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FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS

URVASHI ARORA[1,*], SANJAY KUMAR[2]

[1]Department of Mathematics, Rajdhani College, University of Delhi, Raja Garden, New Delhi-110015, India

[2]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat-131039, Haryana, India

Abstract: The Common fixed point theorem using weakly compatible maps has been proved that satisfies generalized $\Omega -$ weak contraction condition. An application and an example validate our result.

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1. INTRODUCTION

In Banach contraction principle, the constant k was switched with a control function by Boyd and Wong [3]. Different authors have considered different control functions. .Here we use the following control function.

(∅) : Ω: $[0, \infty) \rightarrow [0, \infty)$ is non-decreasing, continuous function with $\Omega(t) < t$ for each $t > 0$. Researchers tried to relax the conditions on commutativity /minimal commutative type mappings, continuity, and contraction along with containing of range of a map into range of other. In 1996 Jungck [8] enlarged concept of compatible mappings to weakly compatible

*Corresponding author
E-mail Address: urvashi.arora@rajdhani.du.ac.in
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mappings. "For metric space(X,d), self-mappings f and g are said to be weakly compatible, whenever$fu = gu$,$u \in X$ gives $fgu = gfu$."

Now we familiarize with the new generalizedΩ −contraction for mappings pairs in the following manner:

Consider a metric space (X,d). Define self-maps A , B, S and T on a metric space (X,d) such that:

$S(X) \subset B(X), T(X) \subset A(X)$;

$$d^3(Sx,Ty) \leq \Omega\, max \begin{Bmatrix} \mathrm{d}^2(\mathrm{Ax},\mathrm{Sx})\mathrm{d}(\mathrm{By},\mathrm{Ty}), \mathrm{d}(\mathrm{Ax},\mathrm{Sx})\mathrm{d}^2(\mathrm{By},\mathrm{Ty}), \\ \mathrm{d}(\mathrm{Ax},\mathrm{Sx})\,\mathrm{d}(\mathrm{Ax},\mathrm{Ty})\,\mathrm{d}(\mathrm{By},\mathrm{Sx}), \\ \mathrm{d}(\mathrm{Ax},\mathrm{Ty})\,\mathrm{d}(\mathrm{By},\mathrm{Ty})\,\mathrm{d}(\mathrm{By},\mathrm{Sx}) \end{Bmatrix} x,y \in X,$$ where function

Ω:[0, ∞) → [0, ∞) is continuous non-decreasing such that$\Omega(t) < t$, $\forall t > 0$.

2. WEAKLY COMPATIBLE MAPPINGS

Now we give main results using generalized Ω −contraction condition as defined above. We shall denote complete metric space (X,d) by X.

Theorem 2.1. Let A ,BS,T be maps from X to X satisfying the following conditions:

(M1) If $B(X)$ contains $S(X)$, and $A(X)$contains $T(X)$;

(M2) $$d^3(Sx,Ty) \leq \Omega\, max \begin{Bmatrix} [d^2(Ax,Sx)d(By,Ty), d(Ax,Sx)d^2(By,Ty)], \\ d(Ax,Sx)\, d(Ax,Ty)\, d(By,Sx), \\ d(Ax,Ty)\, d(By,Sx)\, d(By,Ty) \end{Bmatrix}$$

for all $x,y \in X$, where function Ω : [0, ∞) → [0, ∞) is continuous non-decreasing such that $\Omega(t) < t$,$\forall t > 0$,

(M3) either of AX,X,SX ,TX is complete.

Then there exists only one fixed point of all mappings A,B, S,$T$with condition that pairs(B,T) and (A,S) are weakly compatible.

Proof. Consider an arbitrary point $x_0 \in X$. On using (M1) $x_1$can be found so that $S(x_0) = B(x_1) = y_0$ and for this x_1 one can choose $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. In this fashion construct sequences

$y_{2n} = S(x_{2n}) = B(x_{2n+1})$, $y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$for each $n \geq 0$. (2.1)

For easiness, we write $\beta_{2n} = d(y_{2n}, y_{2n+1})$

In first stage we show that the sequence $\{\beta_{2n}\}$ is a non-increasing sequence tending to zero.

Case I. Consider the case when n is even, then on putting $x = x_{2n}$ and $y = x_{2n+1}$ in (M2), we get

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq \Omega\, max \left\{ \begin{array}{c} d^2(Ax_{2n}, Sx_{2n}) d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Ax_{2n}, Sx_{2n}) d^2(Bx_{2n+1}, Tx_{2n+1}) \\ d(Ax_{2n}, Sx_{2n}) d(Ax_{2n}, Tx_{2n+1}) d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1}) d(Bx_{2n+1}, Sx_{2n}) d(Bx_{2n+1}, Tx_{2n+1}) \end{array} \right\}.$$

Using (2.1), we have

$$d^3(y_{2n}, y_{2n+1}) \leq \Omega\, max \left\{ \begin{array}{c} d^2(y_{2n-1}, y_{2n})\, d(y_{2n}, y_{2n+1}) \\ d(y_{2n-1}, y_{2n}) d^2(y_{2n}, y_{2n+1})' \\ d(y_{2n-1}, y_{2n})\, d(y_{2n-1}, y_{2n+1})\, d(y_{2n}, y_{2n}), \\ d(y_{2n-1}, y_{2n+1})\, d(y_{2n}, y_{2n})\, d(y_{2n}, y_{2n+1}) \end{array} \right\} \quad (2.2)$$

On using $\beta_{2n} = d(y_{2n}, y_{2n+1})$ in (2.2) , we have

$$\beta_{2n}^3 \leq \Omega\, max\{\beta_{2n-1}^2 \beta_{2n}, \beta_{2n-1}\beta_{2n}^2, 0,0\} \quad (2.3)$$

By using property of Ω and triangular inequality, we have

If $\beta_{2n-1} < \beta_{2n}$, then from (2.3) we get

$\beta_{2n}^3 \leq \psi\, \beta_{2n}^3$, a contradiction, therefore, $\beta_{2n} \leq \beta_{2n-1}$.

If we consider n is odd, then one gets $\beta_{2n+1} < \beta_{2n}$.

This shows that the sequence $\{\beta_{2n}\}$ is decreasing.

We consider $\lim_{n\to\infty} \beta_{2n} = l$, for some $l \geq 0$.

If $l > 0$; then from (M2), we have

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq \Omega\, max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} d^2(Ax_{2n}, Sx_{2n}) d(Bx_{2n+1}, Tx_{2n+1}) \\ + d(Ax_{2n}, Sx_{2n}) d^2(Bx_{2n+1}, Tx_{2n+1}) \end{array} \right], \\ d(Ax_{2n}, Sx_{2n}) d(Ax_{2n}, Tx_{2n+1}) d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1}) d(Bx_{2n+1}, Sx_{2n}) d(Bx_{2n+1}, Tx_{2n+1}) \end{array} \right\}.$$

Now taking limits $as\ n \to \infty$, we get

$l^3 \leq \Omega\,(l^3)$, a contradiction, therefore, we have $l = 0$.

Thus when $\lim_{n\to\infty} \beta_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n-1}) = l = 0.$ (2.4).

One can claim that $\{y_n\}$ is Cauchy.

If possible suppose sequence $\{y_n\}$ be not Cauchy in X, so for given $\epsilon > 0$, two positive integer sequences $\{\alpha(t)\}$, $\{\beta(t)\}$ can be constructed $\beta(t) > \alpha(t) > t$ with all positive integers t.

$$d\,(y_{\alpha(t)}, y_{\beta(t)}) \geq \epsilon, \qquad d\left(y_{\alpha(t)}, y_{\beta(t)-1}\right) < \epsilon \qquad (2.5)$$

We have $\epsilon \leq d\left(y_{\alpha(t)}, y_{\beta(t)}\right) \leq d\left(y_{\alpha(t)}, y_{\beta(t)-1}\right) + d(y_{\beta(t)-1}, y_{\beta(t)})$

Letting $t \to \infty$, we get $\lim_{t\to\infty} d\left(y_{\alpha(t)}, y_{\beta(t)}\right) = \epsilon.$

By triangular inequality,

$$\left|d\left(y_{\beta(t)}, y_{\alpha(t)+1}\right) - d\left(y_{\alpha(t)}, y_{\beta(t)}\right)\right| \leq d\left(y_{\alpha(t)}, y_{\alpha(t)+1}\right).$$

Proceeding as $n \to \infty$ then (2.4) and (2.5) combine to give $\lim_{t\to\infty} d\left(y_{\beta(t)}, y_{\alpha(t)+1}\right) = \epsilon.$

Again using triangular inequality, we get

$$\left|d\left(y_{\alpha(t)}, y_{\beta(t)+1}\right) - d\left(y_{\alpha(t)}, y_{\beta(t)}\right)\right| \leq d\left(y_{\beta(t)}, y_{\beta(t)+1}\right).$$

Proceeding as $t \to \infty$, with (2.4) and (2.5),

$$\lim_{t\to\infty} d\left(y_{\alpha(t)}, y_{\beta(t)+1}\right) = \epsilon.$$

Similarly by triangular inequality,

$$\left|d\left(y_{\alpha(t)+1}, y_{\beta(t)+1}\right) - d\left(y_{\alpha(t)}, y_{\beta(t)}\right)\right| \leq d\left(y_{\alpha(t)}, y_{\alpha(t)+1}\right) + d\left(y_{\beta(t)}, y_{\beta(t)+1}\right).$$

Proceeding as $t \to \infty$ and using (2.4) and (2.5), we obtain

$$\lim_{k\to\infty} d\left(y_{\beta(t)+1}, y_{\alpha(t)+1}\right) = \epsilon.$$

On setting $x = x_{\alpha(t)}$ and $y = x_{\beta(t)}$ in (M2), we find

$$d^3(Sx_{\alpha(t)}, Tx_{\beta(t)}) \leq \Omega\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Ax_{\alpha(t)}, Sx_{\alpha(t)})\, d\,(Bx_{\beta(t)}, Tx_{\beta(t)}) \\ +d(Ax_{\alpha(t)}, Sx_{\alpha(t)})\, d^2(Bx_{\beta(t)}, Tx_{\beta(t)}) \end{bmatrix}, \\ d(Ax_{\alpha(t)}, Sx_{\alpha(t)})\, d(Ax_{\alpha(t)}, Tx_{\beta(t)})\, d(Bx_{\beta(t)}, Sx_{\alpha(t)}), \\ d(Ax_{\alpha(t)}, Tx_{\beta(t)})\, d(Bx_{\beta(t)}, Sx_{\alpha(t)})\, d(Bx_{\beta(t)}, Tx_{\beta(t)}) \end{Bmatrix}$$

Using (2.1) and letting $t \to \infty$, we obtain

$\epsilon^3 \leq \Omega\, max \left\{\frac{1}{2}[0+0], 0{,}0\right\} = 0$, which contradicts $\epsilon \geq 0$. Thus in X the sequence $\{y_n\}$ is Cauchy sequence.

Assume AX to be complete subspace of X and we have$z \in X$ such that $Aw = z$. Again $\{y_n\}$ is Cauchy so it has a subsequence $\{y_{2n+1}\}$ which is convergent and sequence $\{y_n\}$ and subsequence $\{y_{2n}\}$ converge to same point. Thus we have $y_{2n} = S(x_{2n}) = B(x_{2n+1}) \to z$ as $n \to \infty$.

On setting $x = w$ and $y = z$ in (M2) we obtain

$$d^3(Sw, Tz) \leq \Omega\, max \begin{Bmatrix} [d(B\,z, T\,z)\, d^2(A\,w, S\,w) + d^2(B\,z, T\,z)\, d(A\,w, S\,w)]/2, \\ d(A\,w, T\,z)\, d(A\,w, S\,w)\, d(B\,z, S\,w), \\ d(B\,z, S\,w) d(A\,w, T\,z) d(B\,z, T\,z) \end{Bmatrix}$$

Therefore, $d^3(Sw, z) \leq \Omega\, max \begin{Bmatrix} [d(z,z)\, d^2(z, Sw) + d^2(z,z)\, d(z, Sw)]/2, \\ d(z, Sw)\, d(z,z)\, d(z, Sw), \\ d(z,z)\, d(z, Sw)\, d(z,z) \end{Bmatrix}$.

This shows Sw equals z and thus Sw = Aw = z. So w becomes a common coincidence point for mappings S and A. Asz belongs to SX and so we can find $v \in X$ so that $z = Bv$.

We claim $Tv = z$. Now putting $x = x_{2n}$ and $y = v$ in (M2), we have

$$d^3(Sx_{2n}, Tv) \leq \Omega max \begin{Bmatrix} \frac{1}{2}[d^2(Ax_{2n}, Sx_{2n}) d(Bv, Tv) + d(Ax_{2n}, Sx_{2n}) d^2(Bv, Tv)], \\ d(Ax_{2n}, Sx_{2n}) d(Ax_{2n}, Tv) d(Bz, Sx_{2n}), \\ d(Ax_{2n}, Tv) d(Bv, Sx_{2n}) d(Bv, Tv) \end{Bmatrix}.$$

Therefore,

$d^3(z, Tv) \leq \Omega\, max \begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix}$, this shows $z = Tv$ and so $z = Tv = Bv$. Therefore v is point of coincidence for B and T. (A, S) and (B, T) are pairs which are weakly compatible, so

$$Sz = S(Aw) = A(\ S\ w\ ) = A\ z$$

$$Tz = T(B\ v) = B(\ T\ v\ ) = B\ z$$

We prove $S(z) = z$ now. For this put $x = z$ and $y = x_{2n+1}$ in (M2), we find

$d^3(Sz, z) \ \le\ \Omega\, max \begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix}$. This gives $Sz = Az = z$.

Next it is asserted that $Tz = z$.Now put $x = x_{2n}$ and $y = z$ in (M2) and get $z = Tz$. So $z = Tz = Bz$. Therefore A, B, S and T have common fixed point z. The proof is similar when one considers BX or SX or TX to be complete.

Uniqueness:

Let S, T, A and B have two common fixed points $p \ne q$.

On putting $x = p$ and $y = q$ in (M2)

$d^3(Sp, Tq) \le \Omega\ max\{0,0,0\}$. This implies $p = q$.

Example 2.1 Let $X = [2\ , 20]$ having usual metric d.Let the self- maps A, B, S ,Tdefined on X be:

$$Ax = \begin{cases} 12 & if \quad 2 < x \le 5 \\ x-3 & if \quad x > 5 \\ 2 & if \quad x = 2. \end{cases}, \qquad Bx = \begin{cases} 2 & if \quad x = 2 \\ 6 & if \quad x > 2 \end{cases}$$

$$Sx = \begin{cases} 6 & if \quad 2 < x \le 5 \\ x & if \quad x = 2 \\ 2 & if \quad x > 5. \end{cases}, \quad Tx = \begin{cases} x & if \quad x = 2 \\ 3 & if \quad x > 2 \end{cases}.$$

Define a continuous, non-decreasing function Ω: [0, ∞) ⟶ [0, ∞) such that$\Omega(t) < t$, $\forall t > 0$. Consider sequence $< x_n >= 5 + \frac{1}{n}$ .It can be noted that (S, A) and (B, T) are pairs of compatible weakly maps. So the requirements of Theorem 2.1 are met, then 2 is the only fixed point which is common for all the maps S, T, A,B

On putting $S = T$ in Theorem2.1, the result follows

Corollary 2.1. Let self-maps A, B,S and T be on metric space (X ,d).

(M4) If $B(X)$ contains $S(X), A(X)$ contains $S(X)$

Also $d^3(Sx, Sy) \le \Omega\ \ max \begin{Bmatrix} [d^2(A\,x, S\,x)d(B\,y, S\,y) + d(A\,x, S\,x)d^2(B\,y, S\,y)]/2, \\ d(\,A\,x, S\,x)\ d(\,A\,x, S\,y)\ d(\,B\,y, S\,x), \\ d(\,A\,x, S\,y)\ d(\,B\,y, S\,x)\ d(\,B\,y, S\,y) \end{Bmatrix}$

for all $x, y \in X$, function Ω: [0, ∞) ⟶ [0, ∞) is continuous, non-decreasing, with $\Omega(t) < t$ for every positive t.

(M5) Assume either of subspaceBX or SXor $AX$is complete.

Then there exists unique common fixed point of S, A and B, provided pairs (A, S) and (B, S) are weakly compatible pairs.

On putting $A = B = I$ in Theorem2.1 , we get the result.

We now prove the above result for weakly compatible mappings, by excluding the condition of complete subspaces in Theorem2.1.

Theorem2.2. Let T, A, S, and B be maps fromX into itself so that (M1), (M2) are satisfied.

The there is unique common fixed point for mappings $T, S, Band\ A$, provided pairs (B,T), (A,S) are weakly compatible.

Proof. For complete metric space, a subspace is complete iff it is closed. So conclusion follows by Theorem 2.1.

3. APPLICATION

An analogous result of Banach contraction principle was given in 2002 by Branciari [3] for a map satisfying contraction condition of integral type.

Theorem 3.1. For a space (X, d) which is complete, if a map P: X → X is so that ∀ x, y in X,

$\int_0^{d(Px,Py)} \xi(t)\, dt \leq c \int_0^{d(x,y)} \xi(t)\, dt$'c' is in [0, 1), ξ : R+ → R+ is "Lebesgue-integrable over R+ function", non-negative, summable over all subsets of R+ which is compact, and so that given any positive ε , $\int_0^{\epsilon} \xi(t)dt > 0$.Then P possesses one and only one fixed point z ∈ X so that, for every x in X, $\lim_{n\to\infty} (P^{n}) = z$.

We now prove application of Theorem2.1.

Theorem3.2. Let A, B, S, T be self-maps on X so that (M1) ,(M2) , (M3) and the following :

$$\int_0^{d^3(Sx,\ Ty)} \varphi(t)\, dt \leq \int_0^{M(x,\ y)} \varphi(t)\, dt$$

$$M(x,y) = \Omega\, max \begin{Bmatrix} [d^2(Ax, Sx)\, d(By, Ty) + d(Ax,\ Sx) d^2(By, Ty)]/2, \\ d(Ax,\ Ty)\ d(Ax,\ Sx)\, d(By,\ Sx), \\ d(By,\ Sx)\ d(Ax,\ Ty)\, d(By,\ Ty) \end{Bmatrix}$$ where,

function Ω: [0, ∞) → [0, ∞) is continuous, non-decreasing, with$\Omega(t) < t$ for every positive t, for all $x, y \in X$and $\emptyset$: $[0, \infty) \to [0, \infty)$ is continuous :$\emptyset\,(t) = 0$ iff $t = 0$ and $\emptyset(t) > 0, \forall\, t > 0$. Also when φ : R+ → R+ is "Lebesgue-integrable over R+ function" that's non-negative, summable on every compact subset of R+ and so that ∀ positive ε ,

$$\int_0^{\epsilon} \varphi\,(\,t\,)\,dt \;>\; 0.$$

Then there is a unique common fixed point for A, B, S and T, provided (A,S), (B,T) are compatible pairs.

Proof. The proof follows by choosing φ (t) = 1, in Theorem3.1.

CONCLUSION

The Common fixed point theorem satisfying generalized $\Omega -$ weak contraction condition has been proved using weakly compatible maps. The result has been validated with an example and an application.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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