A CONSTRUCTION OF BALANCED DEGREE-MAGIC GRAPHS

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Abstract. A graph $G$ is called degree-magic if it admits a labelling of the edges by integers $1, 2, \ldots, |E(G)|$ such that the sum of the labels of the edges incident with any vertex $v$ is equal to $(1 + |E(G)|)\deg(v)/2$. Degree-magic graphs extend supermagic regular graphs. In this paper, a new construction of balanced degree-magic graphs is introduced.

Keywords: supermagic graphs; degree-magic graphs; cycle graphs.

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1. INTRODUCTION

The finite simple graphs and multigraphs without loops and isolated vertices are considered. If $G$ is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of $G$, respectively. Cardinalities of these sets are called the order and the size of $G$. For any integers $p$ and $q$, the set of all integers $z$ satisfying $p \leq z \leq q$ is indicated by $[p, q]$.

Let a graph $G$ and a mapping $f$ from $E(G)$ into the set of positive integers be given. The index-mapping of $f$ is the mapping $f^*$ from $V(G)$ into the set of positive integers defined by

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\[ f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every} \quad v \in V(G), \]

where \( \eta(v,e) \) is equal to 1 when \( e \) is an edge incident with a vertex \( v \), and 0 otherwise. An injective mapping \( f \) from \( E(G) \) into the set of positive integers is called a \textit{magic labelling} of \( G \) for an \textit{index} \( \lambda \) if its index-mapping \( f^* \) satisfies

\[ f^*(v) = \lambda \quad \text{for all} \quad v \in V(G). \]

A magic labelling \( f \) of \( G \) is called a \textit{supermagic labelling} if the set \( \{ f(e) : e \in E(G) \} \) consists of consecutive positive integers. A graph \( G \) is said to be \textit{supermagic} (\textit{magic}) whenever there exists a supermagic (magic) labelling of \( G \).

A bijective mapping \( f \) from \( E(G) \) into \([1, |E(G)|] \) is called a \textit{degree-magic labelling} (or only \textit{d-magic labelling}) of a graph \( G \) if its index-mapping \( f^* \) satisfies

\[ f^*(v) = \frac{1+|E(G)|}{2} \deg(v) \quad \text{for all} \quad v \in V(G). \]

A \( d \)-magic labelling \( f \) of \( G \) is called \textit{balanced} if for all \( v \in V(G) \) it holds

\[ |\{ e \in E(G) : \eta(v,e) = 1, f(e) \leq \lfloor |E(G)|/2 \rfloor \}| \]
\[ = |\{ e \in E(G) : \eta(v,e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor \}|. \]

A graph \( G \) is said to be \textit{degree-magic} (\textit{balanced degree-magic}) (or only \textit{d-magic}) when a \( d \)-magic (balanced \( d \)-magic) labelling of \( G \) exists.

The concept of magic graphs was put forward by Sedláček [10]. Later, supermagic graphs were introduced by Stewart [11]. Besides, a new construction of supermagic complements of some graphs was recommended [9]. Moreover, the notion of degree-magic graphs was then suggested by Bezegová and Ivančo [1] as an extension of supermagic regular graphs. Recently, numerous papers are published on degree-magic and supermagic graphs, see [2, 3, 4, 5, 6, 7, 8] for more comprehensive references.

Let one recall the basic properties of \( d \)-magic graphs that will be used in the next.

**Theorem 1.1.** [1] \textit{Let} \( G \) \textit{be a regular graph. Then} \( G \) \textit{is supermagic if and only if it is} \( d \)-\textit{magic}. 
Theorem 1.2. [1] Let $H_1$ and $H_2$ be edge-disjoint subgraphs of a graph $G$ which form its decomposition. If $H_1$ is $d$-magic and $H_2$ is balanced $d$-magic, then $G$ is a $d$-magic graph. Moreover, if $H_1$ and $H_2$ are both balanced $d$-magic, then $G$ is a balanced $d$-magic graph.

2. Balanced Degree-Magic Graphs

An injective mapping $f$ from $E(G)$ into the set of positive integers is called a single-consecutive labelling (SC-labelling) of a graph $G$ if its index-mapping $f^*$ satisfies

$$f^*(V(G)) = [a, a + |V(G)| - 1] \text{ for some integer } a.$$ 

Let $f_i, i \in \{1, 2\},$ be a SC-labelling of a graph $G_i$. The labellings $f_1$ and $f_2$ are called complementary if $f_1(E(G_1)) \cap f_2(E(G_2)) = \emptyset$ and $f_1(E(G_1)) \cup f_2(E(G_2)) = [1, m]$, where $m = |E(G_1)| + |E(G_2)|$. The complementary labellings $f_1$ and $f_2$ are called balanced if all pairs of vertices $u \in V(G_1), v \in V(G_2)$ satisfy

$$|\{e \in E(G_1) : \eta(u, e) = 1, f_1(e) \leq \lfloor m/2 \rfloor\}|$$

$$+ |\{e \in E(G_2) : \eta(v, e) = 1, f_2(e) \leq \lfloor m/2 \rfloor\}|$$

$$= |\{e \in E(G_1) : \eta(u, e) = 1, f_1(e) > \lfloor m/2 \rfloor\}|$$

$$+ |\{e \in E(G_2) : \eta(v, e) = 1, f_2(e) > \lfloor m/2 \rfloor\}|.$$

Now, one is able to prove the following Proposition.

Proposition 2.1. Let $H_1$ and $H_2$ be spanning subgraphs of a graph $G$ which form its decomposition with vertices $v_1, v_2, \ldots, v_n$. Let $f$ be a SC-labelling of $H_1$ such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and let $g$ be a SC-labelling of $H_2$ such that $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. If $f$ and $g$ are complementary, then $G$ is a supermagic graph.

Proof. Since $f$ is a SC-labelling of $H_1$ such that $f^*(v_i) = f^*(v_i) + (i - 1)$ and $g$ is a SC-labelling of $H_2$ such that $g^*(v_i) = g^*(v_i) - (i - 1)$ for all $i \in [1, n], f^*(v_i) + g^*(v_i) = f^*(v_i) + g^*(v_i)$. Now, consider a mapping $\varphi$ from $E(G)$ into the set of positive integers defined by

$$\varphi(v_iv_j) = \begin{cases} 
    f(v_iv_j) & : v_iv_j \in E(H_1), \\
    g(v_iv_j) & : v_iv_j \in E(H_2).
\end{cases}$$
Obviously, $\varphi^*(v_i) = f^*(v_i) + g^*(v_i) = f^*(v_1) + g^*(v_1)$. Since $\varphi(E(G)) = f(E(H_1)) \cup g(E(H_2))$ and the labellings $f$ and $g$ are complementary, $\varphi$ is a supermagic labelling of $G$. Therefore, $G$ is a desired graph. □

If the graph $G$ in Proposition 2.1 is regular, then $G$ is $d$-magic by Theorem 1.1. For balanced $d$-magic graphs, one can show the following assertion.

**Proposition 2.2.** Let $H_1$ and $H_2$ be spanning subgraphs of a regular graph $G$ which form its decomposition with vertices $v_1, v_2, \ldots, v_n$. Let $f$ be a SC-labelling of $H_1$ such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and let $g$ be a SC-labelling of $H_2$ such that $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. If $f$ and $g$ are (balanced) complementary, then $G$ is a (balanced) $d$-magic graph.

**Proof.** By using the same proof as Proposition 2.1, $G$ is a supermagic graph. Because $G$ is regular, $G$ is $d$-magic by Theorem 1.1. Since $f$ and $g$ are balanced complementary, for each vertex $v_i, i \in [1, n]$, of $G$ it holds

$$
|\{e \in E(G) : \eta(v_i, e) = 1, \varphi(e) \leq \lfloor |E(G)|/2 \rfloor \}| = |\{e \in E(H_1) : \eta(v_i, e) = 1, f(e) \leq \lfloor |E(G)|/2 \rfloor \}| + |\{e \in E(H_2) : \eta(v_i, e) = 1, g(e) \leq \lfloor |E(G)|/2 \rfloor \}|$$

$$= |\{e \in E(H_1) : \eta(v_i, e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor \}| + |\{e \in E(H_2) : \eta(v_i, e) = 1, g(e) > \lfloor |E(G)|/2 \rfloor \}| = |\{e \in E(G) : \eta(v_i, e) = 1, \varphi(e) > \lfloor |E(G)|/2 \rfloor \}|.
$$

Thus, $\varphi$ is a balanced $d$-magic labelling of $G$. That is, $G$ is an expected graph. □

The above two Propositions describe methods to construct supermagic graphs and $d$-magic graphs by using SC-labellings respectively. In order to use Proposition 2.2, one needs reasonable SC-labellings of some graphs.

**Lemma 2.3.** Let $G$ be a cycle graph of order 4 with vertices $v_1, v_2, v_3, v_4$ and let $k, h$ be positive integers. Then there are a SC-labelling $f$ of $G$ such that $f(E(G)) = \{k, k+1, k+4, k+6\}$ and $f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)$ and a SC-labelling $g$ of $G$ such that $g(E(G)) = \{h, h+1, h+3, h+5\}$ and $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)$. Moreover, if $k = 1$ and $h = 3$, then the SC-labellings $f$ and $g$ are balanced complementary.
Proof. Consider a mapping $f$ from $E(G)$ into the set of positive integers given by

$$f(e) = \begin{cases} 
    k & : e = v_1v_3, \\
    k+6 & : e = v_3v_4, \\
    k+1 & : e = v_4v_2, \\
    k+4 & : e = v_2v_1. 
\end{cases}$$

It is easy to see that $f(E(G)) = \{k, k+1, k+4, k+6\}$ and $f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4)$. Hence, $f$ is a desired SC-labelling of $G$. Moreover, consider a mapping $g$ from $E(G)$ into the set of positive integers defined by

$$g(e) = \begin{cases} 
    h+1 & : e = v_1v_3, \\
    h+3 & : e = v_3v_4, \\
    h & : e = v_4v_2, \\
    h+5 & : e = v_2v_1. 
\end{cases}$$

One can see that $g(E(G)) = \{h, h+1, h+3, h+5\}$ and $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4)$. Thus, $g$ is a required SC-labelling of $G$. Now, consider the case $k = 1$ and $h = 3$, one then has

$$f(e) = \begin{cases} 
    1 & : e = v_1v_3, \\
    7 & : e = v_3v_4, \\
    2 & : e = v_4v_2, \\
    5 & : e = v_2v_1, 
\end{cases}$$

and

$$g(e) = \begin{cases} 
    4 & : e = v_1v_3, \\
    6 & : e = v_3v_4, \\
    3 & : e = v_4v_2, \\
    8 & : e = v_2v_1. 
\end{cases}$$

Clearly, $f$ and $g$ are balanced complementary labellings. \qed

**Lemma 2.4.** Let $G$ be a cycle graph of odd order $n \geq 3$ with vertices $v_1, v_2, ..., v_n$ and let $k, h$ be positive integers. Then there exist a SC-labelling $f$ of $G$ such that $f(E(G)) = [k, k+n-1]$ and
$f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and a SC-labelling $g$ of $G$ such that $g(E(G)) = [h, h + n - 1]$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. Moreover, if $k = 1$ and $h = n + 1$, then the SC-labellings $f$ and $g$ are balanced complementary.

Proof. Consider a mapping $f$ from $E(G)$ into the set of positive integers given by

$$f(e) = \begin{cases} 
    k + (n-1)/2 : e = v_n v_1, \\
    k : e = v_1 v_2, \\
    k + (n-1)/2 + 1 : e = v_2 v_3, \\
    k + 1 : e = v_3 v_4, \\
    k + (n-1)/2 + 2 : e = v_4 v_5, \\
    k + 2 : e = v_5 v_6, \\
    \vdots \\
    k + (n-1)/2 - 1 : e = v_{n-2} v_{n-1}, \\
    k + n - 1 : e = v_{n-1} v_n. 
\end{cases}$$

One is able to check that $f(E(G)) = [k, k + n - 1]$ and $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$. Thus, $f$ is a desired SC-labelling of $G$. Besides, consider a mapping $g$ from $E(G)$ into the set of positive integers defined by

$$g(e) = \begin{cases} 
    h + (n-1)/2 : e = v_1 v_n, \\
    h : e = v_n v_{n-1}, \\
    h + (n-1)/2 + 1 : e = v_{n-1} v_{n-2}, \\
    h + 1 : e = v_{n-2} v_{n-3}, \\
    h + (n-1)/2 + 2 : e = v_{n-3} v_{n-4}, \\
    h + 2 : e = v_{n-4} v_{n-5}, \\
    \vdots \\
    h + (n-1)/2 - 1 : e = v_3 v_2, \\
    h + n - 1 : e = v_2 v_1. 
\end{cases}$$

One can get that $g(E(G)) = [h, h + n - 1]$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. Hence, $g$ is a required SC-labelling of $G$. Now, consider the case $k = 1$ and $h = n + 1$, one then gets
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\[
f(e) = \begin{cases} 
1 + (n-1)/2 : e = v_n v_1, \\
1 : e = v_1 v_2, \\
2 + (n-1)/2 : e = v_2 v_3, \\
2 : e = v_3 v_4, \\
3 + (n-1)/2 : e = v_4 v_5, \\
3 : e = v_5 v_6, \\
\vdots \\
(n-1)/2 : e = v_{n-2} v_{n-1}, \\
n : e = v_{n-1} v_n,
\end{cases}
\]

and

\[
g(e) = \begin{cases} 
n + 1 + (n-1)/2 : e = v_1 v_n, \\
n + 1 : e = v_n v_{n-1}, \\
n + 2 + (n-1)/2 : e = v_{n-1} v_{n-2}, \\
n + 2 : e = v_{n-2} v_{n-3}, \\
n + 3 + (n-1)/2 : e = v_{n-3} v_{n-4}, \\
n + 3 : e = v_{n-4} v_{n-5}, \\
\vdots \\
n + (n-1)/2 : e = v_3 v_2, \\
2n : e = v_2 v_1.
\end{cases}
\]

Evidently, \( f \) and \( g \) are balanced complementary labellings. \( \square \)

In the next results, one is able to prove some sufficient conditions for balanced \( d \)-magic graphs.

**Theorem 2.5.** Let \( G \) be a graph which can be decomposable into two spanning cycle subgraphs of order 4. Then \( G \) is a balanced \( d \)-magic graph.

**Proof.** Suppose that two spanning cycle subgraphs of \( G \) have vertices \( v_1, v_2, v_3, v_4 \). Thus, by Lemma 2.3, there are two balanced complementary SC-labellings \( f, g \) of these cycles such that \( f^*(v_1) < f^*(v_2) < f^*(v_3) < f^*(v_4) \) and \( g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4) \). Since these two
cycles are regular and form its decomposition, $G$ is a regular graph. Therefore, according to Proposition 2.2, $G$ is a balanced $d$-magic graph.

Combining Theorem 1.2 and Theorem 2.5, one immediately has

**Corollary 2.6.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of order 4, then $G$ is a balanced $d$-magic graph.

Joining Theorem 1.1 and Corollary 2.6, one absolutely has

**Corollary 2.7.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of order 4, then $G$ is a supermagic graph.

**Theorem 2.8.** Let $G$ be a graph which can be decomposable into two spanning cycle subgraphs of odd order $n \geq 3$. Then $G$ is a balanced $d$-magic graph.

**Proof.** Assume that two spanning cycle subgraphs of $G$ of odd order $n \geq 3$ have vertices $v_1, v_2, \ldots, v_n$. Hence by Lemma 2.4, there are two balanced complementary SC-labellings $f, g$ of these cycles such that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_n)$ and $g^*(v_1) > g^*(v_2) > \cdots > g^*(v_n)$. It is clear that these two cycles are regular and they form its decomposition, so $G$ is a regular graph. Therefore, according to Proposition 2.2, $G$ is a balanced $d$-magic graph.

Combining Theorem 1.2 and Theorem 2.8, one suddenly has

**Corollary 2.9.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of odd order $n \geq 3$, then $G$ is a balanced $d$-magic graph.

Joining Theorem 1.1 and Corollary 2.9, one certainly has

**Corollary 2.10.** For any positive integer $k$, if a graph $G$ can be decomposable into $2k$ spanning cycle subgraphs of odd order $n \geq 3$, then $G$ is a supermagic graph.

Notice that there exist SC-labellings $f$ and $g$ of a cycle graph of order 8 with vertices $v_1, v_2, \ldots, v_8$ such that $f(E(G)) = [k, k+3] \cup \{k+8, k+9, k+11, k+12\}$ and $g(E(G)) = [h, h+3] \cup \{h+6, h+9, h+10, h+11\}$ for any positive integers $h, k$. Moreover, if $k = 1$ and
If $h = 5$, then the SC-labellings $f$ and $g$ are balanced complementary. These SC-labellings $f$ and $g$ are shown as follows.

$$f(e) = \begin{cases} 
  k &: e = v_1v_4, \\
  k+11 &: e = v_4v_6, \\
  k+2 &: e = v_6v_7, \\
  k+12 &: e = v_7v_8, \\
  k+3 &: e = v_8v_5, \\
  k+9 &: e = v_5v_3, \\
  k+1 &: e = v_3v_2, \\
  k+8 &: e = v_2v_1, 
\end{cases}$$

and

$$g(e) = \begin{cases} 
  h+11 &: e = v_1v_4, \\
  h &: e = v_4v_6, \\
  h+9 &: e = v_6v_7, \\
  h+1 &: e = v_7v_8, \\
  h+6 &: e = v_8v_5, \\
  h+2 &: e = v_5v_3, \\
  h+10 &: e = v_3v_2, \\
  h+3 &: e = v_2v_1. 
\end{cases}$$

One can prove that $f^*(v_1) < f^*(v_2) < \cdots < f^*(v_8)$ while $g^*(v_1) > g^*(v_2) > g^*(v_3) > g^*(v_4) > g^*(v_7) > g^*(v_6) > g^*(v_5) > g^*(v_8)$. Furthermore, consider the case $k = 1$ and $h = 5$, one then obtains
\[ f(e) = \begin{cases} 
1 : e = v_1v_4, \\
12 : e = v_4v_6, \\
3 : e = v_6v_7, \\
13 : e = v_7v_8, \\
4 : e = v_8v_5, \\
10 : e = v_5v_3, \\
2 : e = v_3v_2, \\
9 : e = v_2v_1, 
\end{cases} \]

and

\[ g(e) = \begin{cases} 
16 : e = v_1v_4, \\
5 : e = v_4v_6, \\
14 : e = v_6v_7, \\
6 : e = v_7v_8, \\
11 : e = v_8v_5, \\
7 : e = v_5v_3, \\
15 : e = v_3v_2, \\
8 : e = v_2v_1, 
\end{cases} \]

Obviously, \( f \) and \( g \) are balanced complementary labellings. However, by the method of Proposition 2.2, one cannot construct a balanced \( d \)-magic graph by using two balanced complementary labellings of a cycle subgraph of order 8 upwardly because the condition does not hold.

For the last result, two balanced complementary of SC-labellings of some cycle graphs and their associated balanced \( d \)-magic graphs are presented as follows.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Two balanced complementary SC-labellings of a cycle graph \( C_3 \).}
\end{figure}
Figure 2. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_3$.

Figure 3. Two balanced complementary SC-labellings of a cycle graph $C_4$.

Figure 4. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_4$.

Figure 5. Two balanced complementary SC-labellings of a cycle graph $C_5$. 
Figure 6. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_5$.

Figure 7. Two balanced complementary SC-labellings of a cycle graph $C_7$.

Figure 8. A balanced $d$-magic graph constructed by two spanning cycle subgraphs $C_7$. 
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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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