of a topological space \((X, \tau)\), the following properties are equivalent.

i) \(A\) is \(\Lambda_{rs}\)-closed

ii) \(A = T \cap \text{cl}(A)\), where \(T\) is a \(\Lambda_{rs}\)-set.

iii) \(A = \Lambda_{rs} (A) \cap \text{cl}(A)\)

**Proof**

(i) \(\Rightarrow\) (ii) Let \(A = T \cap C\), where \(T\) is a \(\Lambda_{rs}\)-set and \(C\) is a closed set. Since \(A \subseteq C\), we have \(\text{cl}(A) \subseteq C\) and \(A = T \cap \text{cl}(A) \supseteq T \cap \text{cl}(A) \supseteq A\). Therefore \(A = T \cap \text{cl}(A)\).

(ii) \(\Rightarrow\) (iii) Let \(A = T \cap \text{cl}(A)\), where \(T\) is a \(\Lambda_{rs}\)-set. Since \(A \subseteq T\), by lemma 3.3 (ii)
we have \( \Lambda_{rs}(A) \subseteq \Lambda_{rs}(T) = T \) and hence \( A \subseteq \Lambda_{rs}(A) \cap cl(A) \subseteq T \cap cl(A) = A \).

Therefore \( A = \Lambda_{rs}(A) \cap cl(A) \)

(iii) \( \Rightarrow \) (i) Since \( \Lambda_{rs}(A) \) is a \( \Lambda_{rs} \)-set and \( cl(A) \) is a closed set and \( A = \Lambda_{rs}(A) \cap cl(A) \).

Therefore \( A \) is a \( \Lambda_{rs} \)-closed.

Lemma 3.17
Every \( \Lambda_{rs} \)-set is a \( \Lambda_{rs} \)-closed set but not conversely.

Proof
Let \( A \) be a \( \Lambda_{rs} \)-set. Then \( A = A \cap X \), where \( A \) is a \( \Lambda_{rs} \)-set and \( X \) is a closed set. Therefore \( A \) is a \( \Lambda_{rs} \)-closed set.

Example 3.18
Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}\} \), \( \Lambda_{rs} \)-set = \( \{\emptyset, X\} \).

\( \Lambda_{rs} \)-closed set = \( \{\emptyset, X, \{b, c, d\}\} \). Since \( \{b, c, d\} \) is a \( \Lambda_{rs} \)-closed set but it is not a \( \Lambda_{rs} \)-set.

Corollary 3.19
Every regular semi open is a \( \Lambda_{rs} \)-closed set but not conversely.

Example 3.20
In example 3.2 \( \{c\} \) is a \( \Lambda_{rs} \)-closed but \( \{c\} \) is not regular semi open.

Lemma 3.21
Every \( \Lambda_{r} \)-closed set is a \( \Lambda_{rs} \)-closed set but not conversely.

Proof
Obvious.

Example 3.22
In example 3.6 \( \{a, b\} \) is a \( \Lambda_{rs} \)-closed but \( \{a, b\} \) is not \( \Lambda_{r} \)-closed.

Definition 3.23
A subset \( A \) of a topological space \( (X, \tau) \) is said to be \( \Lambda_{rs} \)-open if the complement of \( A \) is
\( \Lambda_{rs} \) -closed. The collection of all \( \Lambda_{rs} \) -open sets in a topological space \((X, \tau)\) is denoted by \( \Lambda_{rs} \text{O}(X, \tau) \).

**Theorem 3.24**

(i) If \( A_k \) is \( \Lambda_{rs} \) -closed for each \( k \in I \), then \( \bigcap_{k \in I} A_k \) is \( \Lambda_{rs} \) -closed.

(ii) If \( A_k \) is \( \Lambda_{rs} \) -open for each \( k \in I \), then \( \bigcup_{k \in I} A_k \) is \( \Lambda_{rs} \) -open.

**Proof**

i) Suppose \( A_k \) is \( \Lambda_{rs} \) -closed for each \( k \in I \). Then for each \( k \), there exist a \( \Lambda_{rs} \) -set \( T_k \) and a closed set \( C_k \) such that \( A_k = T_k \cap C_k \).

We have \( \bigcap_{k \in I} A_k = \bigcap_{k \in I} (T_k \cap C_k) = \left( \bigcap_{k \in I} T_k \right) \cap \left( \bigcap_{k \in I} C_k \right) \).

By proposition 3.10 \( \bigcap_{k \in I} T_k \) is a \( \Lambda_{rs} \) -set and \( \bigcap_{k \in I} C_k \) is a closed set.

Therefore \( \bigcap_{k \in I} A_k \) is \( \Lambda_{rs} \) -closed.

ii) Let \( A_k \) is \( \Lambda_{rs} \) -open for each \( k \in I \).

Then \( X - A_k \) is \( \Lambda_{rs} \) -closed and \( X - \bigcup_{k \in I} A_k = \bigcap_{k \in I} (X - A_k) \).

By (i) \( X - \bigcup_{k \in I} A_k \) is \( \Lambda_{rs} \) -closed and hence \( \bigcup_{k \in I} A_k \) is \( \Lambda_{rs} \) -open.

**Remark 3.25**

The union of \( \Lambda_{rs} \) -closed sets need not be \( \Lambda_{rs} \) -closed and the intersection of \( \Lambda_{rs} \) -open sets need not be \( \Lambda_{rs} \) -open as seen in the following example.

**Example 3.26**

Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\} \), \( \tau^c = \{\emptyset, X, \{a, c\}, \{a, b\}, \{a\}\} \)

\( \Lambda_{rs} \) -closed = \( \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{a\}\} \)

\( \Lambda_{rs} \) -open = \( \{\emptyset, X, \{a, c\}, \{a, b\}, \{b\}, \{c\}, \{b, c\}\} \)

Since \( \{b\} \) and \( \{c\} \) are \( \Lambda_{rs} \) -closed sets.
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But \{b\} \cup \{c\} = \{b, c\} is not \(\Lambda_{rs}\) - closed set.

Also \{a, c\} and \{a, b\} are \(\Lambda_{rs}\) - open sets. But \{a, c\} \cap \{a, b\} = \{a\} is not \(\Lambda_{rs}\) - open set.

Lemma 3.27

Every \(V_{rs}\) - set is \(\Lambda_{rs}\) - open.

Proof

Take \(A \cup \emptyset\), where \(A\) is a \(V_{rs}\) - set and \(\emptyset\) is a open set.

Theorem 3.28

For a subset \(A\) of a topological space \((X, \tau)\) the following are equivalent

i) \(A\) is \(\Lambda_{rs}\) - open.

ii) \(A = T \cup C\), where \(T\) is a \(\Lambda_{rs}\) - set and \(C\) is open

iii) \(A = T \cup \text{Int} (A)\)

iv) \(A = V_{rs}(A) \cup \text{Int} (A)\)

Proof

(i) \(\Rightarrow\) (ii) Suppose \(A\) is \(\Lambda_{rs}\) - open. Then \(X - A\) is \(\Lambda_{rs}\) - closed.

Therefore \(X - A = S \cap T\), where \(S\) is a \(\Lambda_{rs}\) - set and \(T\) is a closed set.

Hence \(A = (X - S) \cup (X - T)\), where \((X - S)\) is a \(V_{rs}\) - set and \((X - T)\) is a open set.

(ii) \(\Rightarrow\) (iii) Let \(A = T \cup C\), where \(T\) is a \(V_{rs}\) - set and \(C\) is open.

Since \(C \subset A\) and ‘C’ is open implies \(C \subset \text{Int} (A)\)

Therefore \(A = T \cup C \subset T \cup \text{Int} A \subset A\)

\(A = T \cup \text{Int} (A)\)

(iii) \(\Rightarrow\) (iv) Let \(A = T \cup \text{Int} (A)\), where \(T\) is \(V_{rs}\) - set.

Since \(T \subset A\) \(\Rightarrow V_{rs}(T) \subset V_{rs}(A)\)

Since \(A = T \cup \text{Int} (A)\). Here \(A \supset V_{rs}(A) \cup \text{Int}(A) \supset V_{rs}(T) \cup \text{Int}(A) = T \cup \text{Int} A = A\)

Therefore \(A = V_{rs}(A) \cup \text{Int} (A)\).
Let \( A = V_r(A) \cup \text{Int} (A) \). Then \( X - A = (X - V_r(A)) \cap (X - \text{Int} A) \).

\[
= \Lambda_r(X - A) \cap \text{cl} (X - A).
\]

Since \( \Lambda_r(X - A) \) is a \( \Lambda_r \)-set and \( \text{cl} (X - A) \) is a closed set.

Therefore \( (X - A) \) is \( \Lambda_r \)-closed. Therefore \( A \) is \( \Lambda_r \)-open.

**Theorem 3.29**

For a subset \( A \) of a topological space \((X, \tau)\) the following holds

\[
i) \quad \Lambda_r (A) \subseteq \Lambda_r (A)
\]

\[
ii) \quad \Lambda_b (A) \subseteq \Lambda_r (A)
\]

\[
iii) \quad A^{\Lambda_r} \subseteq \Lambda_r (A)
\]

\[
iv) \quad \Lambda_{\theta} (A) \subseteq \Lambda_r (A)
\]

**Proof**

\[
i) \quad \text{Suppose} \ x \notin \Lambda_r (A). \text{Then there exist a regular open set} \ G \text{such that} \ A \subseteq G \text{ and} \ x \notin G. \text{But we have every regular open set is regular semi open. Then there exist a regular semi open} \ G \text{such that} \ A \subseteq G \text{ and} \ x \notin G. \text{Hence} \ x \notin \Lambda_{\theta} (A).
\]

\[
ii) \quad \text{Result follows from the fact that every regular semi open is b –open.}
\]

\[
iii) \quad \text{Result follows from the fact that every regular semi open is semi open}
\]

\[
iv) \quad \text{Result follows from the fact that every regular semi open is semi } \theta\text{-open.}
\]

**Theorem 3.30**

\[
i) \quad \text{Every} \ \Lambda_r \text{- set is} \ \Lambda_{\theta} \text{- set.}
\]

\[
ii) \quad \text{Every} \ \Lambda_{\theta} \text{- set is} \ \Lambda_b \text{- set.}
\]

\[
iii) \quad \text{Every} \ \Lambda_r \text{- set is} \ A^{\Lambda_r} \text{- set.}
\]

\[
iv) \quad \text{Every} \ \Lambda_{\theta} \text{- set is} \ \Lambda_{\theta}^{\Lambda_r} \text{- set.}
\]

**Proof**
Let $A$ be a subset of a topological space $(X, \tau)$.

Suppose $A$ is a $\Lambda_r$-set. Then $A = \Lambda_r(A)$

By lemma 3.3 $A \subseteq \Lambda_{rs}(A)$. By previous theorem $\Lambda_{rs}(A) \subseteq \Lambda_r(A) = A$.

Therefore we have $A = \Lambda_{rs}(A)$. Therefore $A$ is a $\Lambda_{rs}$-set.

ii) Suppose $A$ is a $\Lambda_{rs}$-set. Then $A = \Lambda_{rs}(A)$

By lemma 2.1 [1] $A \subseteq \Lambda_b(A)$. By previous theorem $\Lambda_b(A) \subseteq \Lambda_{rs}(A) = A$.

Therefore we have $A = \Lambda_b(A)$. Therefore $A$ is a $\Lambda_b$-set.

iii) Suppose $A$ is a $\Lambda_{rs}$-set. Then $A = \Lambda_{rs}(A)$

By lemma 3.1 [4] $A \subseteq A^{\Lambda^r}$. By previous theorem $A^{\Lambda^r} \subseteq \Lambda_{rs}(A) = A$.

Therefore we have $A = A^{\Lambda^r}$. Therefore $A$ is a $\Lambda^r$-set.

iv) Suppose $A$ is a $\Lambda_{rs}$-set. Then $A = \Lambda_{rs}(A)$

By lemma 2.1 [18] $A \subseteq \Lambda_{\theta}^{\Lambda^r}(A)$. By previous theorem $\Lambda_{\theta}^{\Lambda^r}(A) \subseteq \Lambda_{rs}(A) = A$.

Therefore we have $A = \Lambda_{\theta}^{\Lambda^r}(A)$. Therefore $A$ is a $\Lambda_{\theta}^{\Lambda^r}$-set.

Definition 3.31

Let $A$ be a subset of a topological space $(X, \tau)$. A point $x \in X$ is called a $\Lambda_{rs}$-cluster point of $A$ if for every $\Lambda_{rs}$-open set $U$ containing $x$, $A \cap U \neq \emptyset$. The set of all $\Lambda_{rs}$-cluster points is called the $\Lambda_{rs}$-closure of $A$ and it is denoted by $\Lambda_{rs} - \text{cl} (A)$.

Lemma 3.32

Let $A$ and $B$ be subsets of a topological space $(X, \tau)$. The following properties hold.

i) $A \subseteq \Lambda_{rs} - \text{cl} (A) \subseteq \text{cl}(A)$

ii) $\Lambda_{rs} - \text{cl} (A) = \bigcap \{ F \mid A \subseteq F \text{ and } F \text{ is } \Lambda_{rs} \text{-closed} \}$

iii) If $A \subseteq B$, then $\Lambda_{rs} - \text{cl} (A) \subseteq \Lambda_{rs} - \text{cl} (B)$
iv) \( A \) is \( \Lambda_{rs} \)-closed iff \( A = \Lambda_{rs} - \text{cl} (A) \)

v) \( \Lambda_{rs} - \text{cl} (A) \) is \( \Lambda_{rs} \)-closed.

**Proof**

i) Let \( x \not\in \Lambda_{rs} - \text{cl} (A) \). Then ‘\( x \)’ is not a \( \Lambda_{rs} \)-cluster point of \( A \). Then there exist a \( \Lambda_{rs} \)-open set \( U \) containing ‘\( x \)’ such that \( A \cap U = \emptyset \).

Therefore \( x \not\in A \). The other inclusion follows from the fact that every closed set is \( \Lambda_{rs} \)-closed.

ii) Let \( x \not\in \Lambda_{rs} - \text{cl} (A) \). Then there exist a \( \Lambda_{rs} \)-open set \( U \) containing ‘\( x \)’ such that \( A \cap U = \emptyset \). Take \( F = U^C \). Then \( F \) is a \( \Lambda_{rs} \)-closed set, \( A \subseteq F \) and \( x \not\in F \).

Hence \( x \not\in \{ F / A \subseteq F \text{ and } F \text{ is } \Lambda_{rs} \text{-closed} \} \).

Similarly \( \Lambda_{rs} - \text{cl}(A) \subseteq \{ F / A \subseteq F \text{ and } F \text{ is } \Lambda_{rs} \text{-closed} \} \).

iii) Let \( x \not\in \Lambda_{rs} - \text{cl} (B) \). Then there exist a \( \Lambda_{rs} \)-open set \( U \) containing ‘\( x \)’ such that \( B \cap U = \emptyset \). Since \( A \subseteq B \), \( A \cap U = \emptyset \) and hence ‘\( x \)’ is not a \( \Lambda_{rs} \)-cluster point of \( A \). Therefore \( x \not\in \Lambda_{rs} - \text{cl} (A) \).

iv) Suppose \( A \) is \( \not\in \Lambda_{rs} \)-closed. Let \( x \not\in A \), then \( x \in A^C \) and \( A^C \) is \( \Lambda_{rs} \)-open.

Take \( A^C = U \). Then \( U \) is a \( \Lambda_{rs} \)-open set containing ‘\( x \)’ and \( A \cap U = \emptyset \) and hence \( x \not\in \Lambda_{rs} - \text{cl} (A) \). By (i) we get \( A = \Lambda_{rs} - \text{cl} (A) \).

Conversely suppose that \( A = \Lambda_{rs} - \text{cl} (A) \).

Since \( A = \cap \{ F / A \subseteq F \text{ and } F \text{ is } \Lambda_{rs} \text{-closed} \} \).

By (ii) \( A \) is \( \Lambda_{rs} \)-closed.

v) By (i) and (iii) we have \( \Lambda_{rs} - \text{cl} (A) \subseteq \Lambda_{rs} - \text{cl} (\Lambda_{rs} - \text{cl} (A)) \).
Let $x \in \Lambda_{rs} - \text{cl} (\Lambda_{rs} - \text{cl} (A))$ implies ‘$x$’ is a $\Lambda_{rs}$ - cluster point of $\Lambda_{rs} - \text{cl} (A)$. Therefore for every $\Lambda_{rs}$ - open set $U$ containing ‘$x$’

$$(\Lambda_{rs} - \text{cl} (A)) \cap U \neq \emptyset.$$  

Let $y \in \Lambda_{rs} - \text{cl} (A) \cap U$. Then ‘$y$’ is a $\Lambda_{rs}$ - cluster point of $A$. Therefore for every $\Lambda_{rs}$ - open set $G$ containing ‘$y$’ and $A \cap G \neq \emptyset$. Since $U$ is $\Lambda_{rs}$ - open and $y \in U$, $A \cap U \neq \emptyset$ and hence $x \in \Lambda_{rs} - \text{cl} (A)$.

Hence $\Lambda_{rs} - \text{cl} (A) = \Lambda_{rs} - \text{cl} (\Lambda_{rs} - \text{cl} (A))$.

By (iv) $\Lambda_{rs} - \text{cl} (A)$ is $\Lambda_{rs}$ -closed.

Note 3.33

1) $\emptyset, X$ are both $\Lambda_{rs}$ -open and $\Lambda_{rs}$ -closed.

2) By (ii) and (v) $\Lambda_{rs} - \text{cl}(A)$ is the smallest $\Lambda_{rs}$ -closed set containing $A$.

Theorem 3.34

Every closed set is a $\Lambda_{rs}$ - closed set but not conversely.

Proof

Let $A$ be a closed set. Let $A = X \cap A$, where $X$ is a $\Lambda_{rs}$ - set and $A$ is a closed set. Therefore $A$ is a $\Lambda_{rs}$ -closed set.

Example 3.35

Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

Here $A = \{a, c\}$ is $\Lambda_{rs}$ - closed but $A$ is not closed.

Corollary 3.36

Every $\theta$ - closed, $\delta$-closed, regular closed set is a $\Lambda_{rs}$ - closed but not conversely.

Proof

Since every $\theta$- closed $[19]$, $\delta$-closed $[19]$, regular closed set is closed and by above theorem
Example 3.37

\[ X = \{1, 2, 3, 4\} \quad \tau =\{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\} \]

Closed set = \{\emptyset, X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{4\}\}

\(\theta\)-closed = \{\emptyset, X\}, \(\delta\)-closed = \{\emptyset, X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}\}

Regular closed = \{\emptyset, X, \{2, 3, 4\}, \{1, 3, 4\}\}

Here \(A = \{2, 3\}\) is \(\Lambda_{rs}\)-closed but \(A\) is not \(\theta\)-closed, \(\delta\)-closed, regular closed.

Theorem 3.38

i) Every \(\Lambda_{r}\)-closed is \(\Lambda_{rs}\)-closed

ii) Every \(\Lambda_{rs}\)-closed is \((\Lambda, b)\) closed

But converse not true.

Proof

i) Let \(A\) be a subset of a topological space \((X, \tau)\). Suppose \(A\) is \(\Lambda_{r}\)-closed. Then \(A = S \cap C\), where ‘S’ is a \(\Lambda_{r}\)-set and ‘C’ is a closed set. By theorem 3.30 (i) ‘S’ is a \(\Lambda_{rs}\)-set. Therefore \(A\) is \(\Lambda_{rs}\)-closed.

ii) Let \(A\) be a subset of a topological space \((X, \tau)\). Suppose \(A\) is \(\Lambda_{rs}\)-closed.

Then \(A = S \cap C\), where ‘S’ is a \(\Lambda_{rs}\)-set and ‘C’ is a closed set.

By theorem 3.30 (ii) \(\Lambda_{rs}\)-set is \(\Lambda_{b}\)-set and every closed set is b-closed.

Therefore \(A\) is \(\Lambda_{rs}\)-closed.

Theorem 3.39

Every \(\Lambda_{rs}\)-closed set is semi closed but not conversely.

Proof

Since every regular semi open set is both semi open and semi closed. Also intersection of semi closed set is semi closed and every closed set is semi closed. Therefore by the definition of \(\Lambda_{rs}\)-closed set, every \(\Lambda_{rs}\)-closed set is semi closed.
Example 3.40

Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c, d\}\}$ since $\{a\}$ is semi closed but not $\Lambda_{rs}$-closed.

Theorem 3.41

If $\text{RSO}(X, \tau)$ is indiscrete space, then every $\Lambda_{rs}$-closed is pre closed and $\alpha$-closed.

Proof

If $\text{RSO}(X, \tau) = \{\emptyset, X\}$. Then only regular semi open sets are $\emptyset, X$ only. Let $A$ be any subset of $X$. WKT every closed set is pre closed and $\alpha$-closed. Therefore by the definition of $\Lambda_{rs}$-closed set, every $\Lambda_{rs}$-closed is pre closed and $\alpha$-closed.

Theorem 3.42

If a space $X$ is locally indiscrete, then every locally closed set is $\Lambda_{rs}$-closed.

Proof

By the definition of locally indiscrete we have every open set is closed and by theorem 3.34 every closed set is $\Lambda_{rs}$-closed.

Theorem 3.43

If a subset $A$ of $(X, \tau)$ is regular open, then $\text{PInt}(A)$ and $\text{Scl}(A)$ is $\Lambda_{rs}$-closed.

Proof

WKT $\text{PInt}(A) = A \cap \text{Int cl}(A)$ and $\text{Scl}(A) = A \cup \text{Int cl}(A)$

since $A$ is regular open, we have $A = \text{int cl}(A)$

Therefore $\text{PInt}(A) = A \cap A = A$ and $\text{Scl}(A) = A \cup A = A$.

WKT every regular open set is regular semi open and every regular open set is $\Lambda_{rs}$-set.

Also every $\Lambda_{rs}$-set is $\Lambda_{rs}$-closed.

Therefore $\text{PInt}(A)$ and $\text{Scl}(A)$ is $\Lambda_{rs}$-closed.

Theorem 3.44

If a subset $A$ of $(X, \tau)$ is regular closed, then $\text{Pcl}(A)$ and $\text{SInt}(A)$ is $\Lambda_{rs}$-closed.

Proof

Similar to above
Definition 3.45

A subset $A$ of a space $(X, \tau)$ is called regular weakly closed (briefly rw- closed) [3] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semi open in $X$.

Lemma 3.46

A subset $A$ of a space $(X, \tau)$ is RW closed iff $\text{cl}(A) \subseteq \Lambda_{rs}(A)$.

Theorem 3.47

For a subset $A$ of a topological space $(X, \tau)$ the following conditions are equivalent.

i) $A$ is closed

ii) $A$ is RW closed and $\Lambda_r$-closed

iii) $A$ is RW closed and $\Lambda_{rs}$-closed

Proof

i) $\Rightarrow$ ii) Every closed set is both RW closed and $\Lambda_r$-closed.

ii) $\Rightarrow$ iii) by theorem 3.38 (i)

iii) $\Rightarrow$ i) $A$ is RW closed so by lemma 3.46 $\text{cl}(A) \subseteq \Lambda_{rs}(A)$.

Also $A$ is $\Lambda_{rs}$-closed. By theorem 3.16 we have $A = \Lambda_{rs}(A) \cap \text{cl}(A)$

Hence $A = \text{cl}(A)$. Therefore $A$ is closed.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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