CONTINUOUS CONTROLLED K-FRAME FOR HILBERT $C^\ast$-MODULES

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Abstract. In this paper, we introduce and study the concept of continuous controlled K-frame for Hilbert $C^\ast$-modules which is a generalization of discrete controlled K-frame.

Keywords: controlled frame; controlled K-frame; continuous controlled K-frame; $C^\ast$-algebra; Hilbert $\mathcal{A}$-modules.

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1. INTRODUCTION AND PRELIMINARIES

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [9] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [7] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [11]. Frames have been used in signal processing, image processing, data compression and sampling theory. The concept of a generalization of frames to a family indexed by some locally compact space endowed with a
Radon measure was proposed by G. Kaiser [14] and independently by Ali, Antoine and Gazeau [5]. These frames are known as continuous frames. Gabardo and Han in [10] called these frames associated with measurable spaces, Askari-Hemmat, Dehghan and Radjabalipour in [3] called them generalized frames and in mathematical physics they are referred to as coherent states [5]. In 2012, L. Gavruta [12] introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K. Controlled frames in Hilbert spaces have been introduced by P. Balazs [4] to improve the numerical efficiency of iterative algorithms for inverting the frame operator. Rahimi [17] defined the concept of controlled K-frames in Hilbert spaces and showed that controlled K-frames are equivalent to K-frames due to which the controlled operator C can be used as preconditions in applications. Controlled frames in $C^*$-modules were introduced by Rashidi and Rahimi [15], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space. We extended the results of frames in Hilbert spaces to Hilbert $C^*$-modules (see [13], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29]).

Motivated by the above literature, we introduce the notion of a continuous controlled K-frame in Hilbert $C^*$-modules.

In the following we briefly recall the definitions and basic properties of $C^*$-algebra, Hilbert $\mathcal{A}$-modules. Our references for $C^*$-algebras as [8, 6]. For a $C^*$-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

**Definition 1.1.** [18] Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \ldots \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

1. $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
2. $\langle ax + y, z \rangle_{\mathcal{A}} = a \langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
3. $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^{\ast}$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert $C^*$-module over $\mathcal{A}$. For every $a$ in $C^*$-algebra $\mathcal{A}$, we have $|a| = (a^* a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.
Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T: \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^*: \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $\text{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is abbreviated to $\text{End}^*_{\mathcal{A}}(\mathcal{H})$.

**Lemma 1.2.** [2]. Let $\mathcal{H}$ and $\mathcal{K}$ two Hilbert $\mathcal{A}$-modules and $T \in \text{End}^*_{\mathcal{A}}(\mathcal{H})$. Then the following statements are equivalent:

(i) $T$ is surjective.

(ii) $T^*$ is bounded below with respect to norm, i.e, there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$, $x \in \mathcal{K}$.

(iii) $T^*$ is bounded below with respect to the inner product, i.e, there is $m' > 0$ such that,

$$\langle T^*x, T^*x \rangle_{\mathcal{A}} \geq m' \langle x, x \rangle_{\mathcal{A}}, x \in \mathcal{K}$$

**Lemma 1.3.** [18] Let $\mathcal{H}$ and $\mathcal{K}$ two Hilbert $\mathcal{A}$-modules and $T \in \text{End}^*_{\mathcal{A}}(\mathcal{H})$. Then the following statements are equivalent,

(i) The operator $T$ is bounded and $\mathcal{A}$-linear.

(ii) There exist $0 \leq k$ such that

$$\langle Tx, Tx \rangle_{\mathcal{A}} \leq k \langle x, x \rangle_{\mathcal{A}} \quad x \in \mathcal{H}.$$ 

For the following theorem, $R(T)$ denote the range of the operator $T$.

**Theorem 1.4.** [30] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a $C^*$-algebra $\mathcal{A}$ and let $T, S$ two operators for $\text{End}^*_{\mathcal{A}}(\mathcal{H})$. If $R(S)$ is closed, then the following statements are equivalent:

(i) $R(T) \subset R(S)$.

(ii) $TT^* \leq \lambda^2 SS^*$ for some $\lambda \geq 0$.

(iii) There exists $Q \in \text{End}^*_{\mathcal{A}}(\mathcal{H})$ such that $T = SQ$.

2. **Continuous Controlled K-Frame for Hilbert C*-Modules**

Let $X$ be a Banach space, $(\Omega, \mu)$ a measure space, and $f: \Omega \to X$ a measurable function. Integral of the Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every
Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $C^*$-modules, $\{\mathcal{K}_w : w \in \Omega\}$ is a family of subspaces of $\mathcal{K}$, and $End_{A^*}(\mathcal{H}, \mathcal{K}_w)$ is the collection of all adjointable $A$-linear maps from $\mathcal{H}$ into $\mathcal{K}_w$. We define

$$\bigoplus_{w \in \Omega} \mathcal{K}_w = \{x = \{x_w \}_{w \in \Omega} : \text{for all } w \in \Omega, \int_{\Omega} \|x_w\|^2 d\mu(w) < \infty\}.$$

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $A$-valued inner product is defined by

$$\langle x, y \rangle_A = \int_{\Omega} \langle x_w, y_w \rangle_A d\mu(w),$$

the norm is defined by $\|x\| = \|(x, x)_{A}\|^{\frac{1}{2}}$. Therefore, $\bigoplus_{w \in \Omega} \mathcal{K}_w$ is a Hilbert $C^*$-module (see [14]).

Let $A$ be a $C^*$-algebra, $l^2(\mathcal{A})$ is defined by,

$$l^2(\mathcal{A}) = \{\{a_\omega \}_{w \in \Omega} \subseteq \mathcal{A} : \int_{\Omega} a_\omega a_\omega^* d\mu(\omega) < \infty\}.$$

$l^2(\mathcal{A})$ is a Hilbert $C^*$-module (Hilbert $\mathcal{A}$-module) with pointwise operations and the inner product defined as,

$$\langle \{a_\omega \}_{w \in \Omega}, \{b_\omega \}_{w \in \Omega} \rangle_{\mathcal{A}} = \int_{\Omega} a_\omega b_\omega^* d\mu(\omega), \{a_\omega \}_{w \in \Omega}, \{b_\omega \}_{w \in \Omega} \in l^2(\mathcal{A}),$$

and,

$$\|\{a_\omega \}_{w \in \Omega}\| = (\int_{\Omega} a_\omega a_\omega^* d\mu(\omega))^{\frac{1}{2}}.$$

**Definition 2.1.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital $C^*$-algebra, and $K \in End_{A^*}(\mathcal{H})$. A mapping $F : \Omega \rightarrow \mathcal{H}$ is called a continuous K-Frame for $\mathcal{H}$ if :

- $F$ is weakly-measurable, ie, for any $f \in \mathcal{H}$, the map $w \rightarrow \langle f, F(w) \rangle_{\mathcal{A}}$ is measurable on $\Omega$.

- There exist two strictly positive constants $A$ and $B$ such that

$$(2.1) \quad A \langle K^* f, K^* f \rangle_{\mathcal{A}} \leq \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{A}} \langle F(w), f \rangle_{\mathcal{A}} d\mu(w) \leq B \langle f, f \rangle_{\mathcal{A}}, f \in \mathcal{H}.$$ 

The elements $A$ and $B$ are called continuous K-frame bounds.

If $A = B$ we call this Continuous K-Frame a continuous tight K-Frame, and if $A = B = 1$ it is called a continuous Parseval K-Frame. If only the right-hand inequality of (2.1) is satisfied, we
call $F$ a continuous bessel mapping with Bessel bound $B$.

Let $F$ be a continuous bessel mapping for Hilbert $C^*$-module $\mathcal{H}$ over $\mathcal{A}$.

The operator $T : \mathcal{H} \to l^2(\mathcal{A})$ defined by,

$$T f = \{ \langle f, F(\omega) \rangle_\mathcal{A} \}_{\omega \in \Omega},$$

is called the analysis operator.

There adjoint operator $T^*: l^2(\mathcal{A}) \to \mathcal{H}$ given by,

$$T^* (\{ a_\omega \}_{\omega \in \Omega}) = \int_{\Omega} a_\omega F(\omega) d\mu(\omega),$$

is called the synthesis operator.

By composing $T$ and $T^*$, we obtain the continuous K-frame operator, $S : \mathcal{H} \to \mathcal{H}$ defined by

$$S f = \int_{\Omega} \langle f, F(\omega) \rangle_\mathcal{A} F(\omega) d\mu(\omega).$$

It’s clear to see that $S$ is positive, bounded and selfadjoint (see [5]).

For the following definition we need to introduce, $GL^+(\mathcal{H})$ be the set of all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse.

**Definition 2.2.** Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a unital $C^*$-algebra and $K \in End^*_{\mathcal{A}}(\mathcal{H})$, $C \in GL^+(\mathcal{H})$. A mapping $F : \Omega \to \mathcal{H}$ is called a continuous C-controlled K-Frame in $\mathcal{H}$ if:

- $F$ is weakly-measurable, ie, for any $f \in \mathcal{H}$, the map
  $$w \to \langle f, F(w) \rangle_\mathcal{A}$$
  is measurable on $\Omega$.

- There exists two strictly positive constants $A$ and $B$ such that

  \begin{equation}
  (2.2) \quad A \langle C^{\frac{1}{2}} K^* f, C^{\frac{1}{2}} K^* f \rangle_\mathcal{A} \leq \int_{\Omega} \langle f, F(w) \rangle_\mathcal{A} \langle CF(w), f \rangle_\mathcal{A} d\mu(w) \leq B \langle f, f \rangle_\mathcal{A}, f \in \mathcal{H}.
  \end{equation}

The elements $A$ and $B$ are called continuous C-controlled K-frame bounds.

If $A = B$ we call this continuous C-controlled K-Frame a continuous tight C-Controlled K-Frame, and if $A = B = 1$ it is called a continuous Parseval C-Controlled K-Frame. If only the right-hand inequality of (2.2) is satisfied, we call $F$ a continuous C-controlled bessel mapping with Bessel bound $B$. 
Example 2.3.

\[ H = \mathcal{A} = l^2(\mathbb{C}) \]

\[ = \left\{ \{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C} / \sum_{n=1}^{\infty} |a_n|^2 < +\infty \right\}. \]

\( \mathcal{A} \) is recognized as a Hilbert \( \mathcal{A} \)-Module with the \( \mathcal{A} \)-inner product

\[ < \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} > \mathcal{A} = \{ a_n \overline{b_n} \}_{n=1}^{\infty}. \]

Consider now the borned linear operator

\[ C : H \rightarrow H \]

\[ \{a_n\}_{n=1}^{\infty} \mapsto \{ \alpha a_n\}_{n=1}^{\infty} \]

where \( \alpha \in \mathbb{R}_{+}^* \). Then \( C \) is positive invertible and

\[ C^{-1}(\{a_n\}_{n=1}^{\infty}) = \{ \alpha^{-1} a_n\}_{n=1}^{\infty}. \]

Let \((\Omega, \mu)\) the measure space where \( \Omega = [0, 1] \) and \( \mu \) is the lebesgue measure and let

\[ F : \Omega \rightarrow H \]

\[ w \mapsto F_w = \{ \frac{w}{n} \}_{n=1}^{\infty}. \]

In the author hand, consider the projection

\[ K : H \rightarrow H \]

\[ \{a_n\}_{n=1}^{\infty} \mapsto (a_1, \ldots, a_r, 0, \ldots) \]

where \( r \) is an integer \((r \geq 2)\).

It’s clair that \( K^* = K \) and for each \( f = \{a_n\}_{n=1}^{\infty} \in H = l^2(\mathbb{C}), \) one has

\[
\int_{\Omega} < f, F_w >_\mathcal{A} < CF_w, f >_\mathcal{A} d\mu(w) = \int_{[0,1]} \left\{ \frac{w}{n} a_n \right\}_{n=1}^{\infty} \cdot \left\{ \frac{\alpha w}{n} \overline{a_n} \right\}_{n=1}^{\infty} d\mu(w)
\]

\[ = \int_{[0,1]} \left\{ \frac{\alpha \overline{a_n}}{n} |a_n|^2 \right\}_{n=1}^{\infty} d\mu(w)
\]

\[ = \frac{\alpha}{3} \left\{ \frac{|a_n|^2}{n^2} \right\}_{n=1}^{\infty}. \]

Hence

\[
\int_{\Omega} < f, F_w >_\mathcal{A} < CF_w, f >_\mathcal{A} d\mu(w) \leq \frac{\alpha \pi^2}{18} < \{a_n\}_{n=1}^{\infty}, \{a_n\}_{n=1}^{\infty} >_\mathcal{A}.
\]
Furthermore,

\[
\langle CK^* f, K^* f \rangle_{A} = \langle (\alpha a_1, \ldots, \alpha a_r, 0, \ldots), (a_1, \ldots, a_r, 0, \ldots) \rangle_{A} \\
= (\alpha |a_1|^2, \ldots, \alpha |a_r|^2, 0, \ldots).
\]

Then for \( A = \frac{1}{3r^2} \), one obtain

\[
\frac{\alpha}{3r^2} (|a_1|^2, \ldots, |a_r|^2, 0, \ldots) \leq \left\{ \frac{\alpha |a_n|^2}{3n^2} \right\}_{n=1}^\infty.
\]

The conclusion is

\[
\frac{1}{3r^2} < C^{1/2} K^* f, C^{1/2} K^* f >_{A} \leq \int \langle f, F_w \rangle_{A} \langle CF_w, f \rangle_{A} \, d\mu(w) \leq \frac{\alpha \pi^2}{18} < f, f >_{A}
\]

Let \( F \) be a continuous \( C \)-controlled bessel mapping for Hilbert \( C^* \)-module \( \mathcal{H} \) over \( \mathcal{A} \).

We define the operator frame

\[
S_C : \mathcal{H} \rightarrow \mathcal{H} \text{ by,} \\
S_C f = \int \langle f, F(\omega) \rangle_{A} CF(\omega) \, d\mu(\omega).
\]

**Remark 2.4.** From definition of \( S \) and \( S_C \), we have, \( S_C = CS \).

Using [16], \( S_C \) is \( \mathcal{A} \)-linear and bounded. Thus, it is adjointable.

Since \( \langle S_C x, x \rangle_{A} \geq 0 \), for any \( x \in \mathcal{H} \), it result, again from [16], that \( S_C \) is positive and selfadjoint.

**Theorem 2.5.** Let \( \mathcal{H} \) be a Hilbert \( \mathcal{A} \)-module, \( K \in \text{End}^*_A(\mathcal{H}) \), and \( C \in \text{GL}^+(\mathcal{H}) \). Let \( F : \Omega \rightarrow \mathcal{H} \) a map. Suppose that \( CK = KC \), \( R(C^1) \subset R(K^*C^1) \) with \( R(K^*C^1) \) is closed. Then \( F \) is a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) if and only if there exist two constants \( 0 < A, B < \infty \) such that :

\[
A \| C^1 K^* f \|^2 \leq \left\| \int \langle f, F(w) \rangle_{A} \langle CF(w), f \rangle_{A} \, d\mu(w) \right\| \leq B \| f \|^2, f \in \mathcal{H}.
\]  

**Proof.** \((\Longrightarrow)\) obvious.

For the converse, we suppose that \( 0 < A, B < \infty \) such that :

\[
A \| C^1 K^* f \|^2 \leq \left\| \int \langle f, F(w) \rangle_{A} \langle CF(w), f \rangle_{A} \, d\mu(w) \right\| \leq B \| f \|^2, f \in \mathcal{H}.
\]
We have,

\[
\| \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w) \| = \| \langle Cf, f \rangle_{\mathcal{A}} \|
\]

\[
= \| \langle Csf, f \rangle_{\mathcal{A}} \|
\]

\[
= \| \langle (CS)_{\frac{1}{2}} f, (CS)_{\frac{1}{2}} f \rangle_{\mathcal{A}} \|
\]

\[
= \| (CS)_{\frac{1}{2}} f \|^2.
\]

Since, \( R(C^1) \subset R(K^*C^1) \) with \( R(K^*C^1) \) is closed, then by theorem 1.4, there exists \( 0 \leq m \) such that,

\[
(C^1)(C^1)^* \leq m(K^*C^1)(K^*C^1)^*.
\]

Thus,

\[
\langle (C^1)(C^1)^* f, f \rangle_{\mathcal{A}} \leq m\langle (K^*C^1)(K^*C^1)^* f, f \rangle_{\mathcal{A}}.
\]

Consequently,

\[
\|C^1 f\|^2 \leq m\|K^*C^1 f\|^2.
\]

Then,

\[
A\|C^1 f\|^2 \leq Am\|K^*C^1 f\|^2 \leq m\|(CS)_{\frac{1}{2}} f\|^2.
\]

Hence,

\[
\frac{A}{m} \|C^1 f\|^2 \leq \|(CS)_{\frac{1}{2}} f\|^2.
\]

So,

\[
\sqrt{\frac{A}{m}} \|C^1 f\| \leq \|(CS)_{\frac{1}{2}} f\|.
\]

(2.4)

From lemma 1.2, we have,

\[
\sqrt{\frac{A}{m}} \langle C^1 f, C^1 f \rangle_{\mathcal{A}} \leq \langle C^1 S^1 f, C^1 S^1 f \rangle_{\mathcal{A}}.
\]

Then,

\[
\langle C^1 f, C^1 f \rangle_{\mathcal{A}} \leq \frac{m}{A} \langle Cs f, f \rangle_{\mathcal{A}}.
\]

So,

\[
\langle C^1 f, C^1 f \rangle_{\mathcal{A}} \leq \frac{m}{A} \langle Cs f, f \rangle_{\mathcal{A}}.
\]
One the deduce

\[ \langle C^1K^*f, C^1K^*f \rangle_{\mathcal{A}} \leq \|K^*\|^2 \langle C^1f, C^1f \rangle_{\mathcal{A}} \leq \|K^*\|^2 \sqrt{\frac{m}{A}} \langle Scf, f \rangle_{\mathcal{A}}. \]

Hence,

\[ (2.5) \quad \frac{1}{\|K^*\|^2} \sqrt{\frac{A}{m}} \langle C^1K^*f, C^1K^*f \rangle_{\mathcal{A}} \leq \langle Scf, f \rangle_{\mathcal{A}}. \]

Since \( S_C \) is positive, selfadjoint and bounded \( \mathcal{A} \)-linear map, we can write

\[ \langle S_C^{-\frac{1}{2}}S_C^{-\frac{1}{2}}f, S_C^{-\frac{1}{2}}S_C^{-\frac{1}{2}}f \rangle_{\mathcal{A}} = \langle Scf, f \rangle_{\mathcal{A}} = \int_0^\infty \langle f, F(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w). \]

From lemma 1.3, there exists \( D > 0 \) such that,

\[ \langle S_C^{-\frac{1}{2}}S_C^{-\frac{1}{2}}f, S_C^{-\frac{1}{2}}S_C^{-\frac{1}{2}}f \rangle_{\mathcal{A}} \leq D \langle f, f \rangle_{\mathcal{A}}, \]

hence,

\[ (2.6) \quad \langle Scf, f \rangle_{\mathcal{A}} \leq D \langle f, f \rangle_{\mathcal{A}}. \]

Therefore by (2.5) and (2.6), we conclude that \( F \) is a continuous \( C \)-controlled \( K \)-frame in Hilbert \( C^* \)-module \( \mathcal{H} \) with frame bounds \( \frac{1}{\|K^*\|^2} \sqrt{\frac{A}{m}} \) and \( D \). \( \square \)

**Lemma 2.6.** Let \( C \in GL^+(\mathcal{H}). \) Suppose \( CS_C = S_C C \) and \( R(S_C^{\frac{1}{2}}) \subset R((CS_C)^{\frac{1}{2}}) \) with \( R((CS_C)^{\frac{1}{2}}) \) is closed. Then \( \|S_C^{\frac{1}{2}}f\|^2 \leq \lambda \|(CS_C)^{\frac{1}{2}}f\|^2 \) for some \( \lambda \geq 0. \)

**Proof.** By theorem 1.4, there exists some \( \lambda > 0 \) such that,

\[ (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* \leq \lambda (CS_C^{\frac{1}{2}})(CS_C^{\frac{1}{2}})^*. \]

Hence,

\[ \langle (S_C^{\frac{1}{2}})(S_C^{\frac{1}{2}})^* f, f \rangle_{\mathcal{A}} \leq \lambda \langle (CS_C^{\frac{1}{2}})(CS_C^{\frac{1}{2}})^* f, f \rangle_{\mathcal{A}}. \]

So,

\[ \|S_C^{\frac{1}{2}}f\|^2 \leq \lambda \|(CS_C^{\frac{1}{2}})f\|^2, f \in \mathcal{H}. \] \( \square \)
Theorem 2.7. Let $F : \Omega \to \mathcal{H}$ a map and $C \in GL^+(\mathcal{H})$. Suppose $CS_C = SC_C$ and $R(S_C^2) \subset R((CS_C)^2)$ with $R((CS_C)^2)$ is closed. Then $F$ is a continuous $C$-controlled Bessel mapping with bound $B$ if and only if $U : l^2(\mathcal{A}) \to \mathcal{H}$ defined by $U(\{a_w\}_{w \in \Omega}) = \int_\Omega a_wCF(w)d\mu(w)$ is well defined bounded with $\|U\| \leq \sqrt{B}\|C^{\frac{1}{2}}\|.$

Proof. Assume that $F$ is a continuous $C$-controlled Bessel with bound $B$. Hence ,

$$\|\int_{\Omega} \langle f, F(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w)\| \leq B\|f\|^2, f \in \mathcal{H}.$$ 

So,

$$\|\langle SCf, f \rangle_{\mathcal{A}}\| \leq B\|f\|^2.$$

In the beginning, we show that $U$ is well defined .

For each $\{a_w\}_{w \in \Omega} \in l^2(\mathcal{A})$,

$$\|U(\{a_w\}_{w \in \Omega})\|^2 = \sup_{f \in \mathcal{H}, \|f\|=1} \|U(\{a_w\}_{w \in \Omega}), f\|_{\mathcal{A}}\|^2$$

$$= \sup_{f \in \mathcal{H}, \|f\|=1} \|\int_{\Omega} a_wCF(w)d\mu(w), f\|_{\mathcal{A}}\|^2$$

$$= \sup_{f \in \mathcal{H}, \|f\|=1} \|\int_{\Omega} a_w\langle CF(w), f \rangle_{\mathcal{A}} d\mu(w)\|^2$$

$$\leq \sup_{f \in \mathcal{H}, \|f\|=1} \|\int_{\Omega} \langle f, CF(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w)\| \|\int_{\Omega} a_w^*a_w d\mu(w)\|$$

$$= \sup_{f \in \mathcal{H}, \|f\|=1} \|\int_{\Omega} \langle f, CF(w) \rangle_{\mathcal{A}} CF(w) d\mu(w), f\|_{\mathcal{A}}\| \|\int_{\Omega} a_w^*a_w d\mu(w)\|$$

$$= \sup_{f \in \mathcal{H}, \|f\|=1} \|\langle SCf, f \rangle_{\mathcal{A}}\| \|\int_{\Omega} a_w^*a_w d\mu(w)\|$$

$$= \sup_{f \in \mathcal{H}, \|f\|=1} \|\langle (SC)\frac{1}{2} f, (CS_C)^{\frac{1}{2}} f \rangle_{\mathcal{A}}\| \|\{a_w\}_{w \in \Omega}\|$$

$$\leq \sup_{f \in \mathcal{H}, \|f\|=1} \|\langle C\rangle^{\frac{1}{2}}\| \|\langle SCf \rangle_{\mathcal{A}}\| \|\{a_w\}_{w \in \Omega}\|^2$$

$$\leq B\|\langle C\rangle^{\frac{1}{2}}\|^2 \|\{a_w\}_{w \in \Omega}\|^2.$$

Then,

$$\|U\| \leq \sqrt{B}\|C\|^\frac{1}{2}.$$
Hence $U$ is well defined and bounded.

Now, suppose that $U$ is well defined, and

$$\|U\| \leq \sqrt{B}\|C\|^{\frac{1}{2}}.$$ 

For any $f \in \mathcal{H}$ and $\{a_w\}_{\omega \in \Omega} \in l^2(\mathcal{A})$, we have,

$$\langle f, U(\{a_w\}_{\omega \in \Omega}) \rangle_{\mathcal{A}} = \langle f, \int_{\Omega} a_w CF(w) d\mu(w) \rangle_{\mathcal{A}}$$

$$= \int_{\Omega} \langle a_w^* C f(w) \rangle_{\mathcal{A}} d\mu(w)$$

$$= \int_{\Omega} \langle C f(w) \rangle_{\mathcal{A}} a_w^* d\mu(w)$$

$$= \langle \{\langle C f(w) \rangle_{\mathcal{A}}\}_{\omega \in \Omega}, \{a_w\}_{\omega \in \Omega} \rangle_{\mathcal{A}}.$$

Then, $U$ has an adjoint, and

$$U^* f = \{\langle C f(w) \rangle_{\mathcal{A}}\}_{\omega \in \Omega}.$$ 

Also,

$$\|U\|^2 = \sup_{\|\{a_w\}_{\omega \in \Omega}\| = 1} \|U(\{a_w\}_{\omega \in \Omega})\|^2$$

$$= \sup_{\|\{a_w\}_{\omega \in \Omega}\| = 1, \|f\| = 1} \|\langle U(\{a_w\}_{\omega \in \Omega}), f \rangle_{\mathcal{A}}\|^2$$

$$= \sup_{\|\{a_w\}_{\omega \in \Omega}\| = 1, \|f\| = 1} \|\{a_w\}_{\omega \in \Omega}, U^* f \rangle_{\mathcal{A}}\|^2$$

$$= \sup_{\|f\| = 1} \|U^* f\|^2$$

$$= \|U^*\|^2$$

So,

$$\|U^* f\|^2 = \|\langle U^* f, U^* f \rangle_{\mathcal{A}}\| = \|\langle UU^* f, f \rangle_{\mathcal{A}}\| = \|\langle CS C f, f \rangle_{\mathcal{A}}\|.$$ 

Then,

$$\langle U^* f \rangle_{\mathcal{A}}^2 = \|CS C f\|^{\frac{1}{2}} f\|^{\frac{1}{2}} \leq B\|C\|^{\frac{1}{2}}\|f\|^2.$$ 

(2.7) 

$$\|U^* f\|^2 = \|\langle CS C f\rangle_{\mathcal{A}}^{\frac{1}{2}} f\|^{\frac{1}{2}} \leq B\|C\|^{\frac{1}{2}}\|f\|^2.$$ 

From lemma 2.6, we have,

$$\|\langle SC C f\rangle_{\mathcal{A}}^{\frac{1}{2}} f\|^2 \leq \lambda\|\langle CS C f\rangle_{\mathcal{A}}^{\frac{1}{2}} f\|^2,$$
for some $\lambda > 0$.

Using (2.7) we get,

$$\|(SC)^{\frac{1}{2}}f\|^2 \leq \lambda \|(CSC)^{\frac{1}{2}}f\|^2$$

$$\leq \lambda B\|C^{\frac{1}{2}}\|^2\|f\|^2.$$

Hence F is a continuous C-controlled Bessel mapping with Bessel bound $\lambda B\|C^{\frac{1}{2}}\|^2$. \hfill \qed

**Proposition 2.8.** Let $F$ be a continuous C-controlled $K$-frame for $\mathcal{H}$ with bounds $A$ and $B$. Then:

$$ACKK^* I \leq SC \leq BI.$$

**Proof.** Suppose $F$ is a continuous C-controlled $K$-frame with bounds $A$ and $B$. Then,

$$A\langle C^{\frac{1}{2}}K^* f, C^{\frac{1}{2}}K^* f \rangle_{\mathcal{H}} \leq \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{H}} \langle CF(w), f \rangle_{\mathcal{H}} d\mu(w) \leq B\langle f, f \rangle_{\mathcal{H}}.$$

Hence,

$$A\langle CKK^* f, f \rangle_{\mathcal{H}} \leq \langle SCf, f \rangle_{\mathcal{H}} \leq B\langle f, f \rangle_{\mathcal{H}}.$$

So,

$$ACKK^* I \leq SC \leq BI.$$

\hfill \qed

**Proposition 2.9.** Let $F$ be a continuous C-controlled Bessel mapping for $\mathcal{H}$, and $C \in GL^+(\mathcal{H})$. Then $F$ is a continuous C-controlled $K$-frame for $\mathcal{H}$ if and only if there exists $A > 0$ such that:

$$ACKK^* \leq CS.$$

**Proof.** ($\Rightarrow$) obvious.

($\Leftarrow$) Assume that there exists $A > 0$ such that: $ACKK^* \leq CS$,

then,

$$A\langle CKK^* f, f \rangle_{\mathcal{H}} \leq \langle SCf, f \rangle_{\mathcal{H}}.$$

Hence,

$$A\langle C^{\frac{1}{2}}K^* f, C^{\frac{1}{2}}K^* f \rangle_{\mathcal{H}} \leq \langle SCf, f \rangle_{\mathcal{H}}.$$
Therefore,
\[ A(C^1K^*f, C^1K^*f)_{\mathcal{A}} \leq \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w). \]

Hence \( F \) is a continuous \( C \)-controlled \( K \)-frame.

\[ \square \]

**Proposition 2.10.** Let \( C \in GL^+(\mathcal{H}), K \in \text{End}_{\mathcal{A}}^*(\mathcal{H}) \) and \( F \) be a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with lower and upper frames bounds \( A \) and \( B \) respectively. Suppose \( KC = CK \) and \( R(C^1) \subset R(K^*C^1) \) with \( R(K^*C^1) \) is closed. Then \( F \) is continuous \( K \)-frame for \( \mathcal{H} \) with lower and upper frames bounds \( A\|C^{-1}\|^2\|C\|^2 \) and \( B\|C^{-1}\|^2 \) respectively.

**Proof.** Assume that \( F \) is a continuous \( C \)-controlled \( K \)-frame with lower and upper frames bounds \( A \) and \( B \). From theorem 2.5, we have:
\[ A\|C^1K^*f\|^2 \leq \| \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w) \| \leq B\|f\|^2, f \in \mathcal{H}. \]

Then,
\[ A\|K^*f\|^2 = A\|C^{-1}C^1K^*f\|^2 \leq A\|C^{-1}\|^2\|C^1K^*f\|^2 \leq \|C^{-1}\|^2\| \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{A}} \langle CF(w), f \rangle_{\mathcal{A}} d\mu(w) \|. \]

So,
\[ (2.8) \quad A\|K^*f\|^2 \leq \|C^1\|^2\|\langle Sf, f \rangle_{\mathcal{A}}\|. \]

Moreover,
\[ \langle Sf, f \rangle_{\mathcal{A}} = \langle CSf, f \rangle_{\mathcal{A}} \]
\[ = \langle (CS)^{1/2}f, (CS)^{1/2}f \rangle_{\mathcal{A}} \]
\[ = \| (CS)^{1/2}f \|^2 \]
\[ \leq \|C^{1/2}\|^2\| (S)^{1/2}f \|^2 \]
\[ = \|C^{1/2}\|^2\| \langle S, (S)^{1/2}f \rangle_{\mathcal{A}} \| \]
\[ = \|C^{1/2}\|^2\| \langle f, f \rangle_{\mathcal{A}} \|, \]
then,

$$\langle Scf, f \rangle_{sA} \leq \|(C)^{1/2}\|^2 \langle Sf, f \rangle_{sA}. \tag{2.9}$$

From (2.8) and (2.9), we have,

$$A \|K^* f\|^2 \leq \|C^{-1/2}\|^2 \|(C)^{1/2}\|^2 \langle Sf, f \rangle_{sA}$$

$$= \|C^{-1/2}\|^2 \|(C)^{1/2}\|^2 \int_\Omega \langle f, F(w) \rangle_{sA} \langle F(w), f \rangle_{sA} d\mu(w).$$

Hence,

$$\|C^{-1/2}\|^{-2} \|(C)^{1/2}\|^{-2} A \|K^* f\|^2 \leq \int_\Omega \langle f, F(w) \rangle_{sA} \langle F(w), f \rangle_{sA} d\mu(w).$$

Moreover,

$$\|\int_\Omega \langle f, F(w) \rangle_{sA} \langle F(w), f \rangle_{sA} d\mu(w)\| = \|\langle Sf, f \rangle_{sA}\|$$

$$= \|\langle C^{-1}CSf, f \rangle_{sA}\|$$

$$= \|\langle (C^{-1}CS)^{1/2}f, (C^{-1}CS)^{1/2}f \rangle_{sA}\|$$

$$= \|(C^{-1}CS)^{1/2}f\|^2$$

$$\leq \|C^{-1}\|^2 \|(CS)^{1/2}f\|^2$$

$$= \|C^{-1}\|^2 \langle (CS)^{1/2}f, (CS)^{1/2}f \rangle_{sA}$$

$$= \|C^{-1}\|^2 \langle CSf, f \rangle_{sA}$$

$$\leq \|C^{-1}\|^2 B \|f\|^2.$$

Then $F$ is a continuous $K$-frame for $\mathcal{H}$ with lower and upper frames bounds $A\|C^{-1/2}\|^{-2} \|(C)^{1/2}\|^{-2}$ and $B\|C^{-1/2}\|^2$.

\[\square\]

**Proposition 2.11.** Let $C \in GL^+(\mathcal{H})$ and $K \in \text{End}_{sA}(\mathcal{H})$. We suppose that $KC = CK$, $R(C^{1/2}) \subset R(K^*C^{1/2})$ with $R(K^*C^{1/2})$ is closed and $F$ is a continuous $K$-frame for $\mathcal{H}$ with lower and upper frames bounds $A$ and $B$ respectively.

Then $F$ is continuous $C$-controlled $K$-frame for $\mathcal{H}$ with lower and upper frames bounds $A$ and $\|C\|\|S\|$.
Proof. Assume that $F$ is a continuous $K$-frame for $\mathcal{H}$ with lower and upper frames bounds $A$ and $B$. Then we have:

$$A\langle K^* f, K^* f \rangle_A \leq \int_{\Omega} \langle f, F(w) \rangle_A \langle F(w), f \rangle_A d\mu(w) \leq B\langle f, f \rangle_A,$$

Since $\langle K^* f, K^* f \rangle_A > 0$ and $\langle f, f \rangle_A > 0$ then,

$$A\|K^* f\|^2 \leq \|\int_{\Omega} \langle f, F(w) \rangle_A \langle F(w), f \rangle_A d\mu(w)\| \leq B\|f\|^2.$$

Then for every $f \in \mathcal{H}$,

$$A\|C^{1/2}K^* f\|^2 = A\|K^*C^{1/2} f\|^2 \leq \|\int_{\Omega} \langle C^{1/2} f, F(w) \rangle_A \langle F(w), C^{1/2} f \rangle_A d\mu(w)\|$$

$$= \|\int_{\Omega} \langle C^{1/2} f, F(w) \rangle_A F(w) d\mu(w), C^{1/2} f \rangle_A \|$$

$$= \|\langle C^{1/2}S f, C^{1/2} f \rangle_A \|$$

$$= \|\langle CS f, f \rangle_A \|$$

$$= \|\langle S f, Cf \rangle_A \|$$

$$\leq \|S f\| \|C f\|,$$

then

$$(2.11) \quad A\|C^{1/2}K^* f\|^2 \leq \|\langle S f, f \rangle_A \| \leq \|S\| \|C\| \|f\|^2.$$

By (2.11) and theorem 2.5, we conclude that $F$ is continuous $C$-controlled $K$-frame for $\mathcal{H}$ with lower and upper frames bounds $A$ and $\|C\|\|S\|$.

\[\square\]

**Theorem 2.12.** Let $C \in GL^+(\mathcal{H})$, and $F$ be a continuous $C$-controlled $K$-frame for $\mathcal{H}$ with bounds $A$ and $B$. Let $M, K \in \text{End}^*_{SA}(\mathcal{H})$ such that $R(M) \subset R(K)$, $R(K)$ is closed and $C$ commutes with $M^*$ and $K^*$. Then $F$ is continuous $C$-controlled $M$-frame for $\mathcal{H}$. 
Proof. Assume that $F$ be a continuous $C$-controlled $K$-frame for $\mathcal{H}$ with bounds $A$ and $B$, then,

$$A(C^{\frac{1}{2}}K^* f, C^{\frac{1}{2}}K^* f)_{\mathcal{H}} \leq \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{H}} \langle CF(w), f \rangle_{\mathcal{H}} d\mu(w) \leq B\langle f, f \rangle_{\mathcal{H}}, f \in \mathcal{H}.$$  

Since $R(M) \subseteq R(K)$, by theorem 1.4, there exists some $0 \leq \lambda$ such that

$$MM^* \leq \lambda KK^*.$$ 

Hence,

$$\langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_{\mathcal{H}} \leq \lambda \langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_{\mathcal{H}};$$

then,

$$\frac{A}{\lambda} \langle MM^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_{\mathcal{H}} \leq A \langle KK^* C^{\frac{1}{2}} f, C^{\frac{1}{2}} f \rangle_{\mathcal{H}}.$$ 

By (2.12), we have,

$$\frac{A}{\lambda} \langle M^* C^{\frac{1}{2}} f, M^* C^{\frac{1}{2}} f \rangle_{\mathcal{H}} \leq \int_{\Omega} \langle f, F(w) \rangle_{\mathcal{H}} \langle CF(w), f \rangle_{\mathcal{H}} d\mu(w) \leq B\langle f, f \rangle_{\mathcal{H}}.$$ 

Then $F$ is continuous $C$-controlled $M$-frame for $\mathcal{H}$ with bounds $\frac{A}{\lambda}$ and $B$. \qed

The following results gives the invariance of a continuous $C$-controlled Bessel mapping by an adjointable operator.

**Proposition 2.13.** Let $T \in \text{End}^*_\sigma(\mathcal{H})$ such that $TC = CT$ and $F$ be a continuous $C$-controlled Bessel mapping with bound $D$. Then $TF$ is also a continuous $C$-controlled Bessel mapping with bound $D\|T^*\|$.

Proof. Assume that $F$ is a continuous $C$-controlled Bessel mapping with bound $D$. Hence we have,

$$\int_{\Omega} \langle f, F(w) \rangle_{\mathcal{H}} \langle CF(w), f \rangle_{\mathcal{H}} d\mu(w) \leq D\langle f, f \rangle_{\mathcal{H}}, f \in \mathcal{H}. $$
We have,

\[ \int_{\Omega} \langle f, TF(w) \rangle d\mu(w) = \int_{\Omega} \langle T^* f, F(w) \rangle d\mu(w) \]

\[ \leq D \langle T^* f, T^* f \rangle \]

\[ \leq D \left\| T^* \right\|^2 < f, f >. \]

The result holds.

Now, we study the invariance of a continuous C-controlled K-frame mapping by adjointable operator.

**Theorem 2.14.** Let \( C \in \text{GL}^+(\mathcal{H}) \), and \( F \) be a continuous C-controlled K-frame for \( \mathcal{H} \) with bounds \( A \) and \( B \). If \( T \in \text{End}_A(\mathcal{H}) \) with closed range such that \( R(K^* T^*) \) is closed and \( C, K, T \) commute with each other. Then \( T F \) is a continuous C-controlled K-frame for \( R(T) \).

**Proof.** Assume that \( F \) is a continuous C-controlled K-frame with bounds \( A \) and \( B \). Then,

\[ A \langle C^{1/2} K^* f, C^{1/2} K^* f \rangle \leq \int_{\Omega} \langle f, F(w) \rangle \langle C F(w), f \rangle d\mu(w) \leq B \langle f, f \rangle, f \in \mathcal{H}. \]

Since \( T \) has a closed range, then \( T \) has Moore-Penrose inverse \( T^\dagger \) such that \( TT^\dagger T = T \) and \( T^\dagger TT^\dagger = T^\dagger \), so \( TT^\dagger_{/R(T)} = I_{R(T)} \) and \( (TT^\dagger)^* = I^* = I = TT^\dagger \).

We have,

\[ \langle K^* C^{1/2} f, K^* C^{1/2} f \rangle \leq \langle (TT^\dagger)^* K^* C^{1/2} f, (TT^\dagger)^* K^* C^{1/2} f \rangle \]

\[ = \langle (T^\dagger)^* T^* K^* C^{1/2} f, (T^\dagger)^* T^* K^* C^{1/2} f \rangle. \]

So,

\[ \langle K^* C^{1/2} f, K^* C^{1/2} f \rangle \leq \left\| (T^\dagger)^* \right\|^2 \langle T^* K^* C^{1/2} f, T^* K^* C^{1/2} f \rangle. \]

Therefore,

\[ \left\| (T^\dagger)^* \right\|^2 \langle K^* C^{1/2} f, K^* C^{1/2} f \rangle \leq \langle T^* K^* C^{1/2} f, T^* K^* C^{1/2} f \rangle. \]
Consequently, from theorem 1.4, and \( R(T^*K^*) \subset R(K^*T^*) \), there exists some \( \lambda \geq 0 \) such that,

\[
(T^*K^*C^2 f, T^*K^*C^2 f) \leq \lambda (K^*T^*C^2 f, K^*T^*C^2 f).
\]

Hence, using (2.14) and (2.15) we have,

\[
\int \Omega \langle f, TF(w) \rangle \langle CTF(w), f \rangle d\mu(w) = \int \Omega \langle T^*f, F(w) \rangle \langle TCF(w), f \rangle d\mu(w) = \int \Omega \langle T^*f, F(w) \rangle \langle CF(w), T^*f \rangle d\mu(w) \geq A \langle C^2 K^*T^*f, C^2 K^*T^*f \rangle \geq \frac{A}{\lambda} \langle T^*C^2 K^*f, T^*C^2 K^*f \rangle,
\]

then,

\[
\int \Omega \langle f, TF(w) \rangle \langle CTF(w), f \rangle d\mu(w) \geq \frac{A}{\lambda} \|(T^*)_c\|^2 \langle C^2 K^*f, C^2 K^*f \rangle.
\]

Using (2.16) and proposition 2.13, the result holds.

\[ \square \]

**Theorem 2.15.** Let \( C \in GL^*_c(\mathcal{H}) \) and \( F \) be a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \) with bounds \( A \) and \( B \).

If \( T \in End^*_{af}(\mathcal{H}) \) is a isometry such that \( R(T^*K^*) \subset R(K^*T^*) \) with \( R(K^*T^*) \) is closed and \( C, K, T \) commute with each other, then \( TF \) is a continuous \( C \)-controlled \( K \)-frame for \( \mathcal{H} \).

**Proof.** Using theorem 1.4, there exists some \( \lambda \geq 0 \) such that,

\[
\|T^*K^*C^2 f\|^2 \leq \lambda \|K^*T^*C^2 f\|^2.
\]
Assume $A$ the lower bound for the continuous $C$-controlled $K$-frame $F$ and $T$ is an isometry then,

$$
\frac{A}{\lambda} \|C^1 K^* f\|^2 = \frac{A}{\lambda} \|T^* C^1 K^* f\|^2 \\
\leq A \|K^* T^* C^1 f\|^2 \\
= A \|C^1 K^* T^* f\|^2 \\
\leq \int_{\Omega} \langle T^* f, F(w) \rangle \langle CF(w), T^* f \rangle d\mu(w) \\
= \int_{\Omega} \langle f, TF(w) \rangle \langle TCF(w), f \rangle d\mu(w),
$$

then,

$$
(2.17) \quad \frac{A}{\lambda} \|C^1 K^* f\|^2 \leq \int_{\Omega} \langle f, TF(w) \rangle \langle CTF(w), f \rangle d\mu(w).
$$

Hence, from proposition 2.13 and inequality (2.17), we conclude that $TF$ is a continuous $C$-controlled $K$-frame for $\mathcal{H}$ with bounds $\frac{A}{\lambda}$ and $B \|T^*\|^2$. □

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**